Fast Solvers for Models of Fluid Flow

P. Aaron Lott

MCSD - NIST, Jeff McFadden
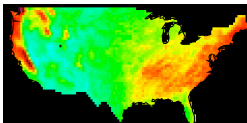AMSC - UMCP, Howard Elman & Anil Deane
MATH - USM, Temple Fay
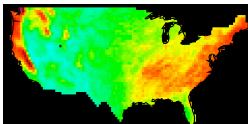
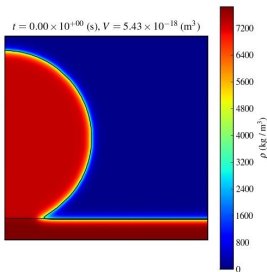Spiral Presentation
June 3, 2009

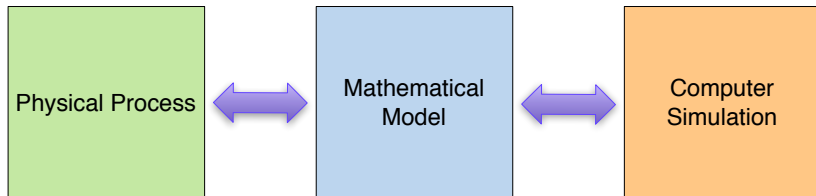# Applications
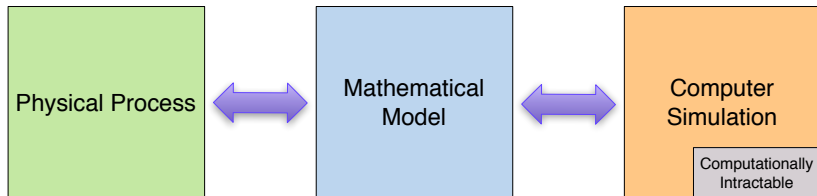
Applications

Prediction

Image by Duffy

Manufacturing Processes
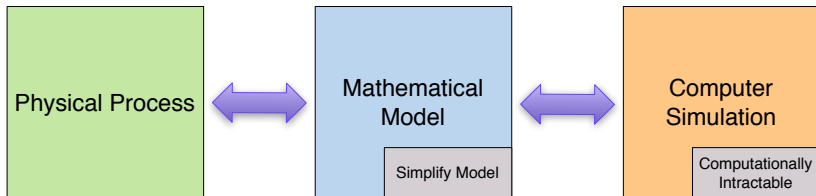
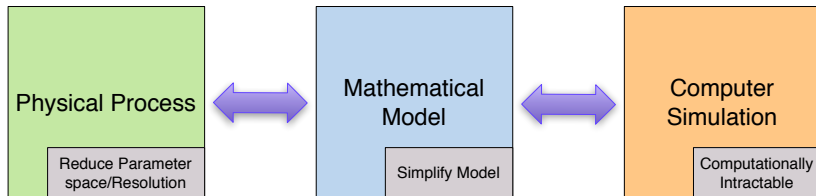Image by Wheeler

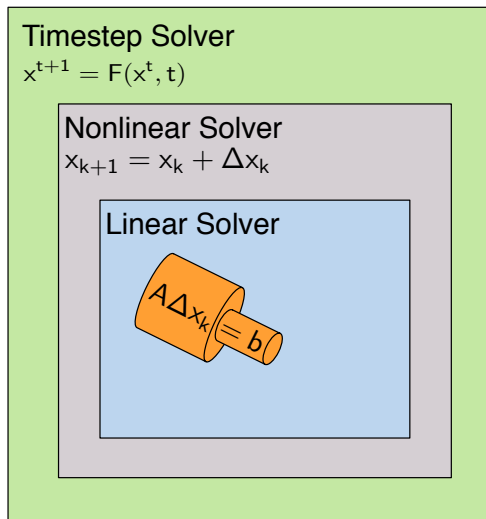# Modeling-Simulation Framework

Modeling-Simulation Framework

Modeling-Simulation Framework

Modeling-Simulation Framework

Bottleneck in Computational Simulation

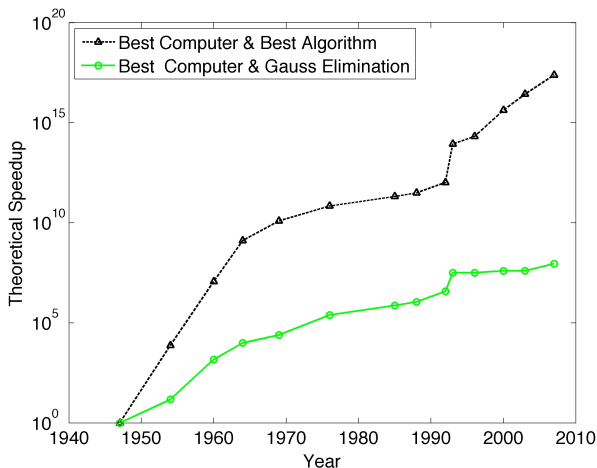# How fast can we solve $Ax = b$?

