Fast Solvers for Models of Fluid Flow

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Background

Computational Methods for Fluid Flow
Need to efficiently compute steady flow states to enable

- Implicit time stepping strategies
- Improved stability analysis
- Classification of flow bifurcations

Fluid Models

Incompressible Navier Stokes

$$\frac{\partial \vec{u}}{\partial t} - \nu\nabla^2\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = f \quad \text{in} \quad \Omega,$$
$$\nabla\cdot\vec{u} = 0 \quad \text{in} \quad \Omega.$$

Advection-Diffusion

$$-\nabla^2 u + (\vec{w}\cdot\nabla)u = g$$

Viscous and Inertial forces occur on disparate scales lead to sharp flow features which:

- require fine numerical grid resolution
- cause poorly conditioned non-symmetric system.

Spatial Discretization

Spectral Element Method
On each element, the solution is expressed via a nodal basis

$$u_e^N(x,y) = \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} u_{ij}\pi_i(x)\pi_j(y). \tag{1}$$

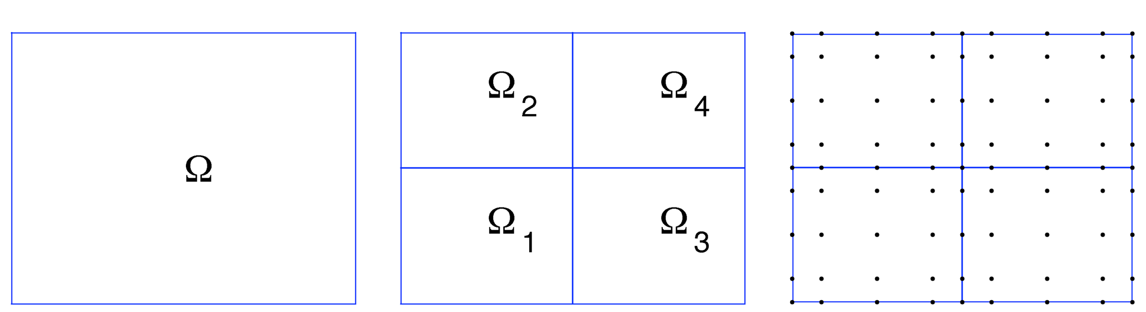

Figure: Simulation domain Ω (left) is divided into elements (middle). In each element grid points based on Gauss-Legendre-Lobatto nodes are chosen (right).

