Concurrence Canonical Decompositions

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Outline

I. Entanglement and Concurrence
II. $G = KAK$ metadecomposition
III. Concurrence Canonical Decomposition
IV. Concurrence Capacities and $XY$ Hamiltonians
V. CCD and 4-qubit Circuits
VI. Conclusions and On-Going Work
Key Ideas

Concurrence canonical decompositions:

- natural extension of two-qubit canonical decomposition
- useful for entanglement theory, entanglement dynamics
- potential applications for quantum logic circuit design
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State Space of Quantum Data, Entanglement

- One-qubit state space: $\mathcal{H}_1 = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \cong \mathbb{C}^2$

- $n$-qubit state space: $\mathcal{H}_n = \otimes_1^n \mathcal{H}_1 \cong \mathbb{C}^N$, convention throughout $N = 2^n$

- Alternately: $\mathcal{H}_n \cong \bigoplus_{\bar{b} \in (\mathbb{F}_2)^n} \mathbb{C}|\bar{b}\rangle$, with $\bar{b} \in (\mathbb{F}_2)^n$ an $n$ bit string

- Entangled $|\psi\rangle$: any $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$
Concurrence Monotone, Concurrence Form

- For Pauli $\sigma^y$, recall that $-i\sigma^y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

- Concurrence Form: $C_n(\ket{\psi}, \ket{\phi}) = \overline{\bra{\psi}(-i\sigma^y)^n\ket{\phi}}$

- Math: $S = (-i\sigma^y)^n$, $x = \ket{\psi}$, $y = \ket{\phi}$, then $C_n(x, y) = x^T S y$

- $C_n(x, y)$ bilinear (linear for each variable)

- Concurrence entanglement monotone: $C_n(\ket{\psi}) = |C_n(\ket{\psi}, \ket{\psi})|$
\[ C_n(\ket{\phi}, \ket{\psi}) = C_n(v\ket{\phi}, v\ket{\psi}), \text{any } v \text{ in } \mathbb{L}\mathbb{U} \]

**Proof:** Proceed as follows.

- For $2 \times 2$ matrix $m$, note that $m^T(-i\sigma^y)m = (\det m)(-i\sigma^y)$
  - $v^T(-i\sigma^y)w$ is $\pm$ area spanned by $v, w$
  - Check equation by considering columns, rows of $m$

- Let $v_1 \otimes v_2 \otimes \cdots \otimes v_n$, each $\det v_j = 1$ by choice global phase

- So $(v_1 \otimes v_2 \otimes \cdots \otimes v_n)^T S(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \otimes_{j=1}^n [v_j^T(-i\sigma^y)v_j] = (-i\sigma^y) \otimes^n$

**Remark:** Similar proof shows $C_n(\ket{\psi_{n-1}} \otimes \ket{\phi_1}) = 0$.  

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Naturality: \textbf{CCD by } $G = KAK$ metadecomposition

- **Metadecomposition:** theorem outputting matrix decompositions

- **Three cascading inputs**, each dependent on last

- **Generalizes canonical dec.:** $SU(4) = [SU(2) \otimes SU(2)] \Delta [SU(2) \otimes SU(2)]$
  
  - $SU(2) \otimes SU(2)$: two-qubit local unitary (LU) group
  
  - $\Delta$: relative phase computations on Bell (or “magic”) basis

- **Some canonical decomposition applications:** 2q control theory, 2q quantum logic circuits, 2q computation times, 2q entanglement theory
Cascading Inputs of $G = KAK$ theorem

1. Lie group $G$, semisimple, $\mathfrak{g} = \text{Lie}(G)$ ($\implies G = \exp \mathfrak{g}$, matrix exponential)

2. Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ (using same term if $G$ compact)
   - $[\theta X, \theta Y] = \theta [X, Y]$
   - $\theta^2 = 1$ (involution)
   - Notation: $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{p} = \mathfrak{g}_{-1}$, $\mathfrak{k} = \mathfrak{g}_{+1}$
   - Encodes generalized polar decomp. of $\mathfrak{g}$

3. (Maximal-Commutative) subspace $\mathfrak{a} \subset \mathfrak{p}$
$G = KAK$ Example:
Singular Value Decomposition

