Recent Advances in Finite Element Methods for Structural Acoustics

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OUTLINE

**STARS3D**: Software infrastructure for modeling Structural Acoustic Radiation and Scattering in 3D.

**Part-I** Overview of STARS3D
- features, applications
- mathematical foundation
- $p$-approximations for numerically dispersive problems

**Part-II** A-posteriori Error Analysis
- sub-domain residual estimator
- effectivity indices

**Part-III** Scalable Parallelization
- stagewise concurrency, parallel multi-frontal, FETI-DP
- accuracy, scalability (efficiency) results on DoD HPC

Closing remarks and feedback
Application Areas

Structural acoustic response of elastic-fluid systems:

- Elastodynamics, eigen-analysis
- Interior noise analysis
- Radiation and scattering of waves exterior to elastic (rigid) structures
- Scattering from buried objects
- Acoustic transmission-loss modeling for sandwiched-honeycomb panels
STARS3D: Technical Capabilities

- 3D domain of arbitrary shape and complexity
- Support for multiple elastic and fluid regions
- Adaptable $hp$-finite/infinite element approximations
- Acoustic finite and infinite elements
- Perfectly Matched Layer (PML) approximations
- Linear isotropic three-dimensional elasticity
- Residual-based $a$-posteriori error estimation
- State-of-the-art parallel multi-frontal solver (NRL, MUMPS)
- Scalable domain-decomposition ($FETI$) algorithms
- Parallel execution in single and multi-frequency setting

General infrastructure supports wide range of applications.
Fluid may fill structure exterior and/or interior.
Model Problem: Strong-form

\[ \sigma_{jk,k} + \rho_s \omega^2 u_j = 0 \quad \text{in } \Omega_s \]

\[ \phi_{,kk} + \frac{\omega^2}{c^2} \phi = 0 \quad \text{in } \Omega_f, \]

\[ \sigma_{kj} n_j = h_k \quad \text{on } \Gamma_h, \]

\[ \sigma_{jk} n_k = -(\phi + \phi_0) n_j \quad \text{on } \Gamma, \]

\[ \frac{\partial (\phi + \phi_0)}{\partial n} = \rho_f \omega^2 u_k n_k \quad \text{on } \Gamma, \quad \text{and} \]

\[ \lim_{r \to \infty} r \left[ \frac{\partial \phi}{\partial r}(r \hat{e}) - i \frac{\omega}{c} \phi(r \hat{e}) \right] = 0, \quad \text{uniformly } \forall |\hat{e}| = 1. \]

\[ i = \sqrt{-1} \]
Abstract Variational Framework

\[ a_s(u, v, \phi) = b_s(v), \]
\[ a_f(\phi, \psi, u) + a_f^+(\phi^+, \psi^+) = b_f(\psi) \]

\[ B_{11}(u, v) + B_{12}(\phi, v) = L_1(v) \]
\[ B_{21}(u, \psi) + B_{22}(\phi, \psi) + B_{22}^+(\psi^+, \psi^+) = L_2(\psi) \]

<table>
<thead>
<tr>
<th></th>
<th>(B_{11})</th>
<th>(B_{22})</th>
<th>(B_{12})</th>
<th>(B_{22}^+)</th>
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<td>Interior Acoustics</td>
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<td>Elastodynamics</td>
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</tr>
</tbody>
</table>
Model Problem: Variational-form

\[ a_s(u, v) = \int_{\Omega_s} \left[ v_{j,k} C_{jklm} u_{l,m} - \omega^2 \rho_s v_k u_k \right] \, d\Omega + \int_{\Gamma} \phi n_k v_k \, d\Gamma \]

\[ a_f(\phi, \psi) = \int_{\Omega} \left[ \phi_{,k} \psi_{,k} - \frac{\omega^2}{c^2} \phi \psi \right] \, d\Omega - \omega^2 \rho_f \int_{\Gamma} \psi n_k u_k \, d\Gamma \]

\[ a_f^+(\phi^+, \psi^+) = \lim_{S \to \infty} \left( \int_{\Omega_{RS}} \left[ \nabla \phi^+ \cdot \nabla \psi^+ - \frac{\omega^2}{c^2} \phi^+ \psi^+ \right] \, d\Omega - i \frac{\omega}{c} \int_{\Gamma^+} \phi^+ \psi^+ \, d\Gamma \right) \]

\[ b_s(v) = -\int_{\Gamma} \phi_0 n_j v_j \, d\Gamma + \int_{\Gamma_h} h_j v_j d\Gamma \]

\[ b_f(\psi) = \int_{\Gamma} \frac{\partial \phi_0}{\partial n} \psi \, d\Gamma \]
Computed Quantities of Interest

