Special values for continuous q-Jacobi polynomials and applications

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Asymptotic Analysis, Orthogonal Polynomials, and Special Functions,
Canadian Mathematical Society 2021 Winter Meeting (online)

Saturday, December 4, 2021

Adopted set notations

- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$;
- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the sets of integers, real numbers and complex numbers respectively
- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\mathbb{C}^\dagger := \{z \in \mathbb{C}^* : |z| < 1\}$;
- $\mathbf{a} := \{a_1, \ldots, a_n\}$, $a_k \in \mathbb{C}$, $n, k \in \mathbb{N}_0$, $0 \le k \le n$:

$$x + \mathbf{a} := \{x + a_1, \ldots, x + a_n\}, \quad y^{x+\mathbf{a}} := \{y^{x+a_1}, \ldots, y^{x+a_n}\},$$

$$\pm a := \{a, -a\},$$

$$\mathrm{e}^{\pm i\theta} := \{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\},$$

$$z^{\pm} := \{z, z^{-1}\},$$

$$\left\{\begin{smallmatrix} a \\ b \end{smallmatrix}\right\} := \{a, b\},$$

$$\left\{\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right\} := \{a, b, c\}.$$

Introduction: special functions related to elliptic PDEs

Special functions arise in the theory of linear partial differential equations through:

- Separation of variables for linear homogeneous partial differential equations $\mathcal{L}\Psi = 0$ in special coordinate systems;
- Functional dependence of **fundamental solutions** for the linear partial differential operators $\mathcal{L}\Phi = \delta$;
- Eigenfunction "expansions" for fundamental solutions in separable coordinate systems

$$\Phi(x) = \sum_{n\in A} \phi_n(x),$$

$$\Phi(x) = \int_{k\in B} \phi(x,k)\mathrm{d}k.$$

- Addition theorems arise which connect these expansions. Askey: *(these) are among the most important facts known about these functions.*

Example: Laplace's equation in three-dimensions

A fundamental solution is the reciprocal distance between two points. The study of the three-variable Laplace equation was pioneered by Bôcher in his 1891 dissertation, *Ueber die Reihenentwickelungen der Potentialtheorie*

- elementary functions (exponential, logarithmic, trigonometric, hyperbolic), Jacobi elliptic functions, elliptic integrals;
- hypergeometric functions (associated Legendre functions, Jacobi/Gegenbauer/Legendre polynomials)
- confluent hypergometric functions (Bessel/Hankel functions, parabolic cylinder functions, Whittaker functions, exponential integrals, error functions, Laguerre polynomials, Hermite polynomials)
- Hill's equation (Mathieu functions, Lamé functions, Lamé-Wangerin functions, ...)
- more general functions (solutions to 2nd order ODEs with 5 regular singular points) ... (ongoing work with Hans Volkmer)

- Generating functions arise as special cases of the eigenfunction expansions which arise (e.g., Ismail & Simeonov (2016))

$$\frac{1}{(z-x)^\nu}=\frac{2^{\mu+\frac12}\Gamma(\mu)e^{i\pi(\mu-\nu+\frac12)}}{\sqrt{\pi}\,\Gamma(\nu)(z^2-1)^{\frac{\nu-\mu}{2}-\frac14}}\sum_{n=0}^\infty(n+\mu)Q_{n+\mu-\frac12}^{\nu-\mu-\frac12}(z)C_n^\mu(x)$$

(5.3) : Theorem 2.1 in Cohl (2013) [5]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\frac{(\beta;q)_n}{(\gamma;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}\beta\gamma^{-1},\beta q^n\\ \gamma q^{n+1}\end{matrix};q,\gamma t^2\right)C_n(x;\gamma|q)$$

(5.4) : q-analogue (continuous q-ultraspherical/Rogers polynomials)

$$\frac{1}{z-x}=\frac{2^{\mu+\frac12}\Gamma(\mu)e^{i\pi(\mu-\frac12)}}{\sqrt{\pi}\,(z^2-1)^{-\frac{\mu}{2}+\frac14}}\sum_{n=0}^\infty(n+\mu)Q_{n+\mu-\frac12}^{-\mu+\frac12}(z)C_n^\mu(x)$$

(7.2) in Durand et al. (1976) [8]

$$\frac{(tqe^{i\theta},tqe^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\frac{(q;q)_n}{(\gamma;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}q\gamma^{-1},q^{n+1}\\ \gamma q^{n+1}\end{matrix};\gamma t^2\right)C_n(x;\gamma|q)$$

(5.4) with $\beta=q$: q-analogue (continuous q-ultraspherical/Rogers polynomials)

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\frac{(\beta;q)_n}{(q;q)_n}t^n\,{}_1\phi_1\left(\begin{matrix}\beta q^n\\ 0\end{matrix};\beta t^2\right)H_n(x|q)$$

(5.6) : q-expansion (continuous q-Hermite polynomials)

$$\frac{1}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\frac{t^n}{(q;q)_n}H_n(x|q)$$

(5.7) : generating function for continuous q-Hermite polynomials

$$\frac{1}{(z-x)^\nu}=\frac{(z^2-1)^{\frac{\nu-1}{2}}}{\Gamma(\nu)e^{i\pi(\nu-1)}}\sum_{n=0}^\infty(2n+1)Q_n^{\nu-1}(z)P_n(x)$$

(5.12) : (13) in Cohl (2013) [5]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\frac{(\beta;q)_n}{(q^{\frac12};q)_n}(tq^{-\frac14})^n\,{}_2\phi_1\left(\begin{matrix}\beta q^{-\frac12},\beta q^n\\ q^{n+\frac32}\end{matrix};q^{\frac12}t^2\right)P_n(x|q)$$

(5.11) : q-analogue (continuous q-Legendre polynomials)

$$\frac{1}{(z-x)^\nu}=\sqrt{\frac{2}{\pi}}\frac{(z^2-1)^{-\frac{\nu}{2}+\frac14}}{e^{i\pi(\nu-\frac12)}\Gamma(\nu)}\sum_{n=0}^\infty\epsilon_nQ_{n-\frac12}^{\nu-\frac12}(z)T_n(x)$$

(5.8) : (3.10) in Cohl & Dominici (2011) [6]