- Notation:
  \[ Gl(n, \mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} ; \det A \neq 0 \} \]
  \[ O(n) = \{ A \in \mathbb{R}^{n \times n} ; A^{-1} = A^T \} \]

- $Gl(n, \mathbb{R}) = O(n)$ \{diag, invertible\} $O(n)$: SVD on invertibles

- $G = Gl(n, \mathbb{R})$, $g = \mathbb{R}^{n \times n}$, with second input $\theta(X) = -X^T$

- $\mathfrak{t} = \{\text{antisymmetric}\}$, $K = \exp \mathfrak{t} = SO(n)$, doubly covered by $O(n)$

- $\mathfrak{p} = \{\text{symmetric}\}$, take a diagonal, $A = \exp \alpha = \{\text{diagonal-invertible}\}$
Example: Bloch Sphere Rotations

- Take $G = SU(2)$ (prefix $SG$: determinant-one subgroup)

- $\mathfrak{su}(2) = \{ A \in \mathbb{C}^{2 \times 2} ; A^\dagger = -A, \text{tr} A = 0 \}$

- $\theta(X) = -X^T = \bar{X}$, fixes $\mathfrak{k} = \mathfrak{so}(2) = \mathfrak{su}(2) \cap \mathbb{R}^{2 \times 2}$

- $\{R_y(t)\} = SO(2) = \exp \mathfrak{so}(2)$, the $Y$-axis Bloch sphere rotations

- Take $a = \mathbb{R}(i|0\rangle\langle 0| - i|1\rangle\langle 1|)$, so $A = \{R_z(t)\}$

- Result: $SU(2) = \{R_y(t)\}\{R_z(t)\}\{R_y(t)\}$, the $YZY$-decomposition
G = KAK  Example: “(Speical) Unitary SVD”

- Almost repeat last slide: take $G = SU(\ell)$

- $\mathfrak{su}(\ell) = \{A \in \mathbb{C}^{\ell \times \ell} ; A^\dagger = -A, \, \text{tr} \, A = 0\}$

- $\theta(X) = -X^T = \bar{X}$, fixes $\mathfrak{k} = \mathfrak{so}(\ell) = \mathfrak{su}(\ell) \cap \mathbb{R}^{\ell \times \ell}$

- $K = SO(\ell) = \exp \mathfrak{so}(\ell)$, determinant one orthogonals

- Take $a = \{i \sum_{j=0}^{\ell-1} t_j |j\rangle\langle j| ; \sum_{j=0}^{N-1} t_j = 0\}$

- Result: $SU(\ell) = SO(\ell) A SO(\ell)$, where $A = \{\text{unitary diagonal, det} = 1\}$
$G = KAK$ Example: 2q Canonical Decomposition

- $G = SU(4)$, with $\mathfrak{g} = \mathfrak{su}(4) = \{ A \in \mathbb{C}^{\ell \times \ell} \;; A = -A^\dagger, \text{tr} A = 0 \}$

- $\theta(X) = (-i\sigma^y) \otimes^2 [-X^T] (-i\sigma^y) \otimes^2 = (-i\sigma^y) \otimes^2 [\bar{X}] (-i\sigma^y) \otimes^2$

- Check: for this $\theta$, in fact $K = SU(2) \otimes SU(2)$
  
  - $+1$ eigenspace of $\theta$ is $[1_2 \otimes \mathfrak{su}(2)] \oplus [\mathfrak{su}(2) \otimes 1_2]$
  
  - Product Rule: $\text{Lie}[SU(2) \otimes SU(2)] = [1_2 \otimes \mathfrak{su}(2)] \oplus [\mathfrak{su}(2) \otimes 1_2]$

- May choose $A = \Delta$ phasing Bell basis so that $\alpha \subset \mathfrak{p}$

- Result: canonical dec. $SU(4) = [SU(2) \otimes SU(2)] \Delta [SU(2) \otimes SU(2)]$
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Definition: Extend Canonical Decomposition

\[ G = KAK \] Inputs from Two Qubits

- **KAK Inputs**: Label \( S = (-i \sigma^y)^\otimes n \), and take \( \theta(X) = S^{-1}[-X^t]S = S^{-1} \tilde{X}S \)