- interior pressure: $\phi$
- wet-surface velocity (normal displacement): $\omega u_n$
- stress ($\sigma_{ij}$)
- far-field pressure form/pattern:

$$\phi_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\Gamma} \left[ \frac{\partial \phi}{\partial n}(y) + I k \mathbf{n}(y) \hat{x} \phi(y) \right] e^{-I k \hat{x} y} d\Gamma(y)$$

- $\Gamma$ is any closed surface enclosing $\Omega_s$,
- $|\hat{x}| = 1$
Infrastructure Overview

- Object-oriented, modular and extensible
- Supports arbitrary, 3D, geometric domains
- General-purpose Problem Solving Env. for PDE’s
Approximation Issues

Why adaptable \(hp\)-FEM?

Error control in FEM approximations:

1. \(h\)-refinement: element size subdivision, algebraic convergence, suitable near singularities and discontinuities

2. \(p\)-refinement: polynomial-degree escalation, exponential convergence, suitable for smooth solutions

3. \(hp\)-refinement: enables exponential convergence for problems with singularities/discontinuities
Approximation Improvement

$h$-refinement

$p$-refinement

$hp$-refinement

Initial approximation

$hp$-adaptability enables:

- feedback-based approximation improvement
- smarter (optimal) error control: low $p$, smaller $h$ near singularities
- preserves exponential convergence for properly designed meshes
Issues in Mid-to-High Frequency regime

How to control dispersion (*pollution*) error?

Dispersion error analysis:

- **wavenumber**: \( k = \frac{2\pi}{\lambda} \)

- **linear** approximation: \( hk < 1 \Rightarrow \frac{1}{h} > \frac{2\pi}{\lambda} \)
  \( \approx 10 \) elements per wavelength

- **high-order** approximation: \( \frac{hk}{2p} < 1 \Rightarrow \frac{1}{h} > \frac{\pi}{p\lambda} \)
  fewer elements per wavelength
**Exponential $p$-convergence: Interior Problem**
Application: ATC Eigen Analysis
Application: Interior acoustics

Point load

Air

Vacuum

SONAX: 50 Hz

250 Hz

STARS3D: 50 Hz

250 Hz
Applications: Honeycomb Beam Predictive Validation

Courtsey: NASA Langley Research Center

Colors not to same scale

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Application: Acoustics Transmission Loss

Model

Frequency-sweep validation against analytic (plate-theory) result

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$p$-Hierarchic Basis Functions

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Topology-Based basis function decomposition

\[ \Phi * \psi = N \]

- \( N = \psi(\xi) \ast \phi(\xi') \)
- \( \phi(\xi') \) associated only with mesh entity (V,E,F,R)
- \( \psi(\xi) \) blends \( \phi(\xi') \) over the element domain
- \( \xi \) - element coordinates, \( \xi' = f(\xi) \) - entity coordinates
Advantages of Topology-Based decomposition

- totally flexible degree specification; $C^0$ by construction
- anisotropic, variable-order $p$-adaptivity
- easy addition of: new basis ($\phi$), new elements ($\psi$)
- independent ($p_1, p_2, p_3$) along ($v_1, v_2, v_3$)

*Flexible, unconstrained $p$-adaptation!*
**A-posteriori error estimation**

- **three-dimensional**, interior acoustics
- **subdomain-based residual estimator**
- estimates of error in $\phi$ and $u_r$ in global $L_2$ and $H^1$ norms
- **effectivity indices** as a function of $ka$ and $p$
**p-Error Analysis Framework**