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\epsilon_n\frac{(\beta;q)_n}{(q;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}\beta,\beta q^n\\ q^{n+1}\end{matrix};t^2\right)T_n(x)$$

(5.10) : q-analogue (Chebyshev polynomial of the first kind)

$$\frac{1}{\sqrt{z-x}}=\frac{\sqrt2}{\pi}\sum_{n=0}^\infty\epsilon_nQ_{n-\frac12}(z)T_n(x)$$

(5.9) : Heine (1881) [15] reciprocal square root identity (1881)

$$\frac{(tq^{\frac12}e^{i\theta},tq^{\frac12}e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\epsilon_n\frac{(q^{\frac12};q)_n}{(q;q)_n}t^n\,{}_2\phi_1\left(\begin{matrix}q^{\frac12},q^{n+\frac12}\\ q^{n+1}\end{matrix};t^2\right)T_n(x)$$

(5.10) with $\beta=q^{\frac12}$: q-analogue (Chebyshev polynomials of the first kind)

$$\frac{1}{(1+t^2-2tx)^\nu}=\sum_{n=0}^\infty t^nC_n^\nu(x)$$

(5.2) : Gegenbauer (1874) [11] generating function

$$\frac{(t\beta e^{i\theta},t\beta e^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty t^nC_n(x;\beta|q)$$

(3.3) : Rogers (1893) [28] generating function

$$\frac{1}{z-x}=\sum_{n=0}^\infty(2n+1)Q_n(z)P_n(x)$$

(5.13) : Heine (1878) [14] Heine's formula

$$\frac{(tqe^{i\theta},tqe^{-i\theta};q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}=\sum_{n=0}^\infty\frac{(q;q)_n}{(q^{\frac12};q)_n}(tq^{-\frac14})^n\,{}_2\phi_1\left(\begin{matrix}q^{\frac12},q^{n+1}\\ q^{n+\frac32}\end{matrix};q^{\frac12}t^2\right)P_n(x|q)$$

(5.11) with $\beta=q$: q-analogue (continuous q-Legendre polynomial)

Wilson polynomial generating function and q-analogue

$$(1-t)^{1-a-b-c-d} \times {}_4F_3\left(\begin{matrix}\frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix, a-ix \\ a+b, a+c, a+d\end{matrix}; -\frac{4t}{(1-t)^2}\right) = \sum_{n=0}^{\infty} \frac{(a+b+c+d-1)_n}{(a+b)_n(a+c)_n(a+d)_n n!} W_n(x^2;a,b,c,d)t^n. \qquad (9.1.15)$$

- Rahman's (1996) q-analogue

$$\frac{(ta_{1234}(qa_p)^{-1};q)_\infty}{(ta_p^{-1};q)_\infty}\,{}_6\phi_5\left(\begin{matrix}\pm(q^{-1}a_{1234})^{\frac{1}{2}}, \pm(a_{1234})^{\frac{1}{2}}, a_pe^{\pm i\theta} \\ \{a_pa_s\}_{s\neq p}, ta_{1234}(qa_p)^{-1}, qa_pt^{-1}\end{matrix}; q, q\right)$$

$$+\frac{(\{ta_s\}_{s\neq p}, q^{-1}a_{1234}, a_pe^{\pm i\theta};q)_\infty}{(\{a_pa_s\}_{s\neq p}, a_pt^{-1}, te^{\pm i\theta};q)_\infty}\,{}_6\phi_5\left(\begin{matrix}\pm ta_p^{-1}(q^{-1}a_{1234})^{\frac{1}{2}}, \pm ta_p^{-1}(a_{1234})^{\frac{1}{2}}, te^{\pm i\theta} \\ \{ta_s\}_{s\neq p}, t^2a_{1234}(qa_p^2)^{-1}, qta_p^{-1}\end{matrix}; q, q\right)$$

$$= \sum_{n=0}^{\infty} \frac{t^n\,(q^{-1}a_{1234};q)_n\,p_n(x;\mathbf{a}|q)}{(q, \{a_pa_s\}_{s\neq p};q)_n}.$$

Generating function for continuous dual Hahn

- generating function for continuous dual Hahn

$$(1-t)^{-\gamma}\,{}_3F_2\left(\begin{matrix}\gamma, a\pm ix\\ a+b, a+c\end{matrix}; \frac{t}{t-1}\right) = \sum_{n=0}^{\infty} \frac{t^n(\gamma)_n S_n(x^2;a,b,c)}{n!(a+b,a+c)_n}$$

- Cohl & Costas-Santos (2022) q-analogue

Theorem 3.8. *Let $\gamma \in \mathbb{C}$. Then one has the following generating function for continuous dual q-Hahn polynomials*

$$\sum_{n=0}^{\infty} \frac{(\gamma;q)_n p_n(x;\mathfrak{a}|q)}{(q,ab,ac;q)_n} t^n = \frac{(a\mathrm{e}^{\pm i\theta};q)_\infty}{(ab,ac;q)_\infty}$$
$$\times\left(\frac{(ab,ac,\gamma t/a;q)_\infty}{(a\mathrm{e}^{\pm i\theta},t/a;q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\gamma, a\mathrm{e}^{\pm i\theta}, 0\\ ab, ac, qa/t\end{matrix}; q, q\right) + \frac{(tb,tc,\gamma;q)_\infty}{(t\mathrm{e}^{\pm i\theta},a/t;q)_\infty}\,{}_4\phi_3\left(\begin{matrix}\gamma t/a, t\mathrm{e}^{\pm i\theta}, 0\\ tb, tc, qt/a\end{matrix}; q, q\right)\right). \quad (89)$$