- For \( \xi \) is \( +1 \)-eigenspace, the \( LU \) algebra \( \text{Lie}[\otimes^n_1 SU(2)] \subsetneq \xi \)

\[ \alpha = \text{span}_\mathbb{R} \left( \{ i|j\rangle\langle j| + i|N - j - 1\rangle\langle N - j - 1| - i|j + 1\rangle\langle j + 1| - i|N - j - 2\rangle\langle N - j - 2| \right) \]
\[ \bigcup \left( \{ i|j\rangle\langle N - j - 1| + i|N - j - 1\rangle\langle j| \right) \text{,} \text{ in case } n \text{ even} \]

- **Concurrence Canonical Decomposition** is \( SU(N) = KAK \)
Interpretation: $K$ Fixes Concurrence

**Theorem (—,GKB):** Let $K = \exp \mathfrak{k}$ for $\mathfrak{k}$ the $+1$-eigenspace of the Cartan involution $\theta(X) = S^{-1}\bar{X}S$, for $S = (-i\sigma^y)^\otimes n$. Then $K$ is the symmetry group of the concurrence form $C_n(\cdot,\cdot)$. Specifically, for $u \in SU(N)$,

$$(u \in K) \iff \left[ C_n(u|\phi\rangle, u|\psi\rangle) = C_n(|\phi\rangle, |\psi\rangle) \quad \text{for every} \quad |\phi\rangle, |\psi\rangle \in \mathcal{H}_n \right]$$

Moreover, for $C_n(y,x) = (-1)^n C_n(x,y)$, and auxiliary constructions demand

- abstract isomorphism $K \cong Sp(N/2)$, a symplectic group, $n$ odd
- abstract isomorphism $K \cong SO(N)$, $n$ even
Algorithms in Even Qubits: Entanglers

- Entangler: Matrix $E$ whose similarity relation exchanges the concurrence canonical decomposition for special unitary $\text{SVD}$, $n$ even
  
  - $E \, \text{SO}(N) \, E^\dagger = K$, with $K$ per CCD
  
  - $E\{\text{diag special unitary}\}E^\dagger = A$
  
  - $[(E\nu E^\dagger = o_1 do_2 \text{ a unitary SVD})] \implies [\nu = (Eo_1 E^\dagger)(EdE^\dagger)(Eo_2 E^\dagger) \text{ a CCD}]$

- Prop: $E$ entangles iff $EE^T = \xi S$, for some global phase $\xi$ with $\xi^N = 1$

- Prop: $(n \text{ odd}) \implies$ no such matrix $E$ exists
A Four-Qubit Entangler: Columns Are GHZ States

$$E_0 = (1/\sqrt{2})$$

$$\begin{pmatrix}
1 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
Algorithm for Unitary SVD, Part 1

- For $v = o_1d o_2$, compute $p = vv^T = o_1d^2 o_1^T$.

- $pp^\dagger = 1_N$, also complex symmetric $p = p^T$

- Given algorithm for $p = o_1d^2 o_1^T$, compute $\sqrt{p} = o_1d o_1^T$

- Compute $o_3 = \sqrt{p^\dagger}v$

- So $v = o_1d(o_1^T) o_3 = o_1d o_2$
Algorithm for Unitary SVD, Part 2

Lemma: Suppose for $p \in \mathbb{C}^{\ell \times \ell}$, both $p = p^T$ and $pp^\dagger = 1$. Then there is an orthogonal matrix $o$ with $opo^T$ some diagonal unitary.