Primary problem:

\[ \mathcal{B}_{11}(u_j^{(ps)}, v_j) + \mathcal{B}_{12}(\phi(p_f), v_j) = \mathcal{L}_1(v_j) \]
\[ \mathcal{B}_{21}(u_j^{(ps)}, \psi) + \mathcal{B}_{22}(\phi(p_f), \psi) = \mathcal{L}_2(\psi) \]

Define error: \[ e_{u_j} = u_j - u_j^{(ps)} \] and \[ e_{\phi} = \phi - \phi(p_f) \]

Residual equations:

\[ \mathcal{B}_{11}(e_{u_j}, v_j) + \mathcal{B}_{12}(e_{\phi}, v_j) = \mathcal{L}_1(v_j) - \mathcal{B}_{11}(u_j^{(ps)}, v_j) - \mathcal{B}_{12}(\phi(p_f), v_j) \]
\[ \mathcal{B}_{21}(e_{u_j}, \psi) + \mathcal{B}_{22}(e_{\phi}, \psi) = \mathcal{L}_2(\psi) - \mathcal{B}_{21}(u_j^{(ps)}, \psi) - \mathcal{B}_{22}(\phi(p_f), \psi) \]
Global Residual Estimator: GRE

\[
\mathcal{B}_{11}(e_{u_j}^{p_s'}, v_j) + \mathcal{B}_{12}(e_{\phi}^{p_f'}, v_j) = \mathcal{L}_1(v_j) - \mathcal{B}_{11}(u_j^{(p_s)}, v_j) - \mathcal{B}_{12}(\phi^{(p_f)}, v_j)
\]

\[
\mathcal{B}_{21}(e_{u_j}^{p_s'}, \psi) + \mathcal{B}_{22}(e_{\phi}^{p_f'}, \psi) = \mathcal{L}_2(\psi) - \mathcal{B}_{21}(u_j^{(p_s)}, \psi) - \mathcal{B}_{22}(\phi^{(p_f)}, \psi)
\]

\[
\mathcal{E}_{u_j,*}(\Omega_s) \overset{\text{def}}{=} \sqrt{\sum_{\tau \in \Delta_h} \| e_{u_j}^{h,p_s',q_s'} \|_{*(\tau)}^2}, \quad \mathcal{E}_{\phi,*}(\Omega_f) \overset{\text{def}}{=} \sqrt{\sum_{\tau \in \Delta_h} \| e_{\phi}^{h,p_f'} \|_{*(\tau)}^2}
\]

- **enriched-subspace**: \( p'_s > p_s \) and \( p'_f > p_f \)
- \( p'_s \to \infty \), and \( p'_f \to \infty \) \( \Rightarrow \) exact error
- **expensive**, but useful verification tool
Subdomain Residual Estimator: \textit{SRE}

Definitions:

- $\varphi_X$ is global piecewise linear basis for vertex $X$,

- support for $\varphi_X$:
  \[ \omega_X = \text{supp}(\varphi_X) = \bigcup_{\tau \in \Delta_h, X \in \partial \tau} \tau \]

- in 3D, $\omega_X$ is the closure of mesh regions connected vertex to $X$

- $\varphi_X$ form a \textit{partition of unity} and vanish on $\partial \omega_X$

- natural setup for a \textit{Dirichlet-type estimator}
Subdomain Residual Estimator: SRE

\[ \mathcal{U}_{0}^{h,p_s',q_s'}(\mathcal{W}_{X}) \overset{\text{def}}{=} \left\{ v_{j} \in \mathcal{U}^{h,p_s',q_s'}(\mathcal{W}_{X}) \mid v_{j}|_{\partial \mathcal{W}_{X,D}} = 0 \right\} \]

\[ \Phi_{0}^{h,p_f'}(\mathcal{W}_{X}) \overset{\text{def}}{=} \left\{ \psi \in \Phi^{h,p_f'}(\mathcal{W}_{X}) \mid \psi|_{\partial \mathcal{W}_{X,D}} = 0 \right\} \]

\[ \mathcal{B}_{11}(\hat{e}_{u_{j}}, v_{j}) + \mathcal{B}_{12}(\hat{e}_{\phi}, v_{j}) = \mathcal{L}_{1}(v_{j}) - \mathcal{B}_{11}(u_{j}^{(p_s)}, v_{j}) - \mathcal{B}_{12}(\phi^{(p_f)}, v_{j}) \]

