Strategies for hp-adaptive refinement

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Abstract. In the hp-adaptive version of the finite element method for solving partial differential equations, the grid is adaptively refined in both h, the size of the elements, and p, the degree of the piecewise polynomial approximation over the element. The selection of which elements to refine is determined by a local *a posteriori* error indicator, and is well established. But the determination of whether the element should be refined by h or p is still open. In this paper, we describe several strategies that have been proposed for making this determination. A numerical example to illustrate the effectiveness of these strategies will be presented.

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The numerical solution of partial differential equations (PDEs) is the most compute-intensive part of a wide range of scientific and engineering applications. Consequently the development and application of faster and more accurate methods for solving PDEs has received much attention in the past fifty years. Determining the best grid and approximation space on which to effect the solution is a central concern in this regard. Thus, developing self-adaptive techniques which lead to optimal resource allocation is critical for future progress in many fields.

Self-adaptive methods have been studied for nearly 30 years now. They are often cast in the context of finite element methods, where the domain of the PDE is partitioned into a mesh consisting of a number of elements (in two dimensions, usually triangles or rectangles), and the approximate solution is a polynomial over each element. Most of the work has focused on h-adaptive methods. In these methods, the mesh size, h, is adapted locally by means of a local error indicator with the goal of placing the smallest elements in the areas where they will do the most good. h-adaptive methods are quite well understood now, and are beginning to be used in practice.

Recently, the research community has begun to focus more attention on hp-adaptive methods. In these methods, one not only locally adapts the size of the mesh, but also the degree of the polynomials, p. The attraction of hp-adaptivity is that the error converges at an exponential rate in the number of degrees of freedom, as opposed to a polynomial rate for fixed p. The new complication is that the local error indicator is no longer sufficient to guide the adaptivity. It tells you which elements should be refined, but it does not indicate whether it is better to refine the element by h or by p. A method for making that determination is called an hp-adaptive strategy. In this paper we present a summary of several strategies that have been proposed in the literature. A numerical example will be given to demonstrate the effectiveness of the strategies.

Although the methods apply to more general elliptic partial differential equations, for simplicity we consider the Poisson equation

$$-\nabla^2 u = f(x,y) \text{ in } \Omega \tag{1}$$
$$u = g(x,y) \text{ on } \partial\Omega \tag{2}$$

where Ω is a bounded, connected, polygonal, open region in R^2.

The PDE is solved approximately by the usual Galerkin finite element method. The finite element space is defined by partitioning Ω into a grid consisting of a set of N_T triangular elements, $\{T_i\}_{i=1}^{N_T}$ with $\bar{\Omega} = \cup_{i=1}^{N_T} \bar{T}_i$, and $\bar{T}_i \cap \bar{T}_j$ is either empty, a common edge, or a common vertex for $i \neq j$. The diameter of the element is denoted h_i. With each element

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we associate an integer degree $p_i \geq 1$. The degree of an edge is the maximum of the degrees of the adjacent elements. The finite element space V_{hp} is the space of continuous piecewise polynomial functions on Ω such that over element T_i it is a polynomial of degree p_i.

If h and p are uniform over the grid, u_{hp} is in the Sobolev space $H^m(\Omega)$, and the other usual regularity assumptions are met, then the *a priori* error bound is [1, 2]

$$||e_{hp}||_{H^1(\Omega)} \leq C \frac{h^\mu}{p^{-(m-1)}} ||u_{hp}||_{H^m(\Omega)} \tag{3}$$

where $\mu = \min(p, m-1)$ and C is a constant that is independent of h, p and u, but depends on m.

This error bound implies that, if the solution is sufficiently smooth, it is better to perform p refinement than h refinement. However, if m is the largest value for which the solution is in $H^m(\Omega)$, it makes no sense to increase p beyond $m-1$. Many hp-adaptive strategies exploit this property locally by estimating the local regularity, m, of the solution over an element, and performing p refinement if m is sufficiently large compared to p_i, and h refinement otherwise.

The simplest strategy is simply to tell the strategy where the solution is irregular, if that information is known. An hp strategy of this type was presented by Ainsworth and Senior [3]. In this approach they simply flag vertices in the initial mesh as being possible trouble spots. During refinement an element is refined by h if it contains a vertex that is so flagged, and by p otherwise.

In another paper by Ainsworth and Senior [4], the regularity is estimated by using three error indicators. An error indicator is computed by approximately solving the differential equation

$$-\nabla^2 e \;=\; f + \nabla^2 u_{hp} \;\text{ in } T_i$$

with appropriate Neumann boundary conditions (see, for example [5]). If T_i has degree p_i, three estimates are computed using approximation spaces of degree p_i+1, p_i+2, and p_i+3. These estimates are then used to determine the three unknown constants in the *a priori* error estimate, one of which is the regularity m.