Transformations for single nonterminating bhs

$$\frac{(c;q)_\infty}{(a,\frac{c}{b},\frac{abz}{c};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right)$$

$$=\frac{(c;q)_\infty}{(a,\frac{c}{b},\frac{bz}{c};q)_\infty}\,{}_2\phi_2^{-1}\left(\begin{matrix}a,\frac{c}{b}\\c,\frac{qc}{bz}\end{matrix};q,q\right)+\frac{(bz;q)_\infty}{(z,\frac{c}{bz},\frac{abz}{c};q)_\infty}\,{}_2\phi_2^{-1}\left(\begin{matrix}z,\frac{abz}{c}\\bz,\frac{qbz}{c}\end{matrix};q,q\right)$$

$$\frac{(c;q)_\infty}{(a,b,\frac{c}{a},\frac{c}{b},\frac{abz}{c};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right)$$

$$=\frac{1}{(a,\frac{c}{b},\frac{bz}{c};q)_\infty}\,{}_3\phi_1^1\left(\begin{matrix}a,b,\frac{abz}{c}\\\frac{qab}{c}\end{matrix};q,q\right)+\frac{1}{(z,c,\frac{c}{a},\frac{c}{b};q)_\infty}\,{}_3\phi_1^1\left(\begin{matrix}\frac{c}{a},\frac{c}{b},z\\\frac{qc}{ab}\end{matrix};q,q\right)$$

$$\frac{(\pm\sqrt{a},\pm\sqrt{qa},\frac{qa}{bc};q)_\infty}{(\frac{qa}{b},\frac{qa}{c};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,b,c\\\frac{qa}{b},\frac{qa}{c}\end{matrix};q,z\right)$$

$$=\frac{(ax^2,\frac{qax}{b},\frac{qax}{c};q)_\infty}{(1/x,\pm x\sqrt{a},\pm x\sqrt{qa},\frac{qax}{bc};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\pm x\sqrt{a},\pm x\sqrt{qa},\frac{qax}{bc}\\qx,\frac{qax}{b},\frac{qax}{c},ax^{\frac{3}{2}}\end{matrix};q,q\right)$$

$$+\frac{(ax,\frac{qa}{b},\frac{qa}{c};q)_\infty}{(x,\pm\sqrt{a},\pm\sqrt{qa},\frac{qa}{bc};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\pm\sqrt{a},\pm\sqrt{qa},\frac{qa}{bc}\\q/x,\frac{qa}{b},\frac{qa}{c},ax\end{matrix};q,q\right).$$

$${}_5W_4(a;b,c;q,z)$$

$$=\left(1-\frac{b^2c^2z^2}{qa}\right)\frac{(qbcz;q)_\infty}{(\frac{bcz}{aq};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\pm\sqrt{qa},\pm q\sqrt{a},\frac{qa}{bc}\\\frac{qa}{b},\frac{qa}{c},\frac{q^2a}{bcz},qbcz\end{matrix};q,q\right)$$

$$+\frac{(qa,\frac{qa}{bc},bz,cz;q)_\infty}{(\frac{qa}{b},\frac{qa}{c},z,\frac{qa}{bcz};q)_\infty}\,{}_5\phi_4\left(\begin{matrix}\pm\frac{bcz}{\sqrt{qa}},\pm\frac{bcz}{\sqrt{a}},z\\bz,cz,\frac{bcz}{a},\frac{b^2c^2z^2}{a}\end{matrix};q,q\right)$$

Jacobi polynomials: $P_n^{(\alpha,\beta)} : \mathbb{C} \to \mathbb{C}$

- Jacobi polynomials are orthogonal polynomials, orthogonal on the real interval $(-1, 1)$. They can be defined for $n \in \mathbb{N}_0$ as

$$P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}; \frac{1-z}{2}\right).$$

- Orthogonality relation:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)n!}\delta_{m,n}.$$

- They satisfy the parity relation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

Integral representation method for Askey–Wilson polynomials to obtain new generating functions Cohl & Costas-Santos (2022)

Theorem 3.1. *Let* $a, b, c, d, f, \sigma \in \mathbb{C}^*$, $\max(|a|, |b|, |c|, |d|, \sigma) < 1$, $q \in \mathbb{C}^\dagger$, $x = \cos\theta \in [-1, 1]$, $z = \mathrm{e}^{i\psi}$. *Then*

$$p_n(x; \mathfrak{a}|q) = \frac{(q, a\mathrm{e}^{\pm i\theta}, b\mathrm{e}^{\pm i\theta}, c\mathrm{e}^{\pm i\theta}; q)_\infty (ab, ac, bc; q)_n}{2\pi(f, \frac{q}{f}, f\mathrm{e}^{2i\theta}, \frac{q}{f}\mathrm{e}^{-2i\theta}, ab, ac, bc; q)_\infty} D_n(x; \mathfrak{a}, f, \sigma|q), \tag{68}$$

where

$$D_n(x; \mathfrak{a}, f, \sigma|q) = \int_{-\pi}^{\pi} \frac{((f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\mathrm{e}^{i\theta}, \frac{q}{f}\mathrm{e}^{-i\theta}, abc)\frac{z}{\sigma}; q)_\infty}{(\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (a, b, c)\frac{z}{\sigma}; q)_\infty} \frac{(d\frac{\sigma}{z}; q)_n}{(abc\frac{z}{\sigma}; q)_n} \left(\frac{z}{\sigma}\right)^n \mathrm{d}\psi \tag{69}$$

$$= \int_{-\pi}^{\pi} \frac{((fabc\,\mathrm{e}^{i\theta}, \frac{q}{f}abc\,\mathrm{e}^{-i\theta})\frac{\sigma}{z}, (f\frac{1}{abc}\mathrm{e}^{i\theta}, \frac{q}{f}\frac{1}{abc}\mathrm{e}^{-i\theta}, 1)\frac{z}{\sigma}; q)_\infty}{(abc\,\mathrm{e}^{\pm i\theta}\frac{\sigma}{z}, (\frac{1}{ab}, \frac{1}{ac}, \frac{1}{bc})\frac{z}{\sigma}; q)_\infty}$$

$$\times \frac{(abcd\frac{\sigma}{z}; q)_n}{(\frac{z}{\sigma}; q)_n} \left(\frac{1}{abc}\frac{z}{\sigma}\right)^n \mathrm{d}\psi. \tag{70}$$

Jacobi polynomial special values

- The Jacobi polynomials have the following **two** special values

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!},$$

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta+1)_n}{n!}.$$

Jacobi polynomial Poisson kernel

- Poisson kernel for Jacobi polynomials: Bailey (1964), p. 102,

$$\sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1, \frac{\alpha+\beta+3}{2})_n}{(\alpha+1, \beta+1, \frac{\alpha+\beta+1}{2})_n} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) t^{2n} = \frac{(1-t^2)}{(1+t^2)^{\alpha+\beta+2}}$$