\[ \mathcal{B}_{21}(\hat{e}_{u_{j}}, \psi) + \mathcal{B}_{22}(\hat{e}_{\phi}, \psi) = \mathcal{L}_{2}(\psi) - \mathcal{B}_{21}(u_{j}^{(p_s)}, \psi) - \mathcal{B}_{22}(\phi^{(p_f)}, \psi) \]

\[ \mathcal{E}^{SRE}_{u_{j},*}(\Omega_{s}) \overset{\text{def}}{=} \sqrt{\sum_{X} \sum_{\tau \in \mathcal{W}_{X}} \| \hat{e}_{u_{j}} \|_{*}^{2}(\tau)} \]

\[ \mathcal{E}^{SRE}_{\phi,*}(\Omega_{f}) \overset{\text{def}}{=} \sqrt{\sum_{X} \sum_{\tau \in \mathcal{W}_{X}} \| \hat{e}_{\phi} \|_{*}^{2}(\tau)} \]
Error estimation: Numerical results

- semi-analytic reference solution\(^1\) \(u_{r,EX}\) and \(\phi_{EX}\)

- effectivity indices

\[
\eta_{u_r,L^2(\Omega_s)}^{EST} = \frac{\mathcal{E}_{u_r,L^2(\Omega_s)}^{EST}}{||u_{r,EX} - u_{r,h,p_s,q_s}||_{L^2(\Omega_s)}}
\]

\[
\eta_{\phi,L^2(\Omega_f)}^{EST} = \frac{\mathcal{E}_{\phi,L^2(\Omega_f)}^{EST}}{||\phi_{EX} - \phi_{(p_f)}||_{L^2(\Omega_f)}}
\]

\(^1\)Thanks to Dr. J. J. Shirron
Resonances in frequency response

- Input data:

\[ c = 320 \text{ms}^{-1}, \quad \rho_s = 7800K \text{gm}^{-3}, \]

\[ \rho_f = 1.25K \text{gm}^{-3}, \quad E = 2.0 \times 10^{11}N \text{m}^{-2}, \quad \nu = 0.3. \]
Effectivity Indices for GRE

<table>
<thead>
<tr>
<th>$k\alpha$</th>
<th>Pressure</th>
<th>Radial displacement</th>
<th>$k\alpha$</th>
<th>Pressure</th>
<th>Radial displacement</th>
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<tr>
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<td>0.9968</td>
<td>9</td>
<td>1.0418</td>
<td>0.6853</td>
</tr>
<tr>
<td>2</td>
<td>0.8030</td>
<td>0.9922</td>
<td>10</td>
<td>0.9242</td>
<td>0.6813</td>
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<tr>
<td>3</td>
<td>0.7721</td>
<td>0.9795</td>
<td>11</td>
<td>0.6510</td>
<td>0.1413</td>
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<tr>
<td>4</td>
<td>0.9163</td>
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<tr>
<td>5</td>
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<td>0.9786</td>
<td>13</td>
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<td>0.6371</td>
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<tr>
<td>6</td>
<td>0.8590</td>
<td>0.9842</td>
<td>14</td>
<td>1.9303</td>
<td>0.0730</td>
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<td>8</td>
<td>0.5115</td>
<td>0.2937</td>
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</tbody>
</table>

- Numbers in red indicate $k\alpha$ is near resonant frequency
Effectivity Indices for Pressure using SRE

<table>
<thead>
<tr>
<th>$ka$</th>
<th>$p_f = 2$</th>
<th>$p_f = 3$</th>
<th>$ka$</th>
<th>$p_f = 2$</th>
<th>$p_f = 3$</th>
</tr>
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<tbody>
<tr>
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<td>0.3946</td>
<td>9</td>
<td>1.4693</td>
<td>0.8138</td>
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<tr>
<td>2</td>
<td>0.5990</td>
<td>0.6848</td>
<td>10</td>
<td>0.9220</td>
<td>1.0748</td>
</tr>
<tr>
<td>3</td>
<td>0.6846</td>
<td>0.4926</td>
<td>11</td>
<td>0.1131</td>
<td>0.0696</td>
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<tr>
<td>4</td>
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<td>1.1176</td>
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</table>
Effectivity Indices for Radial displacement using SRE

