AMSC 660 Homework 5

What to turn in: Turn in a printed copy of well documented matlab code for any of the code you are to write as well as any written work and analysis. Please staple all work, and attach this sheet as a coverpage. Have fun!

Suppose we have a second order non-linear differential equation, with periodic forcing (see eq (1)). We want to find the initial conditions on $x(0)$ and $\dot{x}(0)$ that will yeild periodic solutions. This is important in engineering, when one has a model and needs to know initial conditions to start their system with. This is a difficult problem, in many cases with no analytical solution. However, under certain assumptions of the non-linear ode, this can be solved numerically using what we will refer to as the homotopy method.

Consider the (possibly nonlinear) differential equation

$$\ddot{x}(t) + C\dot{x}(t) + g(t, x) = e(t), \tag{1}$$

where C is a constant; g is continuous, continuously differentiable with respect to x, and is periodic of period P in the variable t; $e(t)$ is continuous and periodic of period P.

Problem 1 [5pts] Convert eq. (1) into a system of first order ordinary differential equations. Use the following initial conditions and functions to solve the corresponding IVP using matlab's ode45 routine for t=[0,P].

$$\begin{cases} \ddot{x} - \dot{x} + 2x + sin(x/\sqrt{2}) = \sqrt{2}\pi + 2sin(t) + cos(cos(\pi/4 - t)), \\ x(0) = 1, \qquad \dot{x}(0) = 0. \end{cases} \tag{2}$$

We are interested in determining initial conditions that guarantee the solution of this equation to also be periodic of period P. To do this, write the solution to the initial value problem corresponding to (1) with initial conditions $(x(0), \dot{x}(0)) = (\alpha, \beta)$ as $x = x(t, v)$ where $v = (\alpha, \beta)^T$, then define

$$f(v) = (x(P, v), \dot{x}(P, v))^T$$

and set

$$F(v) = v - f(v).$$

Problem 2 [5 pts] Show that the desired initial conditions for a periodic solution form a fixed point for f and a zero point for F.

The continuous dependence on initial conditions theorem in [4] shows that the evaluation of our initial value problem is a continuous function of the initial conditions. Thus, both f and F are C^1 differentiable functions with respect to v. Li and Shen [7] show that under the following conditions, F is a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. This guarantees F has a unique zero point and thus we can solve for $(x(0), \dot{x}(0)) = (\alpha, \beta)$.

Li and Shen [7] show if there exist two continuous functions $a(t)$ and $b(t)$ and a positive integer n so that

$$n^2 \le a(t) \le \frac{\partial g}{\partial x}(t, x) \le b(t) \le (n+1)^2 \tag{3}$$

for all values of $t \ge 0$, so that $n^2 < a(t)$, and $b(t) < (n+1)^2$ on a subset of positive Lebesgue measure of the interval $[0, P]$, then they show that there exist unique initial values $x(0) = \alpha^*$ and $\dot{x}(0) = \beta^*$ so that the solution to this initial value problem is periodic of period P and is unique with this property. (The continuity assumption on a and b can be weakened to continuity almost everywhere.) As we will see later, weakening these conditions to the following allows us to have only a local homeomorphism, thus possibly yeilding multiple solutions in $\mathbb{R}^2$.

The weaker condition is that the matrix

$$\begin{pmatrix} 1 - \dfrac{\partial x}{\partial \alpha}(P) & -\dfrac{\partial x}{\partial \beta}(P) \\ \\ -\dfrac{\partial \dot{x}}{\partial \alpha}(P) & 1 - \dfrac{\partial \dot{x}}{\partial \beta}(P) \end{pmatrix} \tag{4}$$

be non-singular in order for $F(v)$ to be a local homeomorphism, and hence for a homotopic path to exist from a given initial condition to the initial condition that yields a periodic solution.