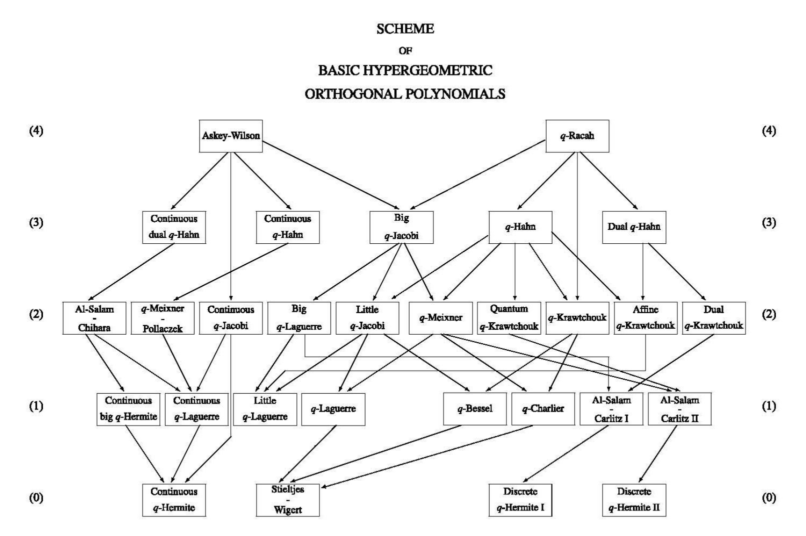
$$\times F_4\left(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}, \alpha+1, \beta+1, \frac{\sin^2\theta\sin^2\phi}{\frac{1}{2}(t+t^{-1})}, \frac{\cos^2\theta\cos^2\phi}{\frac{1}{2}(t+t^{-1})}\right),$$

where

$$F_4(\alpha,\beta;\gamma,\gamma';x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha,\beta)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

and $\sqrt{|x|}+\sqrt{|y|}<1$.

The q-Askey scheme of basic hypergeometric OPs



Continuous q-Jacobi polynomials: $P_n^{(\alpha,\beta)}(x|q)$

- Continuous q-Jacobi polynomials are orthogonal polynomials, orthogonal on the real interval $(-1,1)$. They can be defined for $n \in \mathbb{N}_0$ as

$$P_n^{(\alpha,\beta)}\left(\tfrac{1}{2}(z+z^{-1})|q\right) := \frac{(q^{\alpha+1};q)_n}{(q;q)_n}$$

$$\times\, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+1+n}, q^{\frac{\alpha}{2}+\frac{1}{4}}z^{\pm} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)} \end{matrix}; q, q\right).$$

- Orthogonality relation:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x|q) P_n^{(\alpha,\beta)}(x|q) w_q^{(\alpha,\beta)}(x|q) \mathrm{d}x = h_{n,m}^{(\alpha,\beta)}(q)\delta_{m,n}.$$

- They satisfy the parity relation

$$P_n^{(\alpha,\beta)}(-x|q) = (-q^{\frac{1}{2}(\alpha-\beta)})^n P_n^{(\beta,\alpha)}(x|q).$$

Continuous q-Jacobi polynomial special values

- The continuous q-Jacobi polynomials have the following **four** special values

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{1}{2}\alpha+\frac{1}{4}}+q^{-\frac{1}{2}\alpha-\frac{1}{4}})|q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n},$$

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(q^{\frac{1}{2}\alpha+\frac{3}{4}}+q^{-\frac{1}{2}\alpha-\frac{3}{4}})|q)=q^{-\frac{1}{2}n}\frac{(q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+3)};q)_n}{(q,-q^{\frac{1}{2}(\alpha+\beta+1)};q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{1}{2}\beta+\frac{1}{4}}+q^{-\frac{1}{2}\beta-\frac{1}{4}})|q)=\left(-q^{\frac{\alpha-\beta}{2}}\right)^n\frac{(q^{\beta+1};q)_n}{(q;q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac{1}{2}(q^{\frac{1}{2}\beta+\frac{3}{4}}+q^{-\frac{1}{2}\beta-\frac{3}{4}})|q)=\left(-q^{\frac{\alpha-\beta-1}{2}}\right)^n\frac{(q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+3)};q)_n}{(q,-q^{\frac{1}{2}(\alpha+\beta+1)};q)_n}.$$

Poisson kernel for orthogonal polynomials

- Given an orthogonal polynomial $p_n(x;\mathbf{a})$ over set of parameters $\mathbf{a}$,

$$\int_a^b p_n(x;\mathbf{a})p_m(x;\mathbf{a})w(x;\mathbf{a})\mathrm{d}x = h_n(\mathbf{a})\delta_{m,n}$$

the Poisson kernel with $x, y \in (a,b)$ is defined as

$$\sum_{n=0}^{\infty} h_n^{-1}(\mathbf{a})p_n(x;\mathbf{a})p_n(y;\mathbf{a})t^n.$$

- Note that as $t \to 1$ the Poisson kernel approaches the closure relation

$$\sum_{n=0}^{\infty} h_n^{-1}(\mathbf{a})p_n(x;\mathbf{a})p_n(y;\mathbf{a}) = \frac{\delta(x-y)}{w(x;\mathbf{a})}$$

- which is also the finite value for the Christoffel-Darboux formula

$$\sum_{n=0}^{k} h_n^{-1}(\mathbf{a})p_n(x;\mathbf{a})p_n(y;\mathbf{a})$$

Poisson kernel transformations from z, w special values

- Consider $x = \frac{1}{2}(z + z^{-1})$, $y = \frac{1}{2}(w + w^{-1})$ in the Poisson kernel.
- Then choose $z \in \{a, b, c, d\}$ and $w \in \{a, b, c, d\}$.
- By inserting special values of continuous q-Jacobi polynomials into its Poisson kernel, transformations for single basic nonterminating hypergeometric transformations with arbitrary argument are produced.
- Diagonal transformations produce unique transformations for a single basic nonterminating hypergeometric function.
- Transformation formulae for a single basic nonterminating hypergeometric function are produced in pairs corresponding to the off-diagonal elements.

