Fast Solvers for Models of Fluid Flow

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University of Maryland, College Park

Do we still need better algorithms?

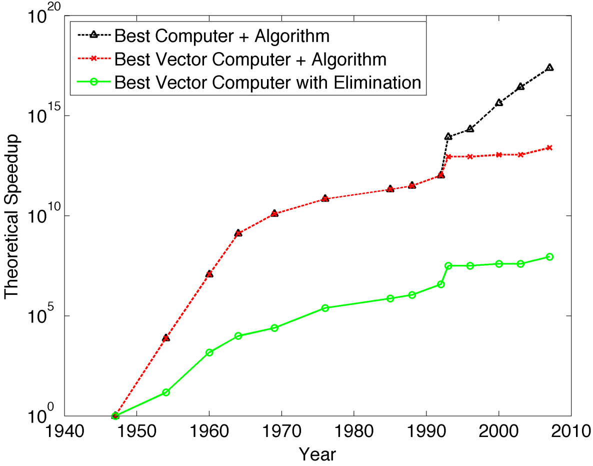

Figure: Evolution of Machines & Algorithms for solving 3D Poisson Equation (Adapted from Deville)

What is scalable?

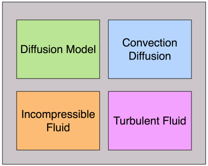

Figure: More difficult problems

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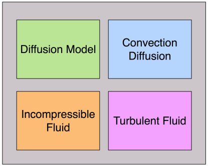

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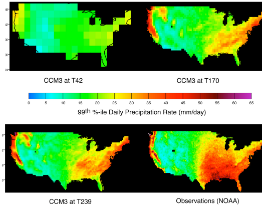

Figure: At increased resolution (Image by Duffy)

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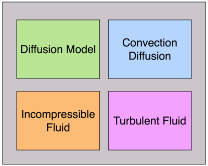

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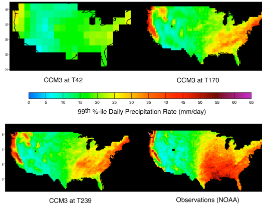

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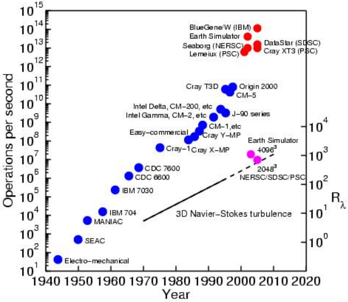


Figure: Solved efficiently as the number of processors increase. (Image by Donzis)

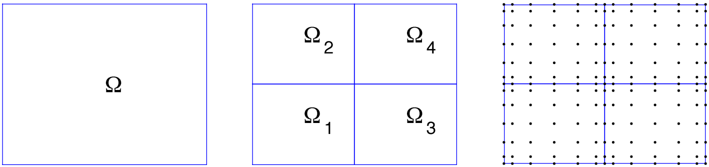
Steady Fluid Model

Incompressible Navier-Stokes Equations

$$\begin{array}{rcl} -\frac{1}{Re}\nabla^2\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p & = & f \\ \nabla\cdot\vec{u} & = & 0 \end{array} \quad \text{in} \quad \Omega$$

$$\vec{u} = \vec{u}_D \quad \text{on} \quad \partial\Omega_D, \qquad \nu\frac{\partial\vec{u}}{\partial n} - \vec{n}p = 0 \quad \text{on} \quad \partial\Omega_N.$$

Spectral Element Method



The solution is expressed via a nodal basis on each element

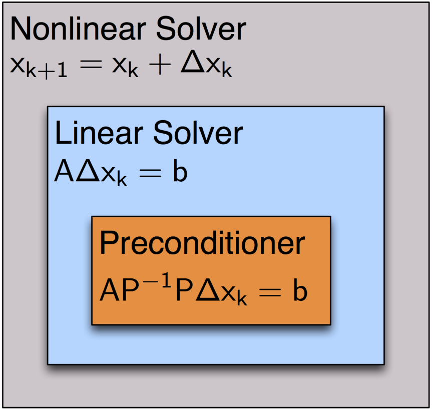
$$u_e^N(x,y) = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} u_{ij}\pi_i(x)\pi_j(y).$$

Discrete Nonlinear Block System

$$\begin{bmatrix} N(u) & -D^T \\ -D & 0 \end{bmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} Mf \\ 0 \end{pmatrix}$$

$N(u)$ - Nonlinear Convection-Diffusion
D^T - Gradient
D - Divergence
M - Mass Matrix