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<th>$p_s = 2$</th>
<th>$p_s = 3$</th>
<th>$ka$</th>
<th>$p_s = 2$</th>
<th>$p_s = 3$</th>
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<td>0.2424</td>
<td>0.0927</td>
<td></td>
<td></td>
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</table>
Brief Aside: Mesh Representation

**AMD**: Adaptable Mesh Database

- non-manifold, polymorphic topology
- on-the-fly application-adaptive topological adjacencies
- high-order geometry representation
- partitioned meshes
- mesh entities mapped to geometric and partition models
- tools to convert legacy (element-node) representations
Brief Aside: Mesh Partitioning

Alternative partitioning: greedy, octree-based, recursive-bisection-based etc.
AMD Mesh Database Example
NASA ATC model (converted from Patran NTL)
17664 mesh regions

Thin plate with inclusion: 9000 mesh regions

4-way partition (METIS)

8-way partition (METIS)

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AMD Mesh Database Example
Mock submarine: 19695 mesh regions

Geometric Model

2-Way Partition

4-Way Partition

8-Way Partition

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STARS3D: Scheme of Computation

**Stage-1**  Compute (cache) element-level matrices/vectors.

**Stage-2**  Assemble and solve global system of equations.

**Stage-3**  Post-process user-requested quantities of interest.

**Single-frequency**  Execute **Stage-1,2,3** once.

**Multi-frequency**  Independent execution of **Stage-1,2,3** for each frequency.
  
  * embarassingly parallel
  
  * Stage-1 cache reusable for fixed $hp$
Each partition:

- Creates its own approximation (elements)
- Computes and caches element integrals for its approximation
- Naturally high scalability
**STARS3D: Stage-2 Computation**

- **Integral Cache**
- **Solve Linear System**
- **Solution Cache**

\[ A \mathbf{x} = \mathbf{b} \]

- Unpartitioned
- Partitioned

- Assembly process deals with (multiple) partition cache(s)
- Linear solve: highly irregular, data-dependent
- Challenging for scalable parallelization
STARS3D: Stage-3 Computation

Example

- Farfield projection
- Norm (error) computation
- Field output

Partitioned domain

- Global integrals broken into sum of partition integrals
- Processors compute for their assigned partition
- Global-reduce operation gets the final result
Parallel Scheme: Multi-Frequency

- Naturally high scalability
- Can use multi-frontal or FETI-DP for Stage-2
Single Frequency Parallel Multi-Frontal Performance

- Stage 1 and Stage 3 scales well
- Stage 2: Matrix-factoring highly irregular; rapidly decreasing concurrent work-load
- Parallel multi-frontal has limited scalability (4 to 8 processors)
- Example: 1.2M dofs, seepdup (efficiency): 3.4/4 (85%), 4.8/8 (60%)
- Utilize scalable DD method: FETI-DP
FETI-DP

- Scalable domain decomposition scheme
- Independent solve of sub-domain problems (“Tearing” phase)
- Sub-domain solutions “interconnected” by Lagrange multipliers
- Two (Fine and Coarse) level iterative-substructuring scheme
  - Coarse problem by defining “corner” DOF at a global level
- Augmented FETI-DP
  - Enforce optional constraint on residuals at each iteration step

FETI-DP: The recipe itself is algebraic in nature

*Large body of literature on FETI and its variants*
FETI-DP Formulation

Subdomain equations:

\[ K^s u^s = f^s \Rightarrow \begin{bmatrix} K^s_{rr} & K^s_{rc} \\ K^s_{cr} & K^s_{cc} \end{bmatrix} \begin{bmatrix} u^s_r \\ u^s_c \end{bmatrix} = \begin{bmatrix} f^s_r \\ f^s_c \end{bmatrix} \]
FETI-DP Formulation (contd.)

