Analysis of Switching in Uniformly Magnetized Bodies

Michael J. DONAHUE and Donald G. PORTER, *Member, IEEE*

Abstract— **A full analysis of magnetization reversal of a uniformly magnetized body by coherent rotation is presented. The magnetic energy of the body in the presence of an applied field H is modeled as $E = (\mu_0/2)\mathbf{M}^T D\mathbf{M} - \mu_0\mathbf{H}^T\mathbf{M}$, where T denotes a matrix transpose. This model includes shape anisotropy, any number of uniaxial anisotropies, and any energy that can be represented by an effective field that is a linear function of the uniform magnetization M. The model is a generalization to three dimensions of the Stoner-Wohlfarth model. Lagrange multiplier analysis leads to quadratically convergent iterative algorithms for computation of switching field, coercive field, and the stable magnetization(s) of the body in the presence of any applied field. Magnetization dynamics are examined as the applied field magnitude $|H|$ approaches the switching field H_s, and it is found that the precession frequency $f \propto (H_s - |H|)^{(1/4)}$ and the susceptibility $\chi \propto (H_s - |H|)^{-(1/2)}$.**

Keywords— **Stoner-Wohlfarth model, uniform rotation, coherent rotation, single-domain particles, standard problems, micromagnetic simulation**

I. Introduction

STANDARD problems have proven useful for verifying the calculations of micromagnetic simulations [1], [2]. Particularly useful are simple problems with solutions that can be determined analytically. For example, it was determined in [1] that an average field method is superior to a sampled field method for calculating self-demagnetizing fields due to its agreement with analysis of an ideal uniformly magnetized body. An assumption of uniform magnetization is one way to bring examination of magnetization switching within the reach of analysis.

The study of magnetization switching in a uniformly-magnetized body by uniform rotation has been a topic of interest in its own right as well. Most widely known is the Stoner-Wohlfarth model [3] that predicts the switching field for a body with anisotropy completely characterized by a single axis. This simple model has been extended in many ways. Most relevant to our work is the extension to an arbitrary shape anisotropy [4] and an additional uniaxial anisotropy [5]. Solutions of these extended models have been determined by tabulation of energies sampled over a two-dimensional space of magnetization directions, and interpolation between tabulated values to find energy minima.

In this paper, we consider and analyze a more general class of uniformly magnetized bodies, any body for which the magnetic energy in the presence of an applied magnetic field $\mathbf{H}$ may be expressed by

$$E = (\mu_0/2)\mathbf{M}^T D\mathbf{M} - \mu_0\mathbf{H}^T\mathbf{M}, \tag{1}$$

where T denotes matrix transpose. Here D is any (3×3) matrix, so this model may include shape anisotropy, any number of uniaxial anisotropies, and indeed any form of magnetic energy that can be represented by an effective field that is a linear function of magnetization. Without loss of generality, D is symmetric, because any asymmetric part contributes nothing to the energy. Exchange energy does not appear due to the assumption of uniform magnetization. This formulation is a three-dimensional form of the Stoner-Wohlfarth model. In this paper we present an analysis that classifies all stationary points of (1) and computes them with a quadratically convergent iterative algorithm over a single parameter.

An even more general three-dimensional model is considered in [6]. Techniques for finding stationary points by geometric construction are described. However, for actual calculation of numerical values, searches over a two-dimensional space are prescribed. For the special case covered by (1) – described in [6] as "biaxial anisotropy of second degree" – we offer a more direct calculation algorithm.

When uniform magnetization is assumed, and the geometric boundaries of the body include sharp corners, the demagnetizing field at the corners diverges. It has been shown that micromagnetic calculations remain valid in that situation [7], [8].

