Finding periodic solutions to nonlinear differential equations using the homotopy method

P. Aaron Lott*

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1 Introduction

Consider the (possibly nonlinear) differential equation

$$\ddot{x}(t) + C\dot{x}(t) + g(t, x) = e(t), \tag{1}$$

where C is a constant; g is continuous, continuously differentiable with respect to x, and is periodic of period P in the variable t; $e(t)$ is continuous and periodic of period P. We are interested in determining initial conditions that guarantee the solution of this equation to also be periodic of period P. By assuming that there exist two continuous functions $a(t)$ and $b(t)$ and a positive integer n so that

$$n^2 \leq a(t) \leq \frac{\partial g}{\partial x}(t, x) \leq b(t) \leq (n+1)^2 \tag{2}$$

for all values of $t \geq 0$, and $n^2 < a(t)$, and $b(t) < (n+1)^2$ on a subset of positive measure of the interval $[0, P]$, then there exist initial values $x(0) = \alpha^*$ and $\dot{x}(0) = \beta^*$ so that the solution to this initial value problem is periodic of period P and is unique with this property. (The continuity assumption on a and b can be weakened to continuity almost everywhere.) Moreover, the proof of this can be made constructive, so that starting at any initial conditions $x(0) = \alpha$ and $\dot{x}(0) = \beta$, we can produce a path of initial values starting at (α, β) in the phase plane and terminating at (α^*, β^*) and a homotopy that continuously deforms the starting solution to the unique periodic solution. This is a result of Li and Shen [3]. We will discuss both the proof of this theorem and a *Mathematica* implementation.

The idea of the proof is as follows. We write the solution to the initial value problem having $(x(0), \dot{x}(0)) = (\alpha, \beta)$ as $x = x(t, v)$ where $v = (\alpha, \beta)^T$. Define

$$f(v) = (x(P, v), \dot{x}(P, v))^T$$

and set

$$F(v) = v - f(v).$$

Observe that the desired initial conditions for a periodic solution form a fixed point for f and a zero point for F. Note both f and F are C^1 differentiable functions with respect to t. We want to show that F is a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. This will guarantee that F has a unique zero and thus the theorem follows.

Linearizing equation (1), we compute its fundamental solution matrix $Y(t)$ and show that

$$\frac{dF}{dt}(v) = I - Y(t),$$

where I is the 2×2 identity matrix. The bounds given in (2) yield a bound on the eigenvalues of the matrix $\frac{dF}{dt}(v)$. This shows that $\frac{dF}{dt}(v)$ is invertible and hence $\|\frac{dF}{dt}(v)^{-1}\|$ exists and is finite. By Hadamard's Theorem [4], this is sufficient to imply that F is a homeomorphism.

2 Invertibility

Linearizing equation (1), we turn it into a system by setting $y = \dot{x}$ and taking partial derivatives we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{\partial g}{\partial x}(t,x) & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

which has the fundamental solution matrix

$$\begin{aligned} Y(t) &= Exp[\int_0^t \begin{pmatrix} 0 & 1 \\ -\dfrac{\partial g}{\partial x}(s,x) & -C \end{pmatrix} ds] \\ &= Exp[\begin{pmatrix} 0 & t \\ -\displaystyle\int_0^t \dfrac{\partial g}{\partial x}(s,x)ds & -Ct \end{pmatrix}]. \end{aligned}$$

Next, we let $u = (x,y)^T$, and observe that

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial u(t,v)}{\partial v}\right) &= \frac{\partial}{\partial v}\dot{u}(t,v) \\ &= \frac{\partial}{\partial v}(R(t,u(t,v)) + E(t)) \\ &= \frac{\partial}{\partial v}(R(t,u(t,v)), \end{aligned}$$

where

$$R(t,u(t,v)) = \begin{pmatrix} y \\ -Cy - g(t,x) \end{pmatrix}$$

and

$$E(t) = \begin{pmatrix} 0 \\ e(t) \end{pmatrix}.$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial v}(R(t,u(t,v)) &= \frac{\partial R}{\partial u} \\ &= \begin{pmatrix} \dfrac{\partial R_1}{\partial x} & \dfrac{\partial R_1}{\partial y} \\ \dfrac{\partial R_2}{\partial x} & \dfrac{\partial R_2}{y} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\dfrac{\partial g}{\partial x}(t,x) & -C \end{pmatrix}, \end{aligned}$$

