# Computing Steady Flow States using Fast Iterative Solvers

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## Background

Fast computation of steady flow states for

- Implicit time stepping strategies
- Efficient stability analysis
- Improved classification of bifurcations

## Steady Fluid Model

**Incompressible Navier Stokes**

$$\begin{aligned} -\nu\nabla^2\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p &= f \\ \nabla\cdot\vec{u} &= 0 \end{aligned} \quad \text{in} \quad \Omega.$$

## Spatial Discretization

**Spectral Element Method**

On each element, the solution is expressed via a nodal basis

$$u_e^N(x,y) = \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} u_{ij}\pi_i(x)\pi_j(y).$$

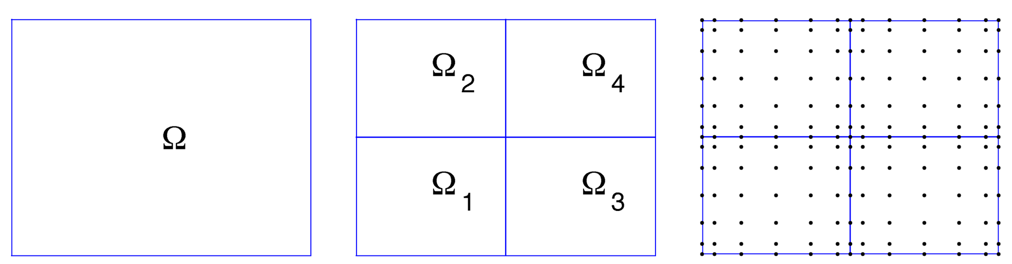



Figure: Illustration of a spectral element discretization.

## Discrete Nonlinear Block System

$$\begin{bmatrix} F(u) & -D^T \\ -D & 0 \end{bmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} Mf \\ 0 \end{pmatrix}$$

$F(u)$ - Nonlinear Convection-Diffusion
$D^T$ - Gradient
$D$ - Divergence
$M$ - Mass Matrix

## Solution Algorithm

Nonlinear Solver (Picard) $x_{k+1} = x_k + \Delta x_k$
Linear Solver (FGMRES) $A\Delta x_k = b$
Preconditioner (LSC+DD) $AP^{-1}P\Delta x_k = b$

## Block Preconditioner

Approximate upper block of the LU factorization

$$\begin{bmatrix} F & -D^T \\ -D & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -DF^{-1} & I \end{bmatrix}\underbrace{\begin{bmatrix} F & -D^T \\ 0 & -S \end{bmatrix}}_{P}.$$

## Least-Squares Commutator (LSC)

Approximating the Schur Complement using LSC provides a computationally efficient operator

$$\hat{S} := (DM^{-1}D^T)(DM^{-1}FM^{-1}D^T)^{-1}(DM^{-1}D^T)$$

$$\hat{S}^{-1} = \underbrace{(DM^{-1}D^T)^{-1}}_{\text{Poisson Solve}}(DM^{-1}FM^{-1}D^T)\underbrace{(DM^{-1}D^T)^{-1}}_{\text{Poisson Solve}}$$

## Domain Decomposition (DD)

We approximate the Convection-Diffusion operator $F$ with an operator $\bar{F}$ based on average wind on each element. This allows the inverse action to be applied efficiently via DD.

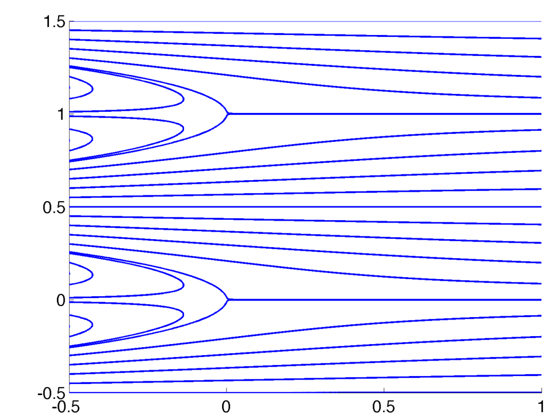
$$\begin{bmatrix} \bar{F}_{II}^1 & 0 & \dots & 0 & \bar{F}_{I\Gamma}^1 \\ 0 & \bar{F}_{II}^2 & 0 & \dots & \bar{F}_{I\Gamma}^2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{F}_{II}^E & \bar{F}_{I\Gamma}^E \\ 0 & 0 & \dots & 0 & \bar{F}_S \end{bmatrix}\begin{pmatrix} u_{I^1} \\ u_{I^2} \\ \vdots \\ u_{I^E} \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} \hat{b}_{I^1} \\ \hat{b}_{I^2} \\ \vdots \\ \hat{b}_{I^E} \\ g_\Gamma \end{pmatrix}.$$

Solving system $\bar{F}_S = \sum_{e=1}^{E}(\bar{F}_{\Gamma\Gamma}^e - \bar{F}_{\Gamma I}^e \bar{F}_{II}^{e^{-1}} \bar{F}_{I\Gamma}^e)$ determines interface values.