Figure: Evolution of Machines & Algorithms for solving 3D Poisson Equation (Adapted from Deville)
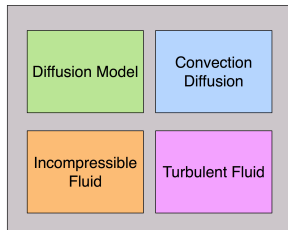
What is scalable?

Figure: More difficult problems
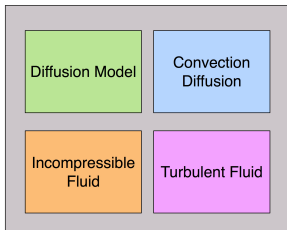
What is scalable?

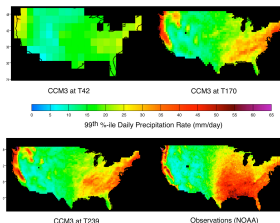
Figure: More difficult problems

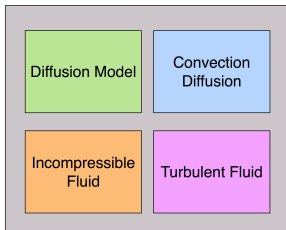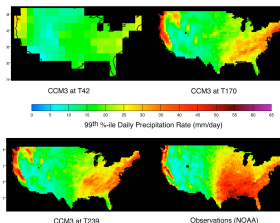Figure: At increased resolution

What is scalable?

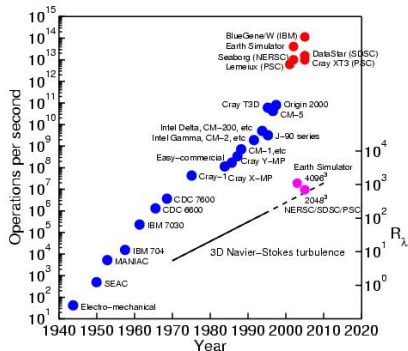Figure: More difficult problems

Figure: At increased resolution

Figure: Solved efficiently as computational power increases. (Images by Duffy & Donzis)

6 / 15

Motivation   Solving $Ax = b$ Close
$\circ\circ\circ\circ\circ$ $\bullet\circ$ $\circ\circ$
                                              $\circ\circ\circ\circ\circ$
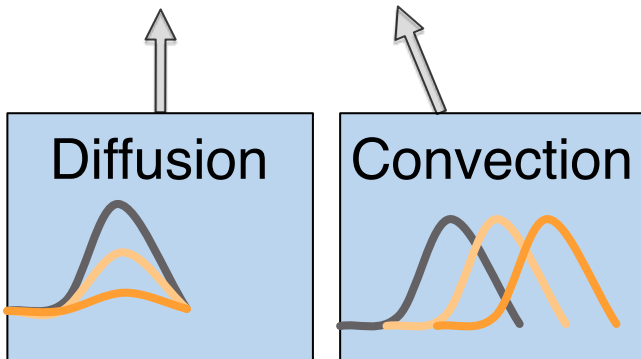
Preconditioning

Use structure of discrete problem to our advantage
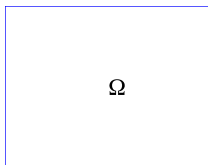
- Parallelism
- Similarities
- Approximations

Design P so that $AP^{-1}Px = b$ is cheaper than $Ax = b$.
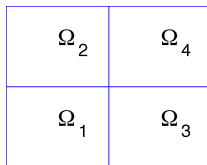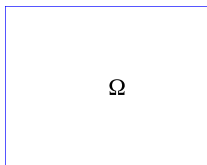
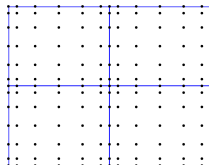## Example: Convection-Diffusion
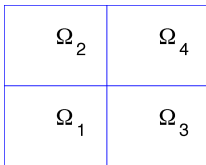
$$- \nabla^2 u + \tilde{w} \cdot u = f$$
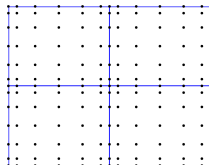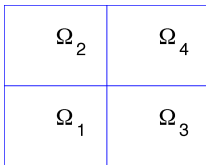
Discretize