Proof: Note that $1_N + 0_{Ni} = pp^\dagger = (a^2 - b^2) + (ba - ab)i$. So $a, b \in \mathbb{R}^{n \times n}$ commute, each is symmetric, and $a$ respects the eigenspaces of $b$. □

Restatement: If a unitary is also complex symmetric, then we may diagonalize the real and imaginary parts simultaneously over the same orthogonal matrix.
Algorithm for Unitary $\text{SVD}$, Part 3

- **Algorithm**: Implement these steps to orthogonally diagonalize complex $p = a + ib$

  - Diagonalize $b = o_1 d o_1^T$

  - Take permutation $\pi$ so that repeated eigenvalues of $d$ occur consecutively: $b = (o_1 \pi)(\pi^T d \pi)(\pi^T o_1^T) = o_2 d_2 o_2^T$

  - $o_2 a o_2^T$ must be block diagonal by lemma

  - Diagonalize each block for $o_3 a o_3^T$, $o_3 b o_3^T$ diagonal

- **Result**: Algorithm for unitary $\text{SVD}$ and even-qubit $\text{CCD}$
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Application to Entanglement Dynamics: Concurrence Capacities, \(n\) even

- Concurrence capacity \(\kappa_n(v) = \max \{|C_n(vx)| \ ; \ C_n(x,x) = 0, x^\dagger x = 1\}\)

- Symmetry: \(\kappa(kv) = \kappa(v), \ k \in K = ESO(N) E^\dagger\)

- Symmetry and max: \(\kappa(vk) = \kappa(v), \ \text{any} \ k \in K\)

- Conclusion: \(\kappa(v = k_1 ak_2) = \kappa(a)\)

- Importance: \(\dim A = N - 1; \ \dim SU(N) = N^2 - 1\)
Concurrence Capacity and Concurrence Spectra

• Choose an entangler so \((E^T E)^2 = 1_N\).
  
  – Possible all even \(n\)
  
  – Example on last slide works in 4 qubits

• Computation: then \(C_n(Ex, Ey) = x^T y\)

• For such an \(E\), label the concurrence spectrum

\[
\lambda_c(v) = \text{spec}[(E^\dagger vE)^T (E^\dagger vE)] = \text{spec}\{E^T v^T [(-i\sigma^y)^\otimes n] vE\}
\]
Concurrence Capacity and Concurrence Spectra Cont.

- If $\lambda_c(v) = \{\lambda_j\}_{j=0}^{N-1}$, note that $|\lambda_j|^2 = 1$

- Using $C_n(Ex, Ey) = x^T y$, we may translate

$$\kappa(v) = \max \left\{ \left| \sum_{j=0}^{N-1} a_j^2 \lambda_j \right| ; \sum_{j=0}^{N-1} a_j^2 = 0, \sum_{j=0}^{N-1} |a_j|^2 = 1 \right\}$$

- Let the convex hull $\text{CH}[\lambda_c(v)]$ be the convex span of $\lambda_c(v)$

- Prop: (— & GKB, cf. Zhang et al) $\kappa(v) = 1$ iff $0 \in \text{CH}[\lambda_c(v)]$
Picture: Concurrence Spectrum and $c_H[\lambda_c(v)]$
Points of $\lambda_c(v)$ always on $\{|z^2| = 1\}$

Convex hull is least convex polygon holding $\lambda_c(v)$

(Maximum $\kappa = 1$) $\iff$ (0 is in convex hull)

Surprising Result: For most randomly chosen $a$ with $n = 2p$ large, the concurrence capacity of any $v = k_1ak_2$ is $\kappa(v) = \kappa(a) = 1$
Concurrence Spectrum and $XY$ Hamiltonians

- Anisotropic Heisenberg $XY$ Hamiltonian: For $g, \gamma \in \mathbb{R}$,

$$H_{XY} = g \sum_{j=1}^{n} (1 - \gamma) \sigma_j^x \sigma_{j+1}^x + \gamma \sigma_j^y \sigma_{j+1}^y$$

- Since Pauli matrices anticommute, $S H_{XY} = H_{XY} S$

- Thus $\theta(iH_{XY}) = S[-iH_{XY}]S = -iH_{XY}S^2 = -iH_{XY}$, so that $iH_{XY} \in \mathfrak{p}$

- Consequence: diagonal Hamiltonian $H_d$, we have $e^{itH_{XY}} = k_1 [E e^{itH_d} E^\dagger] k_1^\dagger$