The Askey–Wilson Poisson kernel

The most general symmetric Poisson kernel for Askey–Wilson polynomials $\mathsf{K}_t(x,y) := \mathsf{K}_t(x,y;\mathbf{a}|q)$, which is given by

$$\mathsf{K}_t(x,y) = \sum_{n=0}^{\infty} \frac{(\frac{abcd}{q}, \pm\sqrt{qabcd}; q)_n \, p_n(x;\mathbf{a}|q) p_n(y;\mathbf{a}|q)\, t^n}{(q, \pm\sqrt{\frac{abcd}{q}}, ab, ac, ad, bc, bd, cd; q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{abcd}{q}, \pm\sqrt{qabcd}, ab, ac, ad; q)_n t^n}{(q, \pm\sqrt{\frac{abcd}{q}}, bc, bd, cd; q)_n a^{2n}} r_n(x;\mathbf{a}|q) r_n(y;\mathbf{a}|q).$$

For $ad = bc$, then the Poisson kernel takes a simplified form

$$K_t(x,y) := \mathsf{K}_t(x,y;a,b,c,\frac{bc}{a}|q)$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{b^2c^2}{q}, \pm q^{\frac{1}{2}} bc, ab, ac; q)_n t^n}{(q, \pm q^{-\frac{1}{2}} bc, \frac{b^2c}{a}, \frac{bc^2}{a}; q)_n a^{2n}} r_n(x;\mathbf{a}|q) r_n(y;\mathbf{a}|q).$$

Askey–Wilson Poisson kernel with $ad = bc$

Gasper & Rahman (1986). Let $q \in \mathbb{C}^\dagger$, $t, \alpha, \beta \in \mathbb{C}$ such that $|t| < 1$. Then the Poisson kernel for Askey–Wilson polynomials with $ad = bc$ is given by

$$K_t = \frac{(t^2, qbct; q)_\infty}{(qt^2, \frac{t}{bc}; q)_\infty} \sum_{n=0}^{\infty} \frac{(-bc, \pm\sqrt{q}bc, \frac{bc}{a}z^\pm, \frac{bc}{a}w^\pm; q)_n q^n}{(q, \frac{bc^2}{a}, \frac{b^2c}{a}, bc, \frac{bc}{a^2}, qbct^\pm; q)_n}$$

$$\times {}_{10}W_9\left(\frac{q^{-n}a^2}{bc}; \frac{q^{1-n}a}{b^2c}, \frac{q^{1-n}a}{bc^2}, q^{-n}, az^\pm, aw^\pm; q, q\right)$$

$$+ \frac{(t, \frac{at}{c}, b^2c^2, aw^\pm, bz^\pm, ctz^\pm, \frac{bct}{a}w^\pm; q)_\infty}{(ab, ac, bc, \frac{a}{c}, \frac{b^2c}{a}, \frac{bc^2t}{a}, \frac{bc}{t}, tz^\pm w^\pm; q)_\infty} \sum_{n=0}^{\infty} \frac{(-t, \pm\sqrt{q}t, \frac{bc^2t}{a}, tz^\pm w^\pm; q)_n q^n}{(q, qt^2, \frac{qt}{bc}, \frac{at}{c}, ctz^\pm, \frac{bct}{a}w^\pm; q)_n}$$

$$\times {}_{10}W_9\left(\frac{q^{n-1}bc^2t}{a}; q^nt, q^{n-1}bct, \frac{q^ntc}{a}, \frac{bc}{a}z^\pm, cw^\pm; q, q\right)$$

$$+ \frac{(t, \frac{ct}{a}, b^2c^2, cw^\pm, \frac{bc}{a}z^\pm, atz^\pm, btw^\pm; q)_\infty}{(ac, bc, \frac{bc^2}{a}, \frac{b^2c}{a}, \frac{c}{a}, abt, \frac{bc}{t}, tz^\pm w^\pm; q)_\infty} \sum_{n=0}^{\infty} \frac{(-t, \pm\sqrt{q}t, abt, tz^\pm w^\pm; q)_n q^n}{(q, qt^2, \frac{qt}{bc}, \frac{ct}{a}, atz^\pm, btw^\pm; q)_n}$$

$$\times {}_{10}W_9\left(q^{n-1}abt; q^nt, q^{n-1}bct, \frac{q^nat}{c}, bz^\pm, aw^\pm; q, q\right).$$

Poisson kernel for continuous q-Jacobi polynomials

- For the continuous q-Jacobi polynomials, $ad = bc$.
- Choosing the values for (a, b, c, d) for continuous q-Jacobi polynomials, the Poisson kernel is given by

$$K_t(x,y) = \sum_{n=0}^{\infty} \frac{(q, q^{\alpha+\beta+1}, q^{\frac{\alpha+\beta+3}{2}}; q)_n t^n}{(q^{\alpha+1}, q^{\beta+1}, q^{\frac{\alpha+\beta+1}{2}}; q)_n q^{(\alpha+\frac{1}{2})n}} P_n^{(\alpha,\beta)}(x|q) P_n^{(\alpha,\beta)}(y|q) t^n.$$

- Gasper & Rahman (1986) derived the following expression for the continuous q-Jacobi polynomials.