Interiors are obtained via Fast Diagonalization
$\bar{F}^{e^{-1}} = \tilde{M}(V_y\otimes V_x)(\Lambda_y\otimes I + I\otimes\Lambda_x)^{-1}(V_y^{-1}\otimes V_x^{-1})\tilde{M}$

## Kovasznay Flow Re=40

$$\begin{aligned} u &= 1 - e^{\lambda x}\cos 2\pi y, \\ v &= \frac{\lambda}{2\pi}e^{\lambda x}\sin 2\pi y, \\ p &= \frac{1}{2}(1 - e^{2\lambda x}), \\ \lambda &:= \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}. \end{aligned}$$



| Polynomial degree N | $\frac{\|u-u_N\|}{\|u\|}$ | $\frac{\|v-v_N\|}{\|v\|}$ | Picard steps | FGMRES steps | $\bar{F}_S^{-1}$ steps | $\hat{S}^{-1}$ steps |
|---|---|---|---|---|---|---|
| 4 | $6.8\times10^{-4}$ | $2.2\times10^{-3}$ | 12 | 42 | 4 | 27 |
| 5 | $2.6\times10^{-5}$ | $8.0\times10^{-5}$ | 12 | 39 | 4 | 35 |
| 6 | $3.6\times10^{-6}$ | $2.0\times10^{-5}$ | 15 | 35 | 3 | 22 |
| 7 | $2.5\times10^{-8}$ | $3.1\times10^{-7}$ | 18 | 32 | 3 | 25 |
| 8 | $1.1\times10^{-9}$ | $1.8\times10^{-8}$ | 23 | 27 | 4 | 28 |
| 9 | $5.1\times10^{-11}$ | $8.8\times10^{-10}$ | 27 | 26 | 5 | 45 |
| 10 | $2.0\times10^{-12}$ | $4.2\times10^{-11}$ | 29 | 24 | 6 | 74 |
| 11 | $8.1\times10^{-14}$ | $1.8\times10^{-12}$ | 30 | 24 | 5 | 58 |
| 12 | $3.1\times10^{-15}$ | $7.0\times10^{-14}$ | 36 | 24 | 10 | 72 |

Table: Convergence results based on p-refinement.

## Flow over a step

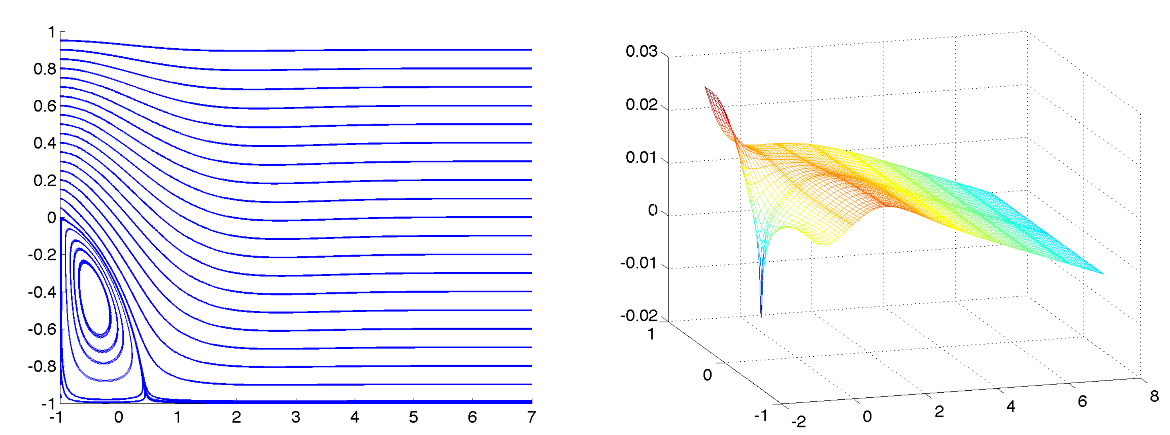


Figure: Streamline plot (left) and pressure plot (right) of flow over a step with $Re = 200$.

| Re | N | Picard steps | FGMRES steps | $\bar{F}_S^{-1}$ steps | $\hat{S}^{-1}$ steps |
|---|---|---|---|---|---|
| 10 | 4 | 4 | 14 | 22 | 66 |
| 100 | 4 | 6 | 22 | 16 | 27 |
| 200 | 4 | 7 | 33 | 14 | 20 |
| 10 | 8 | 4 | 22 | 34 | 59 |
| 100 | 8 | 5 | 31 | 25 | 39 |
| 200 | 8 | 7 | 31 | 23 | 39 |
| 10 | 16 | 4 | 42 | 42 | 106 |
| 100 | 16 | 5 | 57 | 38 | 200 |
| 200 | 16 | 6 | 69 | 34 | 72 |