and thus $\partial R/\partial u$ formally satisfies equation (3). Hence

$$\frac{\partial u}{\partial v} = \begin{pmatrix} \dfrac{\partial x(t,\vec{v})}{\partial \alpha} & \dfrac{\partial x(t,\vec{v})}{\partial \beta} \\ \dfrac{\partial \dot{x}(t,\vec{v})}{\partial \alpha} & \dfrac{\partial \dot{x}(t,\vec{v})}{\partial \beta} \end{pmatrix}.$$

is also a fundamental matrix for equation (3). This permits us to represent $\frac{dF}{dt}(v)$ as

$$I \quad - \quad \begin{pmatrix} \dfrac{\partial x(t,\vec{v})}{\partial \alpha} & \dfrac{\partial x(t,\vec{v})}{\partial \beta} \\ \dfrac{\partial \dot{x}(t,\vec{v})}{\partial \alpha} & \dfrac{\partial \dot{x}(t,\vec{v})}{\partial \beta} \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\partial x(t,\vec{v})}{\partial \alpha} & \dfrac{\partial x(t,\vec{v})}{\partial \beta} \\ \dfrac{\partial \dot{x}(t,\vec{v})}{\partial \alpha} & 1 - \dfrac{\partial \dot{x}(t,\vec{v})}{\partial \beta} \end{pmatrix}.$$

Li and Shen [3] obtain a lower bound for all the eigenvalues of $(\frac{dF}{dt}(v))^2$ by taking the minimum of the four positive values

$$\{4sin^2(\sqrt{\pi A}/2), 4sin^2(\sqrt{2PB}/2), [1-e^{-\pi C}]^2, [1-e^{-A/C}]^2\}$$

where

$$2n^2P < \int_0^P a(t)\,dt = A \leq B = \int_0^P b(t)\,dt < 2(n+1)P$$

This implies $\frac{dF}{dt}(v)$ is invertible.

3 The Analytic Proof

By solving the initial value problem corresponding to each (α, β) defined by the curve $F(v) - (1-\delta)F(v_0) = 0$, we create a sequence of functions $\gamma(\delta)$ starting at $(x(t,v_0), \dot{x}(t,v_0))$ that deforms into $(x_P(t), \dot{x}_P(t))$. Here we prove that $\gamma(\delta)$ forms a continuous deformation to $(x_P(t), \dot{x}_P(t))$.

Proof. The continuous dependence of $(x(t,v), \dot{x}(t,v))^T$ on the initial conditions $v = (\alpha, \beta)$ discussed in [1] allows us to differentiate $x(t)$ and $\dot{x}(t)$ with respect to v. Thus we must have a continuous relation between the initial values of the differential equation and the solution of the differential equation. Now since the linear homotopic path produces a continuous deformation of points $(\alpha, \beta)_i$ to $(\alpha, \beta)^*$, our solutions $x(t, v_i)$ form a continuous deformation to the periodic solution $x_P(t, v^*)$. □

We would like to know if we can find v's that satisfy

$$F(v) - (1-\delta)F(v_0) = 0$$

for all δ between 0 and 1. To do this we take the derivative of $F(v) - (1-\delta)F(v_0)$ with respect to δ and form the initial value problem (4).

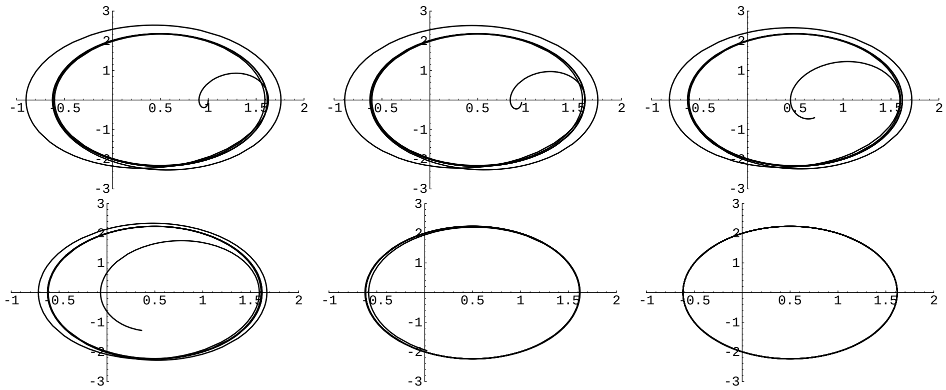

Figure 1: Continuous Deformation of $x(t, v_i)$

$$\begin{cases} v'(\delta) = F'(v)^{-1}F(v_0), \\ v(0) = v_0. \end{cases} \tag{4}$$