Spectral Basis Functions

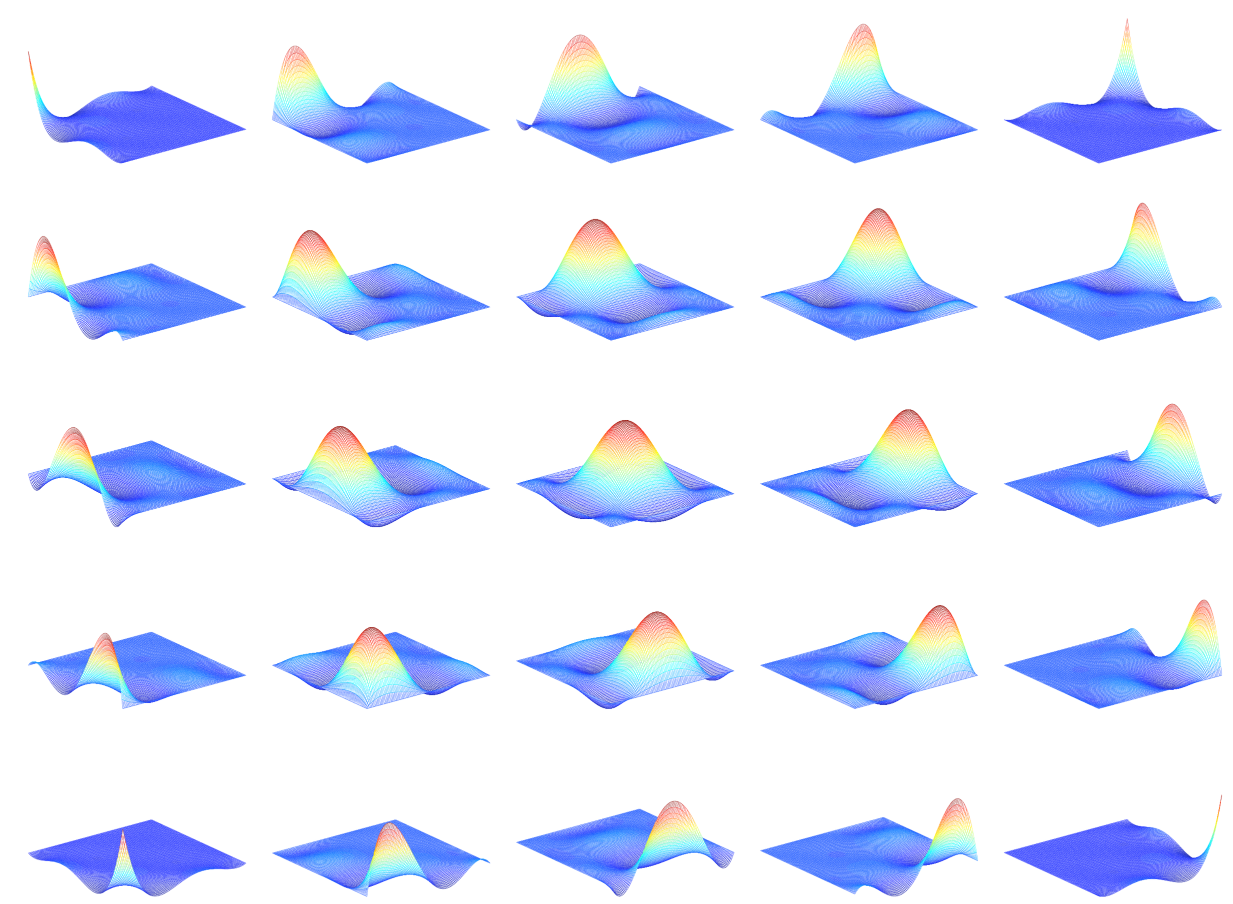

Figure: 4th Order 2D Lagrangian nodal basis functions $\pi_i \otimes \pi_j$ based on the Gauss-Labotto-Legendre points.

Fluid Simulation Layout

Time step (BE)	$x^{n+1} = x^n + \Delta t F(t^{n+1}, x^{n+1})$
Nonlinear Solver (Newton)	$x_{k+1} = x_k + \Delta x_k$
Linear Solver (GMRES)	$A\Delta x_k = b$
Preconditioner (DD)	$AP^{-1}P\Delta x_k = b$

Domain Decomposition System

$$\begin{bmatrix} \bar{P}_{II}^1 & 0 & \dots & 0 & \bar{P}_{I\Gamma}^1 \\ 0 & \bar{P}_{II}^2 & 0 & \dots & \bar{P}_{I\Gamma}^2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{P}_{II}^E & \bar{P}_{I\Gamma}^E \\ 0 & 0 & \dots & 0 & \bar{P}_S \end{bmatrix} \begin{pmatrix} u_{I1} \\ u_{I2} \\ \vdots \\ u_{IE} \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} \hat{b}_{I1} \\ \hat{b}_{I2} \\ \vdots \\ \hat{b}_{IE} \\ g_\Gamma \end{pmatrix}$$

$\bar{P}_S = \sum_{e=1}^{E}(\bar{P}_{\Gamma\Gamma}^e - \bar{P}_{\Gamma I}^e \bar{P}_{II}^{e\,-1} \bar{P}_{I\Gamma}^e)$ represents the Schur complement of the system. The interface u_Γ is obtained via an iterative solve.

Constant Wind Approximation

When the "wind" $\vec{w}$ is constant on each element, then element interiors can be obtained via Fast Diagonalization and $P^{-1} = A^{-1}$.

Otherwise using a constant wind approximation on each element $P^{-1} \approx A^{-1}$.

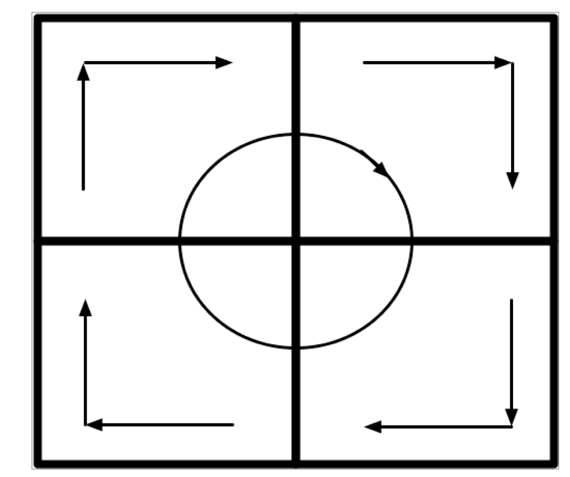

Figure: Illustration of a constant wind approximation

$$\bar{P}^{e\,-1} = \tilde{M}(V_y \otimes V_x)(\Lambda_y \otimes I + I \otimes \Lambda_x)^{-1}(V_y^{-1} \otimes V_x^{-1})\tilde{M}$$