Subdomain interface continuity condition:

\[ u_b^m - u_b^n = 0 \quad \text{on } \partial \Omega^m \cap \partial \Omega^n \quad \Rightarrow \quad \sum_{s=1}^{s=N} B_r^s u_r^s = 0 \]

where \( B_r^s \) is a signed boolean matrix such that \( B_r^s u_r^s = \pm u_b^s \)

- Enforce continuity at the subdomain interfaces by Lagrange Multipliers
FETI-DP Formulation (contd.)

Interface problem by eliminating $u^s_r$ and $u^s_c$:

$$
\begin{align*}
\left( \underbrace{F_{rr}}_{\text{Fine}} + \underbrace{F_{rc}K_{cc}^{-1}F_{rc}^T}_{\text{Coarse}} \right) \lambda &= d_r - F_{rc}K_{cc}^{-1}f_c
\end{align*}
$$

where

$$
\begin{align*}
F_{rr} &= \sum_{s=1}^{s=N} B_r^s K_{rr}^s B_r^s, \quad F_{rc} = \sum_{s=1}^{s=N} B_r^s K_{rc}^s K_{rr}^s B_c^s, \\
K_{cc}^* &= \sum_{s=1}^{s=N} B_c^s K_{cc}^s B_c^s - (K_{rc}^s B_c^s)^T K_{rr}^s K_{rc}^s B_c^s, \\
d_r &= \sum_{s=1}^{s=N} B_r^s K_{rr}^s f_r^s, \quad f_c^* = f_c - \sum_{s=1}^{s=N} B_c^s K_{rc}^s K_{rr}^s f_r^s
\end{align*}
$$
FETI-DP in STARS3D

- Works with higher order hierarchical basis and infinite elements
  - Lagrange multipliers enforce matching of coefficients of $p$-approximation
- Sparse multi-frontal solver to factorize subdomain matrices $K_{rr}^s$
- GMRES and GCR for iterative solution of the interface problem
- Lumped preconditioner: $\sum_{s=1}^{N} B_r^s K_{bb}^s B_r^s T$
- MPI-based implementation

First known application of FETI-DP to $p$-finite+infinite element approximations in 3D
CHSSI Problem Set Overview

Single Frequency:

**Problem 1** Exterior scattering from smooth elastic cylindrical shell
**Problem 2** Exterior scattering from stiffened elastic cylindrical shell

Multiple Frequency:

**Problem 3** Exterior scattering from smooth elastic spherical shell
**Problem 4** Interior acoustics of fluid-filled elastic spherical shell
**Problem 5** Interior acoustics of fluid-filled elastic cylindrical shell

<table>
<thead>
<tr>
<th>Test problem</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td>Problem type</td>
<td>exterior</td>
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<td>exterior</td>
<td>interior</td>
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<tr>
<td>Excitation</td>
<td>plane wave</td>
<td>plane wave</td>
<td>plane wave</td>
<td>elastic traction</td>
<td>elastic traction</td>
</tr>
<tr>
<td>Reference solution</td>
<td>numerical</td>
<td>numerical</td>
<td>analytical</td>
<td>analytical</td>
<td>numerical</td>
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DoD HPC Platforms Used

<table>
<thead>
<tr>
<th></th>
<th>SGI Altix 3900</th>
<th>SGI Origin 3800</th>
<th>Linux-Cluster</th>
<th>IBM p690 SP</th>
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<tbody>
<tr>
<td>OS</td>
<td>GNU/Linux</td>
<td>IRIX 6.5</td>
<td>GNU/Linux</td>
<td>AIX</td>
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<tr>
<td>Processors</td>
<td>IA64</td>
<td>MIPS R14000</td>
<td>IA32</td>
<td>Power 4+</td>
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<tr>
<td>Memory</td>
<td>Shared</td>
<td>Shared</td>
<td>Distributed</td>
<td>Distributed</td>
</tr>
<tr>
<td>Compilers</td>
<td>GNU</td>
<td>MIPS</td>
<td>GNU</td>
<td>IBM</td>
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<tr>
<td></td>
<td>M1</td>
<td>M2</td>
<td>M3</td>
<td>M4</td>
</tr>
</tbody>
</table>