Suli, Houston and Schwab [6] estimate the regularity using two error estimates: $\eta_i(p_i)$ is the error estimate for the current solution, and $\eta_i(p_i-1)$ is the error estimate for a solution that has degree p_i-1 over T_i. Then considering their ratio in conjunction with the *a priori* error bound, m is estimated by

$$m = 1 \;-\; \frac{\log(\eta_i(p_i)/\eta_i(p_i-1))}{\log(p_i/(p_i-1))}$$

Gui and Babuška [7] use a type parameter, $0 < \gamma < 1$, along with the ratio of the two error estimates, $r_i = \eta_i(p_i)/\eta_i(p_i-1)$. p refinement is used if $r_i < \gamma$; otherwise h refinement is used. Adjerid, Aiffa and Flaherty [8] use $\gamma = 0.6$.

Melenk and Wohlmuth [9] compute a predicted error estimate by extrapolation under the assumption that the solution is smooth. The actual error estimate is compared to the predicted error estimate. If it is larger, h refinement is performed because this presumably implies the smoothness assumption is violated. If the actual error estimate is smaller than the predicted error estimate, p refinement is used.

Mavriplis [10] examines the decay rate of the coefficients of the expansion of the solution in Legendre polynomials, $u_{hp} = \Sigma a_i P_i(x)$. The last four coefficients are used in a least squares fit to $Ce^{-\sigma i}$. $\sigma < 1$ indicates h refinement; $\sigma > 1$ indicates p refinement.

Another use of the Legendre expansion is given by Houston, Senior and Suli [11]. This approach uses a root test to estimate the regularity. Let $l_i = \log((2i+1)/(2*a_i^2))/2*\log(i)$. The regularity is estimated by $m = \min(l_p, l_{p-1}) - 1/2$.

Schmidt and Siebert [12] suggest a strategy to estimate whether h or p refinement will reduce the error the most. Two error estimates are computed, one using the enrichment of the finite element space over the element by p refinement, and the other using the enrichment by h refinement. These estimates estimate the amount of change in the solution for each type of refinement. That which gives the largest change is used.

In a series of papers, Demkowicz and his collaborators developed a rather sophisticated hp-adaptive strategy. It is best described in [13]. Given a current grid and solution, a reference solution is computed by refining the grid globally in both h and p, and solving the problem on this grid to get $u_{h/2,p+1}$. The goal is to seek a new optimal grid that maximizes the amount of error reduction (relative to the reference solution) per increase in work, i.e. maximize

$$\frac{|u_{h/2,p+1} - \Pi_{hp} u_{h/2,p+1}|^2_{H^1(\Omega)} \;-\; |u_{h/2,p+1} - \Pi_{hp_{opt}} u_{h/2,p+1}|^2_{H^1(\Omega)}}{N_{hp_{opt}} \;-\; N_{hp}}$$

where $\Pi_{hp} u_{h/2,p+1}$ is a projection-based interpolant of the reference solution onto the current grid, $\Pi_{hp_{opt}} u_{h/2,p+1}$ is a projection-based interpolant of the reference solution onto the optimal grid, N_{hp} is the number of degrees of freedom in the current grid, $N_{hp_{opt}}$ is the number of degrees of freedom in the optimal grid, and $|\cdot|_{H^1(\Omega)}$ is the H^1 seminorm.

Each edge in the grid has the possibility of being refined by p, or being refined by h with new degrees p_1 and p_2 assigned to the children, with $p_1 + p_2 = p + 1$. Each of these possibilities is examined by computing the projection-based interpolant of the reference solution onto the edge. This leads to a determination of which refinement should be performed, or if the edge should be left as is. The h refinement of edges determines the h refinement of elements, and further calculations determine the degree, p_i, of the elements.

The Texas 3 Step strategy of Oden and Patra [14] performs h and p refinements separately. The first step is to perform uniform h refinement until the solution is in the asymptotic convergence range. In the second step they use the *a priori* error estimate locally on each element to compute the number of h refinements to perform on each element such that the global error is reduced to a given tolerance. In the third step, they do the same with p refinements to reduce the error to another given tolerance. The second and third steps may be repeated if necessary.

A completely different approach was suggested by Patra and Gupta [15]. In this strategy, the goal of designing an optimal mesh is formulated as a discrete optimization problem, which is solved using nonlinear programming.

In this talk and the full length paper, we will explain these strategies in more detail, and demonstrate their performance with a numerical example.

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