Test Case: Constant Wind, Pc=400

$$\vec{w} = (0, 200)$$

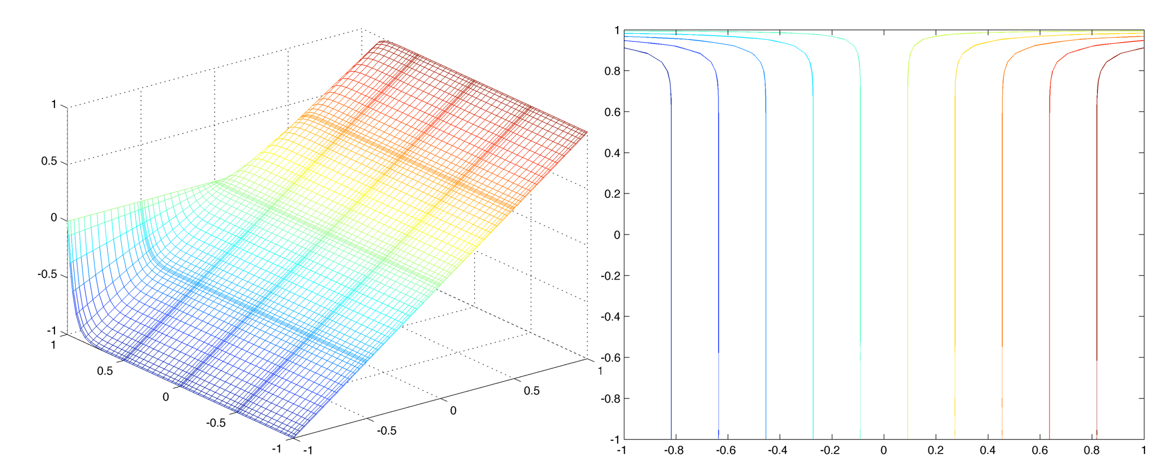

Figure: Steady flow with constant wind exhibiting boundary layer at $y = 1$ using SEM N=16 & E=4x4.

Interface Solver Convergence

Table: Iteration count ($E = 4 \times 4$)

N	Iterations -	Iterations R-R
4	240	44
8	108	42
16	103	43

Table: Iteration count (N=4)

E	Iterations -	Iterations R-R
4×4	240	44
8×8	175	42
16×16	143	50

Robin-Robin preconditioned interface solve (R-R) is invariant to the number of points in the discretization and convergences in significantly fewer steps than the non-preconditioned system (-).

Test Case: Recirculating Wind, Pc=400

$$\vec{w} = 200(y(1-x^2), -x(1-y^2))$$

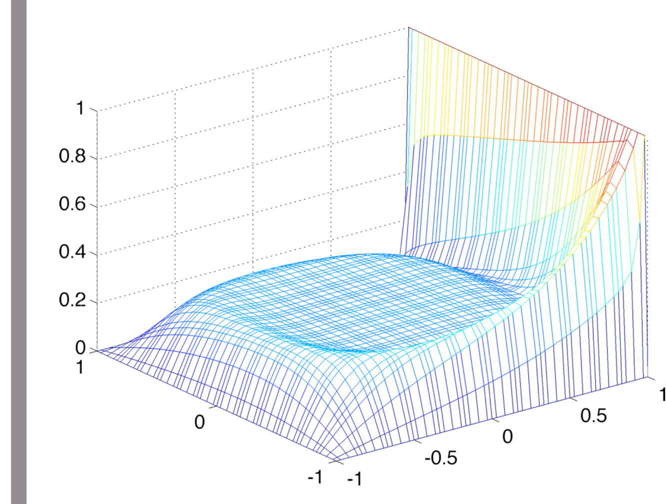

Figure: Computed solution of steady flow with recirculating wind using SEM N=4 & E=12x12.

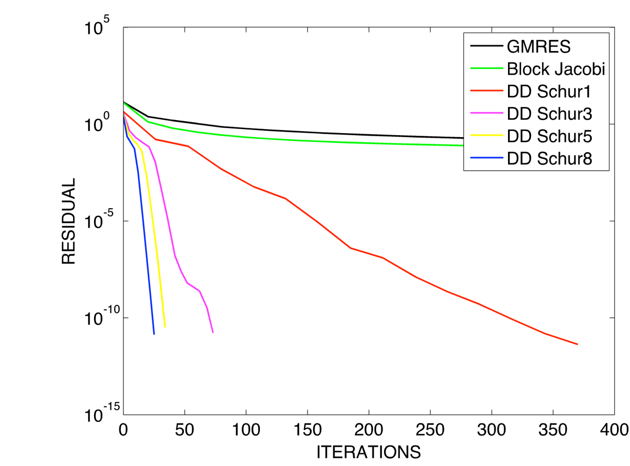

Figure: Comparison of Outer iterations when inner iterations are varied.

Convergence Properties for Refined Meshes

Table: Iteration Count ($E = 4 \times 4$)

N	Number of Outer Iterations
5	52
7	56
9	55
11	53
13	51

Table: Iteration count (N=4)

E	Number of Outer Iterations
10×10	37
11×11	38
12×12	38
13×13	38

Summary & Future Directions

Summary

- Improved simulation efficiency for steady Advection-Diffusion equation

Future Directions

- Improve wind approximation on each element
- Coarse Grid Preconditioner to allow for more elements
- Use Preconditioner in Navier-Stokes simulations
- Apply to realistic fluid simulations

References

- P. A. Lott, "Fast Solvers for Models of Fluid Flow", Ph.D. Thesis University of Maryland College Park, 2008
- P. A. Lott and H. Elman, "Matrix-free preconditioner for the steady advection-diffusion equation with spectral element discretization", in preparation, 2008
- ______, "Matrix-free Block preconditioner for the Navier-Stokes equations", in preparation, 2008