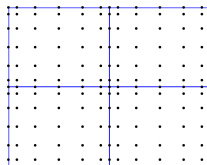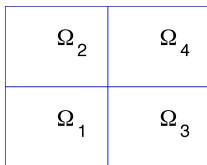# Discretize

Discretize

Discretize

Parallelism

Discretize

Parallelism via Dense interior and Sparse element boundaries

Discrete System operator

$$A = (B \otimes C) + (C \otimes B)$$

$$C \otimes B = \begin{pmatrix} c_{11}B & c_{12}B & \ldots & c_{1l}B \\ c_{21}B & c_{22}B & \ldots & c_{2l}B \\ \vdots & \vdots & & \vdots \\ c_{k1}B & c_{k2}B & \ldots & c_{kl}B \end{pmatrix}.$$

Discrete System operator

$$A = (B \otimes C) + (C \otimes B)$$

$$C \otimes B = \begin{pmatrix} c_{11}B & c_{12}B & \ldots & c_{1l}B \\ c_{21}B & c_{22}B & \ldots & c_{2l}B \\ \vdots & \vdots & & \vdots \\ c_{k1}B & c_{k2}B & \ldots & c_{kl}B \end{pmatrix}.$$

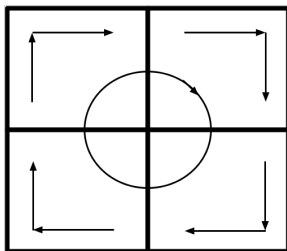Similarity

Discrete System operator

$$A = (B \otimes C) + (C \otimes B)$$

$$C \otimes B = \begin{pmatrix} c_{11}B & c_{12}B & \dots & c_{1l}B \\ c_{21}B & c_{22}B & \dots & c_{2l}B \\ \vdots & \vdots & & \vdots \\ c_{k1}B & c_{k2}B & \dots & c_{kl}B \end{pmatrix}.$$
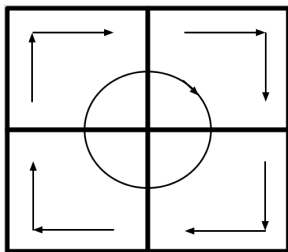
Similarity in Tensor Product Structure

Approximate Model

$$-\nabla^2 u + \underbrace{\vec{w}}_{wind} \cdot \nabla u$$

Approximate Model

$$-\nabla^2 u + \underbrace{\vec{w}}_{wind} \cdot \nabla u$$
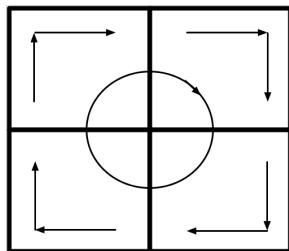
Approximation

Approximate Model
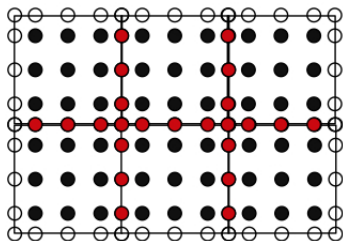
$$-\nabla^2 u + \underbrace{\vec{w}}_{\text{wind}} \cdot \nabla u$$

Approximation gives us fast

inverse operation on each
element

$$\bar{A}^e{}_{\text{interior}}^{-1} = \underbrace{\tilde{M}(V_y \otimes V_x)(\Lambda_y \otimes I + I \otimes \Lambda_x)^{-1}(V_y^{-1} \otimes V_x^{-1})\tilde{M}}_{\text{Diagonalized via } 1D \text{ operators!}}$$