By showing the existence of a solution to the IVP (4) for $0 \leq \delta \leq 1$, we will have also shown the existence of our desired path $\gamma[0,1] \to \mathbb{R}^2$ defined by $\gamma(\delta) = v$ where v satisfies $F(v) - (1-\delta)F(v_0) = 0$. Li and Shen show this in Theorem 2 of [3] by applying an existence and uniqueness theorem to (4) to obtain a solution for $\delta \in [0, \epsilon]$ and using analytic continuation to extend the solution over $[0, 1]$.

4 The Constructive Proof

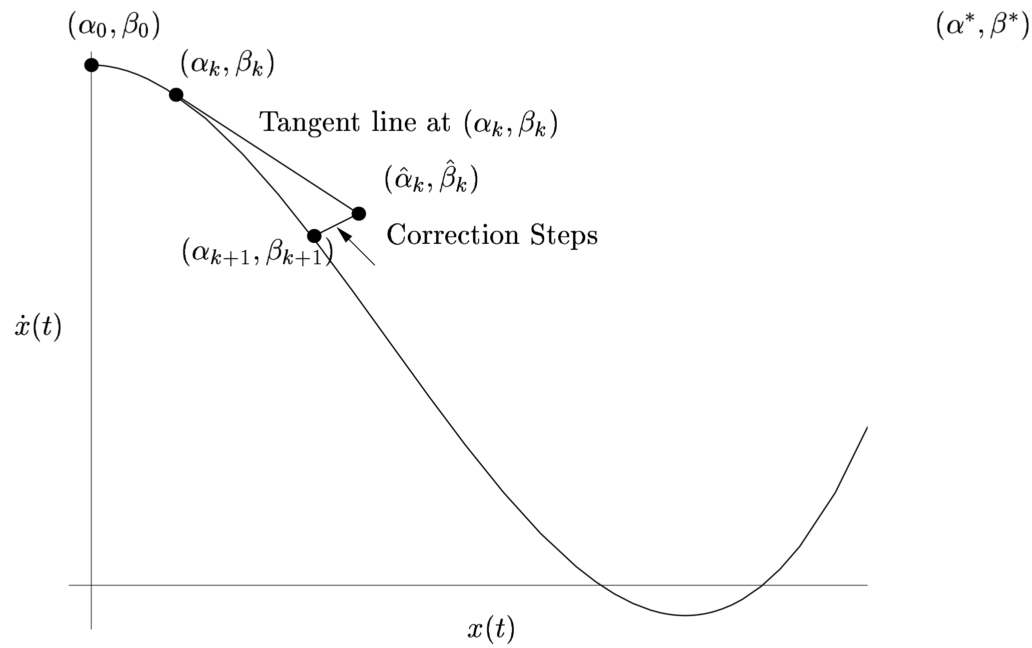

Figure 2: Path of initial positions in the phase plane

Define $\gamma(\delta)$ to be the desired path of initial points in the phase plane where $\delta \in [0,1]$, $\gamma(0) = (\alpha_0, \beta_0)^T$ and $\gamma(1) = (\alpha^*, \beta^*)^T$, and let

$$\Gamma(v,\delta) = F(v) - (1-\delta)F(v_0)$$

Instead of solving (4) to generate $\gamma(\delta)$, we differentiate $\Gamma(v,\delta)$ with respect to the arc length (since we now know the arc exists) and obtain the equivalent initial value problem

$$\begin{cases} d\Gamma(\alpha(s), \beta(s), \delta(s))/ds = 0, \\ \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \quad \delta(0) = 0. \end{cases} \tag{5}$$