Poisson kernel for continuous q-Jacobi polynomials

$$K_t(x,y)=\frac{(q^{2\alpha+1}t^2,-q^{\frac{3\alpha+\beta+5}{2}}t;q)_\infty}{(q^{2\alpha+2}t^2,-q^{\frac{\alpha-\beta-1}{2}}t;q)_\infty}\sum_{n=0}^\infty\frac{(q^{\frac{\alpha+\beta}{2}+\left\{\substack{1\\3/2}\right\}},-q^{\frac{\beta}{2}+\frac34}z^\pm,-q^{\frac{\beta}{2}+\frac34}w^\pm;q)_nq^n}{(q,q^{\beta+1},-q^{\frac{\alpha+\beta+2}{2}},-q^{\frac{\beta-\alpha+1}{2}},-q^{\frac{3\alpha+\beta+5}{2}}t,-q^{\frac{\beta-\alpha+3}{2}}t^{-1};q)_n}$$
$$\times{}_{10}W_9\left(-q^{\frac{\alpha-\beta-2n-1}{2}};q^{-n},-q^{\frac{-\alpha-\beta-2n-1}{2}},q^{-\beta-n},q^{\frac{\alpha}{2}+\frac14}z^\pm,q^{\frac{\alpha}{2}+\frac14}w^\pm;q,q\right)$$
$$+\frac{(q^{\alpha+\beta+2},q^{\alpha+\frac12}t,-q^{\frac{3\alpha-\beta+1}{2}}t,q^{\frac{\alpha}{2}+\frac34}z^\pm,q^{\frac{\alpha}{2}+\frac14}w^\pm,-q^{\alpha+\frac{\beta}{2}+\frac34}tz^\pm,-q^{\alpha+\frac{\beta}{2}+\frac54}tw^\pm;q)_\infty}{(q^{\alpha+1},-q^{\frac{\alpha+\beta}{2}+\left\{\substack{1/2\\1\\3/2}\right\}},-q^{\frac{\alpha-\beta}{2}},q^{\alpha+\beta+\frac32}t,-q^{\frac{\beta-\alpha+1}{2}}t^{-1},q^{\alpha+\frac12}tz^\pm w^\pm;q)_\infty}$$
$$\times\sum_{n=0}^\infty\frac{(\pm q^{\alpha+1}t,q^{\alpha+\beta+\frac32}t,-q^{\alpha+\frac12}t,q^{\alpha+\frac12}tz^\pm w^\pm;q)_nq^n}{(q,q^{2\alpha+2}t^2,-q^{\frac{\alpha-\beta+1}{2}},-q^{\frac{3\alpha-\beta+1}{2}},-q^{\alpha+\frac{\beta}{2}+\frac34}tz^\pm,-q^{\alpha+\frac{\beta}{2}+\frac54}tw^\pm;q)_n}$$
$$\times{}_{10}W_9\left(q^{\alpha+\beta+\frac12+n}t;q^{\alpha+\frac12+n}t,-q^{\frac{\alpha+\beta+2n+1}{2}}t,-q^{\frac{3\alpha+\beta+2n+1}{2}}t,-q^{\frac{\beta}{2}+\frac34}z^\pm,-q^{\frac{\beta}{2}+\frac14}w^\pm;q,q\right)$$
$$+\frac{(q^{\alpha+\beta+2},q^{\alpha+\frac12}t,-q^{\frac{\alpha+\beta+1}{2}}t,-q^{\frac{\beta}{2}+\frac34}z^\pm,-q^{\frac{\beta}{2}+\frac14}w^\pm,-q^{\frac{3\alpha}{2}+\frac34}tz^\pm,-q^{\frac{3\alpha}{2}+\frac54}tw^\pm;q)_\infty}{(q^{\beta+1},-q^{\frac{\alpha+\beta}{2}+\left\{\substack{1/2\\1\\3/2}\right\}},-q^{\frac{\beta-\alpha}{2}},q^{2\alpha+\frac32}t,-q^{\frac{\beta-\alpha+1}{2}}t^{-1},q^{\alpha+\frac12}tz^\pm w^\pm;q)_\infty}$$
$$\times\sum_{n=0}^\infty\frac{(\pm q^{\alpha+1}t,-q^{\alpha+\frac12}t,q^{2\alpha+\frac32}t,q^{\alpha+\frac12}tz^\pm w^\pm;q)_nq^n}{(q,q^{2\alpha+2}t^2,-q^{\frac{\alpha+\beta+1}{2}}t,-q^{\frac{\alpha-\beta+1}{2}}t,q^{\frac{3\alpha}{2}+\frac34}tz^\pm,q^{\frac{3\alpha}{2}+\frac54}tw^\pm;q)_n}$$
$$\times{}_{10}W_9\left(q^{2\alpha+\frac12+n}t;q^{\alpha+\frac12+n}t,-q^{\frac{3\alpha+\beta+2n+1}{2}}t,-q^{\frac{3\alpha-\beta+2n+1}{2}}t,q^{\frac{\alpha}{2}+\frac34}z^\pm,q^{\frac{\alpha}{2}+\frac14}w^\pm;q,q\right).$$

Generating functions from special values

- Generating functions are produced by replacing $z \in \{a, b, c, d\}$ or $w \in \{a, b, c, d\}$.

Generating functions

- $w=a$ generating function

$$\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta+1},q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q^{\beta+1},q^{\frac{\alpha+\beta+1}{2}};q)_n}P_n^{(\alpha,\beta)}(x|q)t^n$$

$$=\frac{(q^{2\alpha+1}t^2,-q^{\frac{3\alpha+\beta+5}{2}}t,-q^{\frac{\alpha+\beta+2}{2}}t,q^{\alpha+\frac{\beta}{2}+\frac{7}{4}}tz^{\pm};q)_\infty}{(q^{2\alpha+2}t^2,-q^{\alpha+1}t,-q^{\alpha+\beta+\frac{5}{2}}t,q^{\frac{\alpha}{2}+\frac{1}{4}}tz^{\pm};q)_\infty}$$

$$\times\,{}_8W_7\left(-q^{\alpha+\beta+\frac{3}{2}}t;q^{\frac{\beta-\alpha}{2}},q^{\frac{\alpha+\beta+3}{2}},-q^{\alpha+\frac{3}{2}}t,-q^{\frac{\beta}{2}+\frac{3}{4}}z^{\pm};q,-q^{\alpha+\frac{1}{2}}t\right)$$

- q to 1 limit

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1,\frac{\alpha+\beta+3}{2})_n}{(\beta+1,\frac{\alpha+\beta+1}{2})_n}P_n^{(\alpha,\beta)}(x)t^n=\frac{1-t^2}{(1+t^2-2tx)^{\frac{\alpha+\beta+3}{2}}}$$

$$\times\,{}_2F_1\left(\begin{matrix}\frac{\beta-\alpha}{2},\frac{\alpha+\beta+3}{2}\\ \beta+1\end{matrix};\frac{-2t(1+x)}{1+t^2-2tx}\right)$$