Overview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$x_{k+1} = x_k + \Delta x_k$
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Block Preconditioner

Linearize & take upper block of LU factorization

$$\underbrace{\begin{bmatrix} F & -D^T \\ -D & 0 \end{bmatrix}}_{A} = \begin{bmatrix} I & 0 \\ -DF^{-1} & I \end{bmatrix} \begin{bmatrix} F & -D^T \\ 0 & -S \end{bmatrix}.$$

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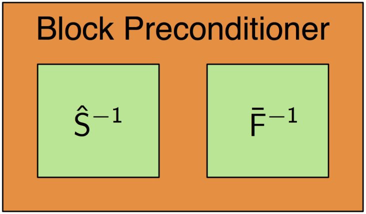
Approximate F and S to make computationally efficient [1]

$$\underbrace{\begin{bmatrix} \bar{F} & -D^T \\ 0 & -\hat{S} \end{bmatrix}}_{P}.$$



$\hat{S}$: Least-Squares Commutator

$$S := DF^{-1}D^T = (DF^{-1}M)(M^{-1}F)(M^{-1}D^T) \approx DM^{-1}D^T F_p M_p$$

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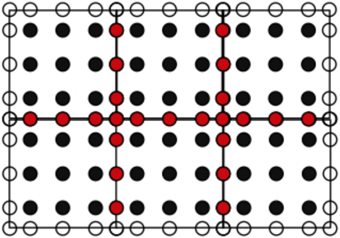
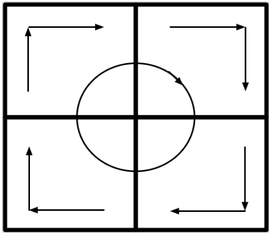
$$\hat{S}^{-1} = \underbrace{(DM^{-1}D^T)^{-1}}_{\textit{Poisson Solve}}(DM^{-1}FM^{-1}D^T)\underbrace{(DM^{-1}D^T)^{-1}}_{\textit{Poisson Solve}}$$

$\bar{F}$: Exploit High Volume-Surface ratio



$$\begin{bmatrix} \bar{F}^1_{II} & 0 & \dots & 0 & \bar{F}^1_{I\Gamma} \\ 0 & \bar{F}^2_{II} & 0 & \dots & \bar{F}^2_{I\Gamma} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{F}^E_{II} & \bar{F}^E_{I\Gamma} \\ 0 & 0 & \dots & 0 & \bar{F}_S \end{bmatrix} \begin{pmatrix} u_{I^1} \\ u_{I^2} \\ \vdots \\ u_{I^E} \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} \hat{b}_{I^1} \\ \hat{b}_{I^2} \\ \vdots \\ \hat{b}_{I^E} \\ g_\Gamma \end{pmatrix}.$$

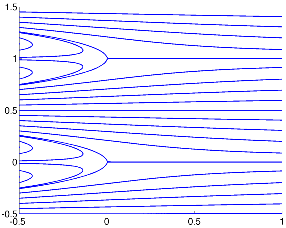
Make volume calculations Fast [2]

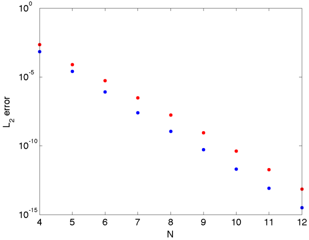


$$\bar{F}_{interface} = \sum_{e=1}^{E} \left(\bar{F}^{e}_{\Gamma\Gamma} - \bar{F}^{e}_{\Gamma I}\, {\bar{F}^{e}_{II}}^{-1}\, \bar{F}^{e}_{I\Gamma}\right)$$

$$ {\bar{F}^{e}_{interior}}^{-1} = \underbrace{\tilde{M}(V_y \otimes V_x)(\Lambda_y \otimes I + I \otimes \Lambda_x)^{-1}(V_y^{-1} \otimes V_x^{-1})\tilde{M}}_{\textit{Diagonalized via } 1D \textit{ operators!}}$$

Polynomial Degree depenedence case study: Kovasznay Flow





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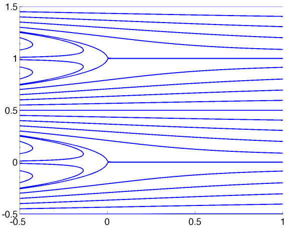
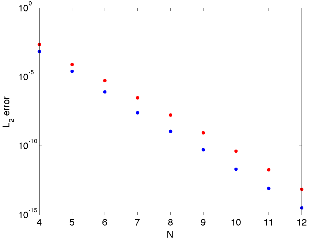
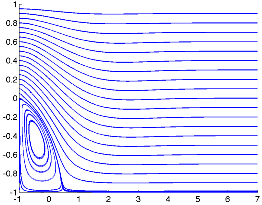




Polynomial degree N	Nonlinear steps	Linear steps	$\bar{F}^{-1}$ steps	$\hat{S}^{-1}$ steps
4	11	53	4	33
8	13	42	5	32
12	16	29	6	54

Table: Linear steps decrease with N.