II. Lagrange Analysis

To analyze the switching properties of the model described by (1), we seek the values of $\mathbf{M}$ that minimize E for fixed $|\mathbf{M}| = M$. This is a constrained optimization problem. Let $\mathbf{M} = M\mathbf{m}$ and $\mathbf{H} = H\mathbf{h}$, where $|\mathbf{m}| = |\mathbf{h}| = 1$, so we may consider the magnitudes and directions of $\mathbf{M}$ and $\mathbf{H}$ separately. M and $\mathbf{h}$ are fixed quantities. The applied field sweeps over a fixed axis by variation of H. The magnetization is free to rotate by variation of $\mathbf{m}$. By choice of the coordinate system, $D = \mathrm{diag}(D_x, D_y, D_z)$ is a diagonal matrix with $D_x < D_y < D_z$. Introduce the Lagrange multiplier $(\mu_0\lambda)$ on the constraint, and solve for the condition

$$\nabla_{\mathbf{M}} E = \mu_0 D\mathbf{M} - \mu_0\mathbf{H} = (\mu_0\lambda)\mathbf{M} \tag{2}$$

to be met by stationary values of $\mathbf{M}$. Solving for $\mathbf{M}$ yields

$$\mathbf{M}(\lambda) = M\mathbf{m}(\lambda) = (D - \lambda I)^{-1}\mathbf{H}, \tag{3}$$

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Authors are with the National Institute of Standards and Technology, Gaithersburg, Maryland 20899 USA (e-mail: michael.donahue@nist.gov, d.porter@ieee.org).

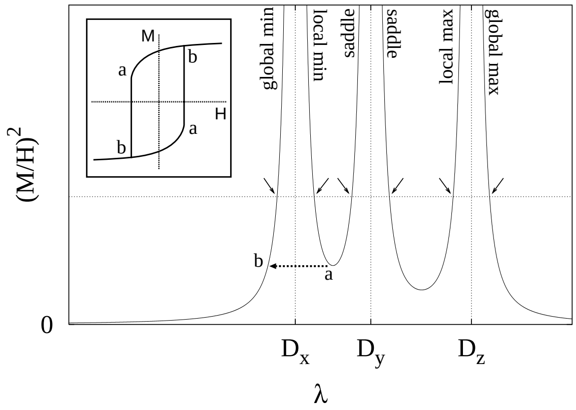

Fig. 1. Plot of $g(\lambda)$. $D_x < D_y < D_z$. $h_\nu \neq 0, \nu = x, y, z$

and substitution back into the constraint yields

$$g(\lambda) \triangleq \frac{h_x^2}{(D_x-\lambda)^2} + \frac{h_y^2}{(D_y-\lambda)^2} + \frac{h_z^2}{(D_z-\lambda)^2} = \left(\frac{M}{H}\right)^2. \tag{4}$$

Fig. 1 contains a plot of $g(\lambda)$ as a function of λ. Each value of H determines a horizontal line in Fig. 1. Solutions of (4) for a given H are the intersection of that line with $g(\lambda)$, indicated by the arrows. The number of solutions varies between 2 and 6, depending on the value of H.

III. Classification of stationary points

Given H, let λ^* be a particular solution to (4), and $\mathbf{m}^* = \mathbf{m}(\lambda^*)$ be the corresponding stationary point of (1). If $\mathbf{m}$, $|\mathbf{m}| = 1$ is a small perturbation from $\mathbf{m}^*$, then the difference in energy is

$$E(\mathbf{m}) - E(\mathbf{m}^*) = \frac{1}{2}\mu_0 M^2\, \Phi(\mathbf{m} - \mathbf{m}^*) \tag{5}$$

where

$$\Phi(\mathbf{m} - \mathbf{m}^*) \triangleq (\mathbf{m} - \mathbf{m}^*)^T (D - \lambda^* I)(\mathbf{m} - \mathbf{m}^*).$$