Table: Convergence results based on p-refinement.

| Re | E | Picard steps | FGMRES steps | $\bar{F}_S^{-1}$ steps | $\hat{S}^{-1}$ steps |
|---|---|---|---|---|---|
| 10 | 16 | 4 | 14 | 22 | 66 |
| 100 | 16 | 6 | 22 | 16 | 27 |
| 200 | 16 | 7 | 33 | 14 | 20 |
| 10 | 64 | 4 | 16 | 56 | 47 |
| 100 | 64 | 5 | 22 | 34 | 24 |
| 200 | 64 | 7 | 28 | 32 | 28 |
| 10 | 256 | 4 | 28 | 109 | 124 |
| 100 | 256 | 5 | 31 | 94 | 48 |
| 200 | 256 | 6 | 34 | 86 | 38 |

Table: Convergence results based on h-refinement.

## Lid-Driven Cavity

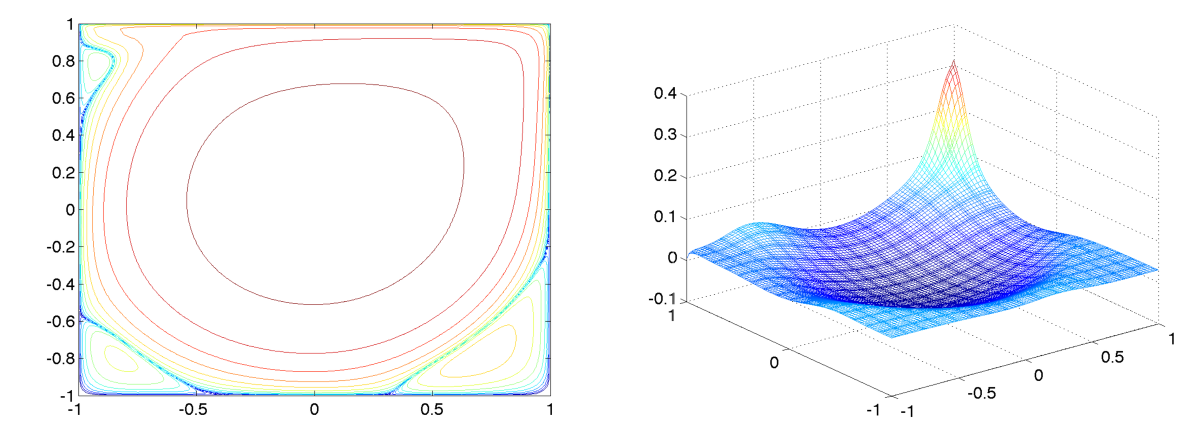


Figure: Streamline plot (left) and pressure plot (right) of Lid-Driven Cavity with $Re = 2000$.

| Re | Picard steps | FGMRES steps | $\bar{F}_S^{-1}$ steps | $\hat{S}^{-1}$ steps |
|---|---|---|---|---|
| 10 | 5 | 30 | 186 | 200 |
| 100 | 7 | 42 | 141 | 200 |
| 500 | 9 | 57 | 130 | 200 |
| 1000 | 11 | 96 | 119 | 200 |
| 2000 | 15 | 194 | 145 | 200 |
| 5000 | 20 | 240 | 107 | 200 |

Table: Convergence results based on Re, N=8 E=256.

| $N$ | Picard steps | FGMRES steps | $\bar{F}_S^{-1}$ steps | $\hat{S}^{-1}$ steps |
|---|---|---|---|---|
| 2 | 16 | 20 | 14 | 8 |
| 4 | 9 | 32 | 16 | 60 |
| 8 | 8 | 38 | 24 | 139 |
| 16 | 8 | 54 | 33 | 200 |

Table: Convergence results based on p-refinement, Re=100.

| $E$ | Picard steps | FGMRES steps | $\bar{F}_S^{-1}$ steps | $\hat{S}^{-1}$ steps |
|---|---|---|---|---|
| 16 | 16 | 20 | 14 | 8 |
| 64 | 12 | 26 | 32 | 22 |
| 256 | 9 | 37 | 94 | 45 |
| 1024 | 8 | 55 | 200 | 87 |

Table: Convergence results based on h-refinement, Re=100.

## Summary

- Developed fast solver for computing steady flows
- Used block preconditioner based on LSC & DD
- Convergence is mildly dependent on mesh refinement and the Reynolds number

## Future Directions

- Accelerate $\hat{S}$ solves
- Develop coarse-grid preconditioner
- Use Newton for nonlinear solve
- Extend to Bousinesq flows & 3D
- Apply to realistic fluid simulations