Generating functions

- $z=d$ generating function

$$\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta+1},\pm q^{\frac{\alpha+\beta+3}{2}};q)_n}{(q^{\beta+1},\pm q^{\frac{\alpha+\beta+1}{2}};q)_n}P_n^{(\alpha,\beta)}(x|q)t^n$$

$$=\frac{(q^{\alpha+\beta+2}t^2,q^{\alpha+\beta+3}t;q)_\infty}{(q^{\alpha+\beta+3}t^2,t;q)_\infty}\,{}_5\phi_4\left(\begin{matrix}q^{\frac{\alpha+\beta+2}{2}},\pm q^{\frac{\alpha+\beta+3}{2}},q^{\frac{\alpha}{2}+\frac14}w^{\pm}\\ q^{\alpha+1},-q^{\frac{\alpha+\beta+1}{2}},q^{\alpha+\beta+3}t,qt^{-1}\end{matrix};q,q\right)$$

$$+\frac{(q^{\alpha+\beta+2},q^{\alpha+1}t,-q^{\frac{\alpha+\beta+1}{2}}t,-q^{\frac{\alpha+\beta+2}{2}}t,q^{\frac{\alpha}{2}+\frac14}w^{\pm};q)_\infty}{(q^{\alpha+1},-q^{\frac{\alpha+\beta+1}{2}},-q^{\frac{\alpha+\beta+2}{2}},t^{-1},q^{\frac{\alpha}{2}+\frac14}tw^{\pm};q)_\infty}$$

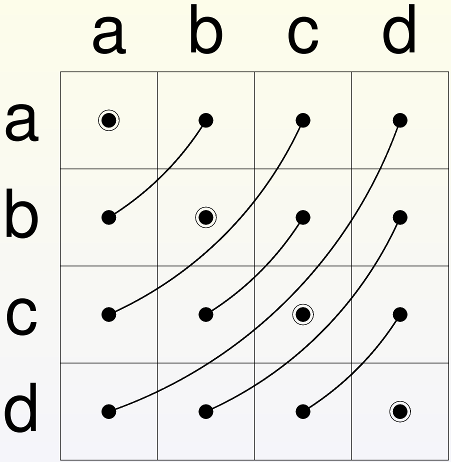
$$\times\,{}_5\phi_4\left(\begin{matrix}q^{\frac{\alpha+\beta+2}{2}}t,\pm q^{\frac{\alpha+\beta+3}{2}}t,q^{\frac{\alpha}{2}+\frac14}tw^{\pm}\\ q^{\alpha+1}t,q^{\alpha+\beta+3}t^2,-q^{\frac{\alpha+\beta+1}{2}}t,qt\end{matrix};q,q\right)$$

Generating functions

- q to 1 limit

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1, \frac{\alpha+\beta+3}{2})_n}{(\alpha+1, \frac{\alpha+\beta+1}{2})_n} P_n^{(\alpha,\beta)}(y) t^n = \frac{1-t^2}{(1-t)^{\alpha+\beta+3}} \times {}_2F_1\left(\begin{matrix} \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2} \\ \alpha+1 \end{matrix}; \frac{2t(y-1)}{(1-t)^2}\right)$$

Single nonterminating transformations from special values



$$a = q^{\frac{\alpha}{2}+\frac{1}{4}},\ b = q^{\frac{\alpha}{2}+\frac{3}{4}},\ c = -q^{\frac{\beta}{2}+\frac{1}{4}},\ d = -q^{\frac{\beta}{2}+\frac{3}{4}}.$$

Alternate continuous q-Jacobi polynomials

- Another q-analogue of the Jacobi polynomials is given by

$$P_n^{(\alpha,\beta)}(x;q) = q^{-n\alpha}\frac{(-q^{\alpha+\beta+1};q)_n}{(-q;q)_n}P_n^{(\alpha,\beta)}(x|q^2).$$

This follows from the quadratic transformation

$$\begin{aligned}{}_4\phi_3&\left(\begin{matrix}q^{-2n},q^{2n}a^2,qb^2,c^2\\ -a,-qa,q^2b^2c^2\end{matrix};q^2,q^2\right)\\ &\qquad= (bc)^n\frac{(-q,-\frac{a}{bc};q)_n}{(-a,-qbc;q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^na,\frac{qb}{c},\frac{c}{b}\\ -q,-\frac{a}{bc},qbc\end{matrix};q,q\right).\end{aligned}$$

and therefore has the following representation in terms of Askey–Wilson polynomials

$$P_n^{(\alpha,\beta)}(x;q) = \frac{q^{n/2}}{(q,-q,-q;q)_n}p_n(x;q^{\frac12},-q^{\frac12},q^{\alpha+1},-q^{\beta+1}|q).$$

- Poisson kernel for quadratic q-Jacobi unavailable as of yet.

Alternate continuous q-Jacobi polynomial special values

- The continuous q-Jacobi polynomials have the following **four** alternate special values

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\frac12}+q^{-\frac12})|q^2)=(q^\alpha)^n\,\frac{(q^{\alpha+1},-q^{\beta+1};q)_n}{(q,-q^{\alpha+\beta+1};q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\frac12}+q^{-\frac12})|q^2)=(-q^\alpha)^n\,\frac{(-q^{\alpha+1},q^{\beta+1};q)_n}{(q,-q^{\alpha+\beta+1};q)_n},$$

$$P_n^{(\alpha,\beta)}(\tfrac12(q^{\alpha+\frac12}+q^{-\alpha-\frac12}|q^2)=\frac{(q^{\alpha+1},-q^{\alpha+1};q)_n}{(q,-q;q)_n},$$

$$P_n^{(\alpha,\beta)}(-\tfrac12(q^{\beta+\frac12}+q^{-\beta-\frac12})|q^2)=\left(-q^{\alpha-\beta}\right)^n\frac{(q^{\beta+1},-q^{\beta+1};q)_n}{(q,-q;q)_n}.$$

Quadratic continuous q-ultraspherical polynomials

$$C_n(x;q^{2\alpha+1}|q^2)=\left(q^{-\alpha-\frac12}\right)^n\frac{(\pm q^{2\alpha+1};q)_n}{(\pm q^{\alpha+1};q)_n}P_n^{(\alpha,\alpha)}(x|q^2).$$

$$C_n(-x;\beta|q)=(-1)^nC_n(x;\beta|q).$$

$$C_n(\tfrac12(q^{\frac12}+q^{-\frac12});q^{2\alpha+1}|q^2)=q^{-\frac{n}{2}}\frac{(q^{2\alpha+1};q)_n}{(q;q)_n},$$

$$C_n(-\tfrac12(q^{\frac12}+q^{-\frac12});q^{2\alpha+1}|q^2)=\left(-q^{-\frac12}\right)^n\frac{(q^{2\alpha+1};q)_n}{(q;q)_n},$$

$$C_n(\tfrac12(q^{\alpha+\frac12}+q^{-\alpha-\frac12});q^{2\alpha+1}|q^2)=q^{-(\alpha+\frac12)n}\frac{(q^{2\alpha+1},-q^{2\alpha+1};q)_n}{(q,-q;q)_n},$$

$$C_n(-\tfrac12(q^{\alpha+\frac12}+q^{-\alpha-\frac12});q^{2\alpha+1}|q^2)=\left(-q^{-(\alpha+\frac12)}\right)^n\frac{(q^{2\alpha+1},-q^{2\alpha+1};q)_n}{(q,-q;q)_n}.$$