For $\lambda^* < D_x$, all diagonal values of $D - \lambda^* I$ are positive, so $\Phi(\mathbf{m} - \mathbf{m}^*) \geq 0$ for all $\mathbf{m}$ and $\mathbf{m}^*$ is a global minimum. Likewise, for $\lambda^* > D_z$, $\mathbf{m}^*$ is a global maximum. To classify $\mathbf{m}^*$ as a local minimum, we need to show that $\Phi(\mathbf{m}-\mathbf{m}^*) \geq 0$ for all $\mathbf{m}$ in a sufficiently small neighborhood of $\mathbf{m}^*$. Let $\mathbf{u}$ be the projection of $\mathbf{m} - \mathbf{m}^*$ onto the orthogonal complement of $\mathbf{m}^*$. Then if $|\mathbf{m}-\mathbf{m}^*|$ is small, the difference between $\Phi(\mathbf{m} - \mathbf{m}^*)$ and $\Phi(\mathbf{u})$ is $\mathcal{O}(|\mathbf{m} - \mathbf{m}^*|^3)$. For our purposes we only need expansions of E to second order, so it suffices to consider Φ restricted to the 2-dimensional subspace $\mathbf{m}^{*\perp}$.

Moreover, since $\Phi(r\mathbf{u}) = r^2\Phi(\mathbf{u})$, it suffices to direct our attention to finding extremal values of Φ on the closed, bounded set $|\mathbf{u}| = 1$, $\mathbf{u}^T\mathbf{m}^* = 0$. Let a_1 and a_2 be respectively the minimum and maximum values of Φ on this set. Solving the constrained optimization problem leads to the relations

$$a_1 a_2 = \frac{H^2}{2M^2}(D_x - \lambda^*)(D_y - \lambda^*)(D_z - \lambda^*) g'(\lambda^*) \tag{6}$$

and

$$a_1 + a_2 = D_x + D_y + D_z - 3\lambda^* - \frac{\mathbf{H}^T\mathbf{m}^*}{M}. \tag{7}$$

Combining these two relations leads to a quadratic equation in either a_1 or a_2. There are two real roots; the smaller is a_1, and the larger is a_2.

However, explicit formulae for the a_i's are not needed to classify the stationary points of (1). These values are also the eigenvalues associated with restriction of the bilinear form Φ to $\mathbf{m}^{*\perp}$. Using the "interlacing property" [9], [10] (also known as the Sturmian separation theorem), it follows that

$$D_x - \lambda^* \leq a_1 \leq D_y - \lambda^* \leq a_2 \leq D_z - \lambda^*. \tag{8}$$

Now consider the case $D_x < \lambda^* < D_y$. Here $a_2 > 0$, so it follows from (6) that a_1 has the sign opposite that of $g'(\lambda^*)$. Therefore, if $g'(\lambda^*) < 0$, then $a_1 > 0$ and thus $\mathbf{m}^*$ is a local minimum. If $g'(\lambda^*) > 0$, then $a_1 < 0$ and $\mathbf{m}^*$ is a saddle point. A similar analysis for $D_y < \lambda^* < D_z$ confirms the remaining classifications indicated in Fig. 1.

IV. Calculation of Switching Field and Minima

A sketch of a hysteresis loop is depicted in the inset of Fig. 1. The points on the hysteresis loop are minima of (1). Saturation corresponds to $\lambda = -\infty$. Remanence corresponds to $\lambda = D_x$ where magnetization is aligned with the easy axis. The switching event is the irreversible transition from a to b that occurs when the applied field magnitude $|H|$ reaches the switching field H_s, where the local minimum ceases to exist. The switch from a to b is also illustrated on the plot of $g(\lambda)$ where the point a is located at the coordinates $(\lambda_s, (M/H_s)^2)$, defined by $D_x < \lambda_s < D_y$ and $g'(\lambda_s) = 0$. Numeric determination of λ_s is straightforward because λ_s is known to be in an interval on which g' is strictly increasing. From an initial estimate of $\lambda_1 = (D_x + D_y)/2$, the quadratically convergent iteration derived from Newton's method is

$$\begin{aligned} \lambda_{n+1} &= \lambda_n - g'(\lambda_n)/g''(\lambda_n) \\ &= \lambda_n - \frac{1}{3}\frac{\mathbf{h}^T (D - \lambda_n I)^{-3}\,\mathbf{h}}{\mathbf{h}^T (D - \lambda_n I)^{-4}\,\mathbf{h}}. \end{aligned} \tag{9}$$

Any iteration that produces a λ_{n+1} outside the known interval containing the solution can be immediately replaced by its average with λ_n until λ_{n+1} returns to the valid interval. Once λ_n is sufficiently close to λ_s, one can calculate $g(\lambda_n)$ with (4) to find H_s. The error in H_s is proportional to $(\lambda_n - \lambda_s)^2$. Two iterations of this algorithm are sufficient to determine the switching field in the small particle limit for μMAG Standard Problem No. 2 to 16 digit precision[1].