There is a constructive proof of this result. That is, starting at any initial conditions $x(0) = \alpha_0$ and $\dot{x}(0) = \beta_0$, we can produce a path γ of initial values starting at (α, β) in the phase plane and terminating at (α^*, β^*), and we can produce a homotopy that continuously deforms the starting solution to the unique periodic solution. This is done by forming the homotopy

$$\Gamma(v, \delta) = F(v) - (1 - \delta)F(v_0), \tag{5}$$

and solving the equation

$$\Gamma(v, \delta) = 0 \tag{6}$$

for all δ between 0 and 1. This way, when $\delta = 0$, $v = v_0$, and $\Gamma(v, 0) = 0$, and when $\delta = 1$, $\Gamma(v, 1) = F(v) = 0$. Thus, by starting at some point $v_0 \in \mathbb{R}^2$, and $\delta = 0$, we construct a path of v_n's that satisfy $\Gamma(v_n, \delta_n) = 0$, and we reach our desired point v^* when $\delta_n = 1$.

Your goal will be to follow the path $\Gamma(v, \delta) = 0$ for an increasing sequence of δ_n starting at 0 and ending at 1. Li and Shen [7] show that 6 exists for all $\delta \in [0, 1]$ as long as $\|\frac{dF}{dv}(v)^{-1}\| < M < \infty$. Which in our examples will be true. The predictor corrector method to follow $\Gamma(v, \delta) = 0$ is illustrated in figure 1.

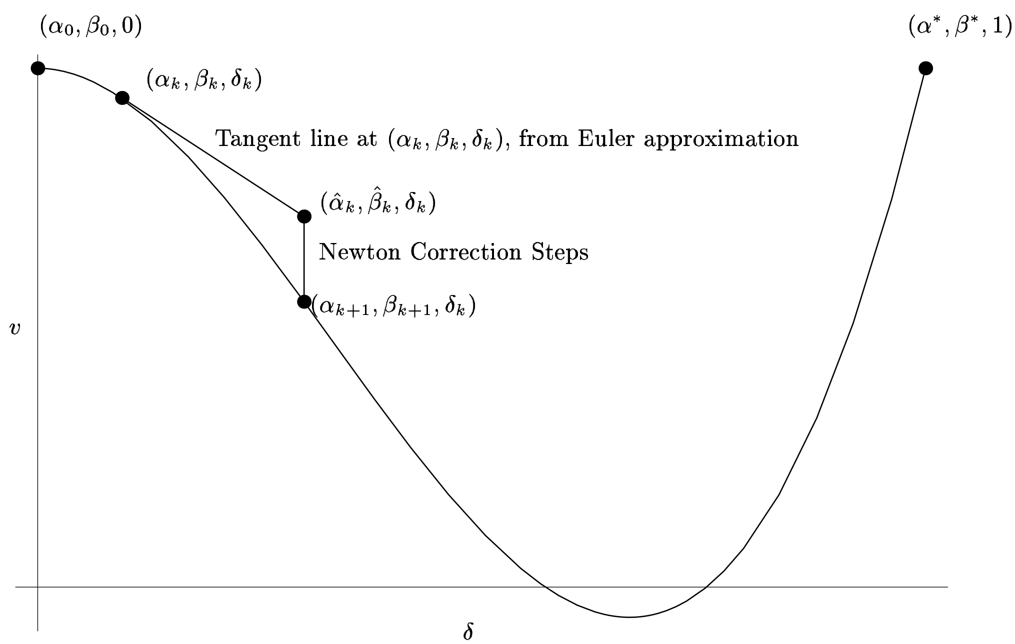

Figure 1: Path of $\Gamma(v, \delta) = 0$

In order to construct this method, we re-parameterize $\Gamma(v, \delta)$ with respect to arc length, and turn equation (6) into the solution of an initial value problem by differentiating with respect to arclength.

$$\begin{cases} d\Gamma(\alpha(s), \beta(s), \delta(s))/ds = 0, \\ \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \quad \delta(0) = 0. \end{cases} \tag{7}$$

where we have the additional arc length constraint

$$\| (\alpha'(s), \beta'(s), \delta'(s)) \| = 1. \tag{8}$$

Define for each δ, $\gamma(\delta) = v(\delta)$, where $v(\delta)$ is the corresponding value of the solution to (7), and thus satisfies equation (6), then γ defines a path with $\gamma(0) = v_0$ and $\gamma(1) = v^* = (\alpha^*, \beta^*)^T$. $\gamma(\delta)$ provides a continuous deformation of $x(t, v_0)$ to $x(t, v^*)$ by considering $x(t, \gamma(\delta))$ as δ ranges from 0 to 1.
Problem 3 [5 pts]. Construct a matlab homotopy evaluation function that returns $\Gamma(v, \delta)$ at a given v, δ.