Preconditioning Algorithm

$$
\begin{bmatrix}
\bar{A}_{II}^1 & 0 & \ldots & 0 & \bar{A}_{I\Gamma}^1 \\
0 & \bar{A}_{II}^2 & 0 & \ldots & \bar{A}_{I\Gamma}^2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \bar{A}_{II}^E & \bar{A}_{I\Gamma}^E \\
0 & 0 & \ldots & 0 & \bar{A}_S
\end{bmatrix}
\begin{pmatrix}
u_{I1} \\
u_{I2} \\
\vdots \\
u_{IE} \\
u_{\Gamma}
\end{pmatrix}
=
\begin{pmatrix}
\hat{b}_{I1} \\
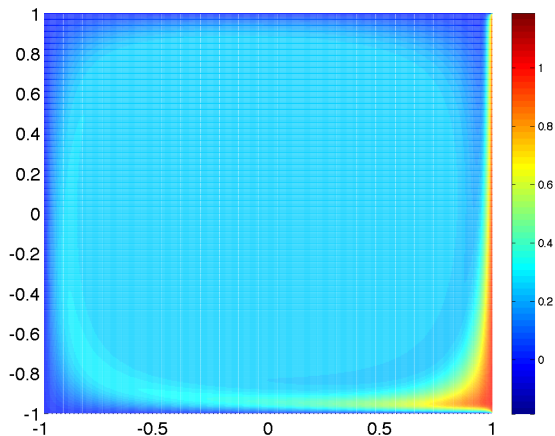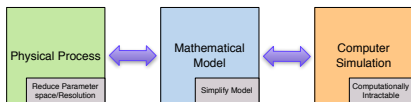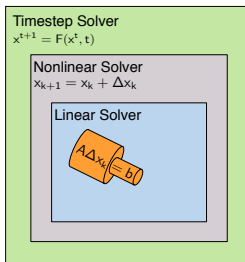\hat{b}_{I2} \\
\vdots \\
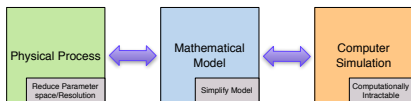\hat{b}_{IE} \\
g_{\Gamma}
\end{pmatrix}.
$$

Sample Results

Figure: Computed Solution for Double Glazing Problem

Recap

Recap

Recap
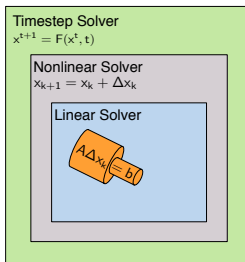
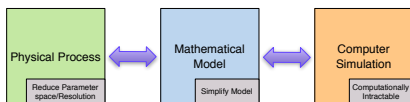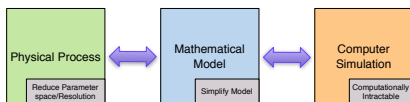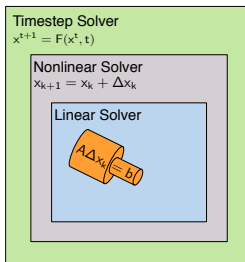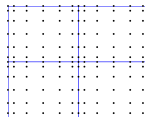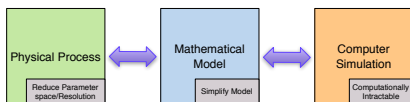$$AP^{-1}Px = b$$

Recap

Parallelism

$$AP^{-1}Px = b$$

Recap

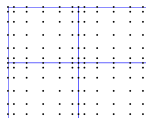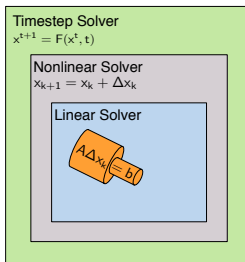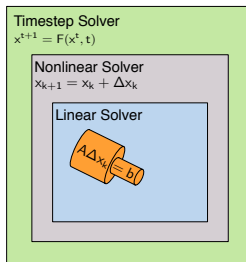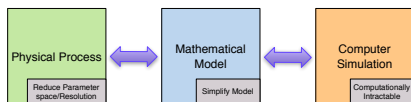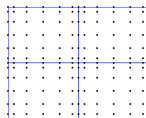Parallelism

Similarity

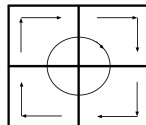$$(B \otimes A) + (A \otimes B)$$

$$AP^{-1}Px = b$$

Recap

Parallelism

Similarity

$$(B \otimes A) + (A \otimes B)$$

Approximation

$$AP^{-1}Px = b$$

Current Work

- Convection in Binary Alloys with Jeff McFadden @ NIST
- Reactive Wetting with Daniel Wheeler @ NIST
- Climate Model with Kate Evans @ ORNL