**STARS3D** easily ports to any platform that has:

- *Unix*-like OS, GNU-make, bash
- ANSI C, F77 and (optionally) F90 compilers
- MPI and (optionally) OpenMP support
## Multi-frequency (multi-frontal) Parallel Scalability

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>2.0 (1.00)</td>
<td>2.0 (1.00)</td>
<td>2.0 (1.00)</td>
<td>2.0 (1.00)</td>
</tr>
<tr>
<td>4</td>
<td>3.8 (0.95)</td>
<td>3.3 (0.81)</td>
<td>4.0 (1.00)</td>
<td>3.4 (0.84)</td>
</tr>
<tr>
<td>8</td>
<td>7.5 (0.94)</td>
<td>6.7 (0.83)</td>
<td>7.6 (0.95)</td>
<td>7.5 (0.93)</td>
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<tr>
<td>16</td>
<td>14.2 (0.87)</td>
<td>16.1 (1.00)</td>
<td>15.9 (0.99)</td>
<td>14.3 (0.90)</td>
</tr>
<tr>
<td>32</td>
<td>19.9 (0.62)</td>
<td>20.3 (0.63)</td>
<td>29.4 (0.92)</td>
<td>23.4 (0.73)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.0 (1.00)</td>
<td>2.0 (1.00)</td>
<td>2.0 (1.00)</td>
<td>2.0 (1.00)</td>
</tr>
<tr>
<td>4</td>
<td>4.0 (1.00)</td>
<td>3.9 (0.97)</td>
<td>2.8 (0.69)</td>
<td>3.4 (0.84)</td>
</tr>
<tr>
<td>8</td>
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<td>7.8 (0.97)</td>
<td>4.8 (0.60)</td>
<td>6.5 (0.81)</td>
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<tr>
<td>16</td>
<td>15.9 (0.99)</td>
<td>15.0 (0.94)</td>
<td>14.3 (0.89)</td>
<td>12.1 (0.75)</td>
</tr>
<tr>
<td>32</td>
<td>31.0 (0.97)</td>
<td>29.0 (0.91)</td>
<td>18.7 (0.58)</td>
<td>24.0 (0.75)</td>
</tr>
</tbody>
</table>

Speedup (Efficiency)
## Single-frequency Parallel Scalability (FETI-DP)

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>P1 M1</th>
<th>M3</th>
<th>M4</th>
<th>P2 M1</th>
<th>M3</th>
<th>M4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.00 (1.00)</td>
<td>X</td>
<td>X</td>
<td>1.00 (1.00)</td>
<td>X</td>
<td>1.00 (1.00)</td>
</tr>
<tr>
<td>4</td>
<td>1.05 (1.04)</td>
<td>1.00</td>
<td>1.00 (1.00)</td>
<td>1.04 (1.01)</td>
<td>X</td>
<td>0.95 (0.84)</td>
</tr>
<tr>
<td>8</td>
<td>0.96 (0.86)</td>
<td>1.09</td>
<td>0.79 (0.81)</td>
<td>0.91 (0.91)</td>
<td>X</td>
<td>0.66 (0.60)</td>
</tr>
<tr>
<td>16</td>
<td>0.84 (0.81)</td>
<td>1.32</td>
<td>0.62 (0.63)</td>
<td>0.75 (0.74)</td>
<td>X</td>
<td>0.50 (0.48)</td>
</tr>
<tr>
<td>32</td>
<td>0.66 (0.65)</td>
<td>0.90</td>
<td>0.52 (0.52)</td>
<td>0.47 (0.46)</td>
<td>X</td>
<td>0.31 (0.26)</td>
</tr>
</tbody>
</table>

- fixed total work (32 partitions)
- ’X’: lack of enough memory per node
Closing Remarks

- STARS3D: general purpose, yet efficient for linear PDE’s
- \( p \)-version superior for wave-dominated problems
- Error-analysis a must for reliable verification
- Domain-decomposition (FETI-DP) type approach leads to better scalability
- Ongoing and future efforts: transient problems, adaptivity
- new application/interests: seismic-acoustics, Schrödinger-equation


Contact: dey [at] pa.nrl.navy.mil, 202-767-7321 Thank you! Questions?