Problem 4 [5 pts] Re-write equation (7) in matrix form (9).

$$\begin{cases} \left(\dfrac{\partial F(v)}{\partial v} | F(v_0)\right) X = 0 \\ \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \quad \delta(0) = 0. \end{cases} \tag{9}$$

Where $X = (\alpha'(s), \beta'(s), \delta'(s))^T$. Hint: The arclength of $\Gamma(v, \delta) = 0$ depends on α, β, and δ., use chain rule.
Problem 5 [5 pts] Construct a matlab function that returns the matrix $\dfrac{\partial F(v)}{\partial v}$ and the vector $(\alpha'(s), \beta'(s), \delta'(s))^T$ when passed a value for v and δ. Hint: When computing partial derivatives of x and $\dot{x}$ with respect to α and β use a second order forward difference scheme. Also, you should store $F(v_0)$ so that you don't have to compute it each time. Precaution: Be sure to evaluate you partial derivatives at time t = P when evaluating the partial derivatives.

Problem 6 [5 pts] Write a matlab function that receives (α, β, δ) and returns a true/false flag and the vector $v_{new} = (\alpha_{new}, \beta_{new}, \delta)$, that satisfies $\Gamma(\alpha_{new}, \beta_{new}, \delta) = 0$. Use Newton's method to solve this iteratively. You will need to set a value HOME_EPSILON so that if $\Gamma(\alpha_{new}, \beta_{new}, \delta) <$ HOME_EPSILON then we will say you are on the path. When solving for v_{new}, if Γ is not getting closer to zero, then return a false flag, and the current values for α and β, this implies that Euler's method made too big of a step in δ and the step size should be reduced and the prediction should be re-computed with a new value for δ.

Problem 7 [5 pts] Using the above routines, write a matlab function that implements the homotopy method as an adaptive predictor corrector method to solve for periodic solutions for the following ODE's. Try to obtain 10^{-6} accuracy in your results by adjusting the value of HOME_EPSILON, the accuracy of ode45, and the distance $\epsilon = \|\delta - 1\|$.

$$\begin{cases} \ddot{x} + x - \dfrac{x^3}{6} = \dfrac{1}{3} cos(3/5t), \\ x(0) = \alpha, \qquad \dot{x}(0) = \beta. \end{cases} \tag{10}$$

$$\begin{cases} \ddot{x} - \dot{x} + 2x + sin(x/\sqrt{2}) = \sqrt{2}\pi + 2sin(t) + cos(cos(\pi/4 - t)), \\ x(0) = \alpha, \qquad \dot{x}(0) = \beta. \end{cases} \tag{11}$$

Problem 8 [5 pts] Plot $x(t)$ vs. $\dot{x}(t)$ for t=[0,P] and evaluate $x(0) - x(P)$ and $\dot{x}(0) - \dot{x}(P)$ for each of the above solutions.

References

[1] Eugene Allgower and Kurt Georg. Simplicial and continuation methods for approximating fixed points and solutions to systems of equations. *SIAM Review*, 22(1):28–85, January 1980.

[2] Eugene Allgower and Kurt Georg. *Numerical Continuation Methods: An Instruction.* Springer-Verlag, New York, 1990.

[3] William E. Boyce and Richard C. DiPrima. *Elementary Differential Equations.* John Wiley and Sons, Inc., New York, 6^{th} edition, 1997.

[4] Earl A. Coddington and Norman Levinson. *Theory of Ordinary Differential Equations.* Robert E. Krieger Publishing Company, New York, 2^{nd} edition, 1984.

[5] Tien-Yein Li. Lecture on the numerical method of finding solutions of system of nonlinear equations. Hsinchu, Taiwan, 1988. Chiao-Tung University.

[6] Tien-Yien Li and James A. Yorke. A simple reliable numerical algorithm for following homotopy paths. In Stephen M. Robinson, editor, *Analysis and Computation of Fixed Points*, pages 73–91, New York, 1980. University of Wisconsin, Academic Press.

[7] Weiguo Li and Zuhe Shen. Constructive proof on the existence of periodic solution of Duffing equation. *Chinese Sci. Bull.*, 42(22):1870–1874, 1997.