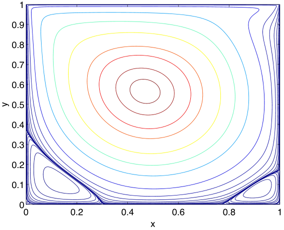
Element dependence case study: Flow over a step.



E	Nonlinear steps	Linear steps	$\bar{F}^{-1}$ steps	$\hat{S}^{-1}$ steps
16	7	33	14	20
64	7	28	32	28
256	6	34	86	38

Table: $\bar{F}^{-1}$ is mildly dependent on E

Reynolds number dependence case study: Lid-Driven Cavity



Re	Nonlinear steps	Linear steps	$\bar{F}^{-1}$ steps	$\hat{S}^{-1}$ steps
10	5	30	186	200
100	7	42	141	200
1000	11	96	119	200
5000	20	240	107	200

Table: Works well. Can do better by improving $\hat{S}^{-1}$ solve.

Review

Nonlinear Solver

$$x_{k+1} = x_k + \Delta x_k$$

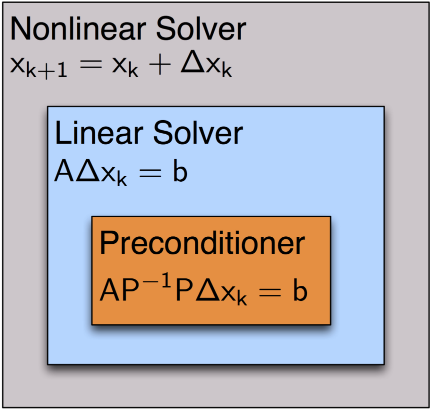
Linear Solver

$$A \Delta x_k = b$$

Preconditioner

$$A P^{-1} P \Delta x_k = b$$

Review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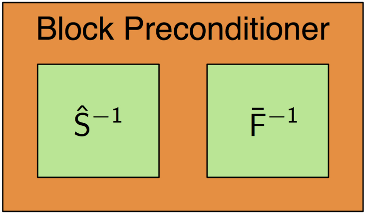
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$\hat{S}^{-1}$
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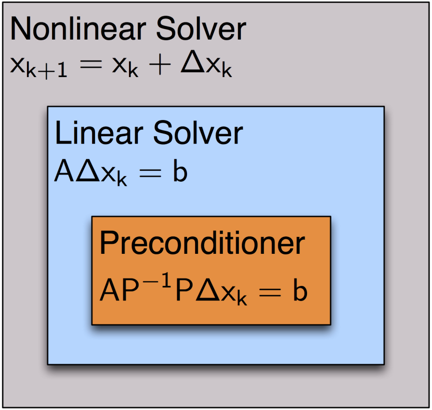
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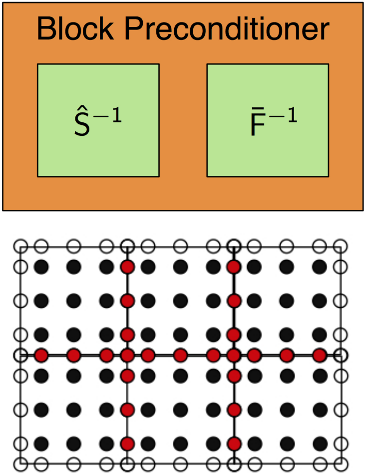
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Conclusions

Summary

- Developed fast steady flow solver via spectral elements [3]
- Based on block preconditioner using LSC & DD
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Current Work

- Accelerating $\hat{S}$ Poisson Solves with DD
- Extending to Binary Alloys with Jeff McFadden @ NIST
- Applying to Climate Model with Kate Evans @ ORNL

References

H. C. ELMAN, V. E. HOWLE, J. SHADID, D. SILVESTER, AND R. TUMINARO, *Least squares preconditioners for stabilized discretizations of the Navier-Stokes equations*, SIAM Journal on Scientific Computing, 30 (2007), pp. 290–311.

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——, *Fast iterative solvers for incompressible Navier-Stokes equations with spectral elements*, In preparation.