- If $tH_d = \sum_{j=0}^{N-1} td_j |j\rangle \langle j|$, then $\lambda_c[e^{itH_{XY}}] = \{e^{2itd_j}\}_{j=0}^{N-1}$
Concurrence Dynamics: $\kappa[e^{itH_{XY}}]$ vs. $t$
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Quantum Logic Circuit Design Philosophy

- Philosophically, two (three?) major steps in quantum circuit design
  - Choose some matrix decomposition
  - Work hard to get efficient (logic?) circuits on each factor
  - (?) Adapt to implementation, i.e. physical system, DFS, & ECC

- Some matrix decompositions are better than others
  - Respects tensor (Kronecker) product structure in some way
  - Implicit CNOT or other conditioned structure? (e.g. Cosine-Sine)
4q Concurrence Canonical Decomp. & Circuits?

- 4q Concurrence canonical decomposition has some advantages
  
  - $v = a \otimes b \otimes c \otimes d$ demands some $v = k_1 1_{16} 1_{16}$
  
  - **Efficient** circuits for diagonal computations (---, Markov) $\iff$ reasonably good circuit for $E\{\text{diag}\}E^\dagger$

- **Problem:** $EoE^\dagger$ implementation, $o \in SO(16)$

  - Possibility: controlled real two-qubit computations
  
  - Could we instead **recurse** $SO(16) = K_2 A_2 K_2$?
Limitations of Recursive $SO(2^{2p}) = KAK$

**THM: (—, private notes):** Let $n = 2p$ even, and choose any $E$ an entangler in $n$-qubits. Then there exists no Cartan involution $\theta : so(N) \rightarrow so(N)$ fixing $\mathfrak{k}$ with $\exp(\mathfrak{k}) = K \subset SO(N)$ so that $E^\dagger \left( \bigotimes_1^n SU(2) \right) E \subset K$.

- Attempt instead to split locals across $p, \mathfrak{k} \in so(N)$?

- Can matrix decomposition be forced to recognize tensor structure?

- Similar result holds in less intuitive odd-qubit cases.
$4q$ Entangler, Real then Imaginary Columns

$$E_1 = \left( \frac{1}{\sqrt{2}} \right)$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
Interesting Property of Entangler $E_1$

- Switch to using entangler $E_1$ of last slide

- Say $v = a \otimes b \otimes c \otimes d = [\otimes_{j=1}^{4} R_y(\theta_j)][\otimes_{j=1}^{4} R_z(\alpha_j)][\otimes_{j=1}^{4} R_y(\theta^2_j)]$

- $R_y$ tensor real $\implies E_1[\otimes R_y]E_1^\dagger$ are block diagonal

- Compute $E_1[\otimes R_z]E_1^\dagger$ are block anti-diagonal, moreover cosine-sine
Interesting Property of Entangler $E_1$, Cont.

- Some recognition of tensor structure for $v = a \otimes b \otimes c \otimes d$:
  - Take $\text{CCD}$ by $v = (v) 1_{16} 1_{16}$
  - Cosine-Sine matrix decomposition of orthogonal $E_1 v E_1^\dagger$:
    \[
    E_1 v E_1^\dagger = \{ E_1 \{ \otimes_{j=1}^{4} R_y(\theta_j) \} E_1^\dagger \} \{ E_1 \{ \otimes_{j=1}^{4} R_z(\alpha_j) \} E_1^\dagger \} \{ E_1 \{ \otimes_{j=1}^{4} R_y(\theta'_j) \} E_1^\dagger \}
    \]
- Can this be leveraged for efficient circuit diagrams?
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Conclusions

Concurrence canonical decompositions:

- Natural extension by \textit{KAK inputs} of two-qubit canonical decomposition
- Useful for entanglement theory, entanglement dynamics esp. $\kappa[\nu(t)]$
- Applications for quantum logic circuit design using cosine-sine?
Ongoing Work

- Further work on circuit design

- Algorithms for odd qubit CCD
  - Similarity matrix (fingered $F$) to $SU(N) = Sp(N/2) \times Sp(N/2)$, where $\dim A = N/2 - 1$, $A$ repeated diagonal
  - Latter computable by symplectic Jacobi algorithm?

- Further work on entanglement dynamics

- Numerical tests of nonlocality of concurrence 0 states