The advantage of considering (5) is that arc length gives us $\|(\alpha'(s), \beta'(s), \delta'(s))\| = 1$, thus allowing us to solve the 2×3 system:

$$\frac{d\Gamma}{ds} = \frac{\partial \Gamma}{\partial v}\frac{dv}{ds} + \frac{\partial \Gamma}{\partial \delta}\frac{d\delta}{ds} = \frac{\partial F}{\partial \alpha}\alpha'(s) + \frac{\partial F}{\partial \beta}\beta'(s) + F(v_0)\delta'(s) = 0.$$

$$\begin{cases} \begin{pmatrix} \dfrac{\partial \Gamma_1}{\partial \alpha} & \dfrac{\partial \Gamma_1}{\partial \beta} & \dfrac{\partial \Gamma_1}{\partial \delta} \\[2ex] \dfrac{\partial \Gamma_2}{\partial \alpha} & \dfrac{\partial \Gamma_2}{\partial \beta} & \dfrac{\partial \Gamma_2}{\partial \delta} \end{pmatrix} \begin{pmatrix} \alpha'(s) \\[2ex] \beta'(s) \\[2ex] \delta'(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\[2ex] \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \quad \delta(0) = 0. \end{cases} \tag{6}$$

Evaluating these partial derivatives we obtain the system

$$\begin{cases} \begin{pmatrix} 1 - \dfrac{\partial x}{\partial \alpha} & -\dfrac{\partial x}{\partial \beta} & \alpha_0 - x(p, \alpha_0, \beta_0) \\[2ex] -\dfrac{\partial \dot{x}}{\partial \alpha} & 1 - \dfrac{\partial \dot{x}}{\partial \beta} & \beta_0 - \dot{x}(p, \alpha_0, \beta_0) \end{pmatrix} \begin{pmatrix} \alpha'(s) \\[2ex] \beta'(s) \\[2ex] \delta'(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\[2ex] \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \quad \delta(0) = 0. \end{cases} \tag{7}$$

We can compute $x(t, \alpha, \beta)$ numerically using Mathematica's `NDSolve`, and use central difference derivative approximations to compute $\frac{\partial x}{\partial \alpha}, \frac{\partial x}{\partial \beta}, \frac{\partial \dot{x}}{\partial \alpha}$, and $\frac{\partial \dot{x}}{\partial \beta}$. Once we have $(\dot{\alpha}, \dot{\beta}, \dot{\delta})$ we use Eulers method to find $(\alpha_1, \beta_1, \delta_1)$. This point, however, may not satisfy $\Gamma(v,\delta) = F(v) - (1-\delta)F(v_0)$ due to Euler's low order approximation. We can correct our approximation with a sequence of Newton's steps by fixing δ

and obtaining points of the form $v_n = v_{n-1} - \Gamma'(v_{n-1},\delta)^{-1}\Gamma(v_{n-1},\delta)$, with

$$\Gamma'(v_{n-1}) = \begin{pmatrix} 1 - \dfrac{\partial x}{\partial \alpha} & -\dfrac{\partial x}{\partial \beta} \\ \\ -\dfrac{\partial \dot{x}}{\partial \alpha} & 1 - \dfrac{\partial \dot{x}}{\partial \beta} \end{pmatrix}$$

until our Newtons method sequence converges, giving us $\Gamma(v_n,\delta) = F(v_n) - (1-\delta)F(v_0)$. After finding a new point on the path we make our next Euler's Approximation and continue until $\tau = 1$.

In certain parts of the path Newton's method may fail to converge for Euler's step size. We insure convergence by halving the step size for Euler's method whenever $\|\Gamma(v_{n-1})\|/\|\Gamma(v_n)\| < 10$ for two points along the Newton sequence. This way, Newton's method must produce points v_n such that $\|\Gamma(v_n)\| \to 0$. On the other hand, we double our step size after Newton's method converges, thus allowing us to speed up our algorithm when the Γ is well behaved.

5 Implementation

We have implemented this algorithm using the computer algebra system Mathematica, using it's differential equation solver `NDSolve` with working precision 26 to emulate double precision arithmetic for our computations. We found single precision arithmetic to work for linear cases, but in the nonlinear cases double precision arithmetic was necessary.

References

[1] Earl A. Coddington and Norman Levinson. *Theory of Ordinary Differential Equations.* Robert E. Krieger Publishing Company, New York, 2^{nd} edition, 1984.

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