Once we know λ_s and H_s, we can use a similar iterative technique to find the local minimum for any given applied field H, $|H| < H_s$. In this case we want to find λ satisfying $g(\lambda) - (M/H)^2 = 0$ on the interval $D_x < \lambda < \lambda_s$. The Newton iterate is

$$\lambda_{n+1} = \lambda_n - \left(g(\lambda_n) - (M/H)^2\right)/g'(\lambda_n)$$

$$= \quad \lambda_n - \frac{1}{2}\frac{\mathbf{h}^T\left(D-\lambda_n I\right)^{-2}\mathbf{h}-(M/H)^2}{\mathbf{h}^T\left(D-\lambda_n I\right)^{-3}\mathbf{h}}. \quad (10)$$

Again, the known interval containing the solution can be used to assure proper convergence. Once λ_n has converged, use (3) to determine the corresponding $\mathbf{M}$. This method can also be used to find the global minimum, where the relevant interval is $\lambda < D_x$.

V. Coercive field

Typically during the switching event, the magnetization in the direction of the applied field, $\mathbf{M}\cdot\mathbf{H}$, passes through zero, i.e., the coercive field H_c is equal to the switching field H_s. However, if the field is applied at an angle sufficiently removed from the easy axis, then there can exist a local minimum such that $\mathbf{M}\cdot\mathbf{H}_c = 0$ with $0 < H_c < H_s$. In this circumstance it is possible to solve for H_c in closed form. From (3),

$$\frac{\mathbf{M}\cdot\mathbf{H}_c}{H_c^2} = \frac{h_x^2}{D_x-\lambda} + \frac{h_y^2}{D_y-\lambda} + \frac{h_z^2}{D_z-\lambda} = 0. \quad (11)$$

One can obtain a quadratic equation in λ by multiplying (11) by $\det(D-\lambda I) = (D_x-\lambda)(D_y-\lambda)(D_z-\lambda)$. There are two roots, λ_1, λ_2, satisfying $D_x < \lambda_1 < D_y < \lambda_2 < D_z$. The coercive field corresponds to the smaller root, $\lambda_c = \lambda_1$. Substitution into (4) and (3) yields computed values for H_c and $\mathbf{M}$ respectively.

VI. Precession Frequency

Away from equilibrium, the magnetization evolves under Landau-Lifshitz dynamics,

$$\frac{d\mathbf{M}}{dt} = -|\gamma|\left(\mathbf{M}\times\mathbf{H}_{\text{eff}}\right) - \frac{|\alpha\gamma|}{M}\left(\mathbf{M}\times\left(\mathbf{M}\times\mathbf{H}_{\text{eff}}\right)\right), \quad (12)$$

where γ and α are the Landau-Lifshitz gyromagnetic ratio and damping coefficients, respectively. If $\mathbf{m}$ is a small perturbation from equilibrium position $\mathbf{m}^*$, then the magnetization precesses around and gradually decays towards $\mathbf{m}^*$. The precession frequency f depends upon the curvature of the energy surface in the neighborhood of the equilibrium[11], [2]:

$$f = \frac{\gamma\sqrt{1+\alpha^2}M\sqrt{a_1 a_2}}{2\pi}. \quad (13)$$

The product $a_1 a_2$ is a function (6) of the applied field magnitude $|H|$ and λ^*. If λ^* corresponds to a local minimum, it is natural to ask how f varies as $|H|$ increases towards the switching field, H_s.

Expanding (4) in λ about λ_s, one finds

$$g'(\lambda) = 2M\sqrt{\frac{g''(\lambda_s)(H_s-|H|)}{H_s^3}} + \mathcal{O}(H_s-|H|), \quad (14)$$

which combines with (6) and (13) to yield

$$f(H) = \frac{\gamma}{2\pi}\sqrt{\left(1+\alpha^2\right)M\det\left(D-\lambda_s I\right)}\sqrt[4]{g''(\lambda_s)H_s(H_s-|H|)} + \mathcal{O}\left(\left(H_s-|H|\right)^{3/4}\right) \quad (15)$$

as $|H| \uparrow H_s$. This expression reveals the dependence of the ring-down precession frequency f on the applied field magnitude $|H|$ is $f \propto (H_s-|H|)^{(1/4)}$ immediately preceding the switching event. Similar analysis establishes that the susceptibility $\chi \propto (H_s-|H|)^{-(1/2)}$ just prior to switching as well [2], [12].

VII. Special Cases

In all the preceding analysis it has been assumed that $D_x < D_y < D_z$ and $h_x \neq 0$, $h_y \neq 0$, $h_z \neq 0$. If any of these assumptions fail, then one or more terms of (4) is removed and analysis of a simpler problem is possible. If either $h_y = 0$ or $h_z = 0$, or if any two of D_x, D_y, D_z are equal, then the model defined by (1) simplifies to the Stoner-Wohlfarth model with its known solutions. In terms of the plot in Fig. 1, one of the poles is removed, and the saddle points disappear from the analysis.

In the case that $D_x = D_y = D_z$, the problem further simplifies to be equivalent to magnetization reversal in a sphere, with no anisotropy at all.

VIII. Summary

A three-dimensional generalization of the Stoner-Wohlfarth model has been defined for the purpose of analyzing magnetization reversal of a uniformly magnetized body by coherent rotation. Analysis takes the form of solving a constrained optimization problem by use of Lagrange multiplier techniques. For any body represented by the model and any applied field axis, algorithms for computing the switching and coercive fields have been derived. The stable magnetization direction(s) for any applied field may also be calculated using the techniques presented here. The iterative algorithms are simple and converge quickly and reliably. The behavior of precession frequency and susceptibility as the switching field is approached have also been determined.

References

[1] M. J. Donahue, D. G. Porter, R. D. McMichael, and J. Eicke, *J. Appl. Phys.*, vol. 87, no. 8, pp. 5520–5522, April 2000.

[2] R. D. McMichael, M. J. Donahue, D. G. Porter, and Jason Eicke, *J. Appl. Phys.*, vol. 89, pp. 7603–7605, June 2001.

[3] E. C. Stoner and E. P. Wohlfarth, *Phil. Trans. Royal Soc. London*, vol. A240, pp. 599–642, 1948.

[4] C. E. Johnson, *J. Appl. Phys.*, vol. 33, no. 8, pp. 2515–2517, August 1962.

[5] R. W. Cross, J. O. Oti, S. E. Russek, and T. Silva, *IEEE Trans. Magn.*, vol. 31, no. 6, pp. 3358–3360, November 1995.

[6] A. Thiaville, *Physical Review B*, vol. 61, no. 18, pp. 12221–12232, May 2000.

[7] W. Rave, K. Ramstöck, and A. Hubert, *Journal of Magnetism and Magnetic Materials*, vol. 183, pp. 329–333, 1998.

[8] A. Thiaville, D. Tomáš, and J. Miltat, *Phys. Stat. Sol. (A)*, vol. 170, pp. 125–135, 1998.

[9] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, pp. 101–102, Oxford : Clarendon Press, 1965.

[10] G. H. Golub and C. F. Van Loan, *Matrix Computations*, p. 269, Johns Hopkins University Press, 1983.

[11] J. Smit and H. G. Beljers, Tech. Rep., Philips Research Reports, 1955.

[12] A. Hubert and W. Rave, *Phys. Stat. Sol. (B)*, vol. 211, pp. 815–829, 1999.