Poisson kernel for quadratic q-ultraspherical polynomials

$$\sum_{n=0}^{\infty}\frac{(\pm q,\pm q^{\alpha+\frac32};q)_n}{(\pm q^{2\alpha+1},\pm q^{\alpha+\frac12};q)_n}C_n(x;q^{2\alpha+1}|q^2)t^n=\frac{(t^2,-q^{\alpha+2}t;q)_\infty}{(qt^2,-q^{-\alpha-1}t;q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(q^{\alpha+1},\pm q^{\alpha+\frac32},-q^{\alpha+\frac12}z^{\pm},-q^{\alpha+\frac12}w^{\pm};q)_nq^n}{(q,-q^{2\alpha+1},\pm q^{\alpha+1},-q^{\alpha},-q^{\alpha+2}t^{\pm};q)_n}\,{}_{10}W_9\Big(-q^{-\alpha-n};q^{-\alpha-n},-q^{-2\alpha-n},q^{-n},q^{\frac12}z^{\pm},q^{\frac12}w^{\pm};q,q\Big)$$

$$+\frac{(t,q^{2\alpha+2},q^{-\alpha}t,-q^{\frac12}z^{\pm},q^{\frac12}w^{\pm},q^{\alpha+\frac12}tz^{\pm},-q^{\alpha+\frac12}tw^{\pm};q)_\infty}{(-q,\pm q^{\alpha+1},q^{\alpha+1},q^{-\alpha},-q^{2\alpha+1}t,-q^{\alpha+1}t^{-1},tz^{\pm}w^{\pm};q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(-t,\pm q^{\frac12}t,-q^{2\alpha+1}t,tz^{\pm}w^{\pm};q)_nq^n}{(q,qt^2,\pm q^{-\alpha}t,q^{\alpha+\frac12}tz^{\pm},-q^{\alpha+\frac12}tw^{\pm};q)_n}\,{}_{10}W_9\Big(-q^{2\alpha+n}t;q^nt,\pm q^{\alpha+n}t,-q^{\alpha+\frac12}z^{\pm},q^{\alpha+\frac12}w^{\pm};q,q\Big)$$

$$+\frac{(q^{2\alpha+2},t,q^{\alpha}t,-q^{\alpha+\frac12}z^{\pm},q^{\alpha+\frac12}w^{\pm},q^{\frac12}tz^{\pm},-q^{\frac12}tw^{\pm};q)_\infty}{(-q^{2\alpha+1},\pm q^{\alpha+1},q^{\alpha+1},q^{\alpha},-qt,-q^{\alpha+1}t^{-1},tz^{\pm}w^{\pm};q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(-t,-qt,\pm q^{\frac12}t,tz^{\pm}w^{\pm};q)_nq^n}{(q,qt^2,q^{\alpha}t,-q^{-\alpha}t,q^{\frac12}tz^{\pm},-q^{\frac12}tw^{\pm};q)_n}\,{}_{10}W_9\Big(-q^nt;q^nt,-q^{\alpha+n}t,q^{n-\alpha}t,-q^{\frac12}z^{\pm},q^{\frac12}w^{\pm};q,q\Big).$$

Quadratic continuous q-ultraspherical polynomials from its Poisson kernel

- generating function $w = q^{\frac{1}{2}}$

$$\sum_{n=0}^{\infty} \frac{(-q, \pm q^{\alpha+\frac{3}{2}}; q)_n}{(-q^{2\alpha+1}, \pm q^{\alpha+\frac{1}{2}}; q)_n} C_n(x; q^{2\alpha+1}|q^2) t^n = \frac{(q^{2\alpha+2}, qt, \pm\sqrt{q}t, -q^{\alpha+\frac{1}{2}}t, -q^{\alpha+\frac{5}{2}}t, q^{\alpha+\frac{3}{2}}tz^{\pm}; q)_\infty}{(\pm q^{\alpha+1}, \pm q^{\alpha+\frac{3}{2}}, -q^{2\alpha+2}t, q^2t^2, tz^{\pm}; q)_\infty}$$
$$\times\, {}_8W_7\left(-q^{2\alpha+1}t; qt, q^{\alpha-\frac{1}{2}}, q^{\alpha+\frac{3}{2}}, -q^{\alpha+\frac{1}{2}}z^{\pm}; q, qt\right). \quad (92)$$

- transformation $(z = w = q^{\frac{1}{2}})$

$${}_4\phi_3\left(\begin{matrix} -q, \pm qa, a^2 \\ \pm a, -a^2 \end{matrix}; q, z\right)$$
$$= \frac{(q^2z^2, -q^{\frac{5}{2}}az, -q^{\frac{1}{2}}az, q^2az, qaz; q)_\infty}{(q^3z^2, -q^{\frac{3}{2}}a^2z, -q^{\frac{3}{2}}z, qz, z; q)_\infty}\, {}_8W_7\left(-q^{\frac{1}{2}}a^2z; q^{\frac{3}{2}}z, -aq^{\pm\frac{1}{2}}, aq^{\pm}; q, q^{\frac{3}{2}}z\right).$$

Future directions of work for this powerful **machine** for constructing standard generating functions. And transformations for single nonterminating basic hypergeometric functions with arbitrary argument

- Write down and carry through analysis for the 8 generating functions and 16 transformations and q to 1 limits for continuous q-Jacobi polynomial.
- Askey–Wilson polynomial Poisson kernel (these exist in the literature, but there are some typographical errors which need to be fixed).
- Quadratic continuous q-Jacobi polynomial Poisson kernel (2 free parameters).
- Higher multi-linear generating functions for Askey–Wilson polynomials (4 free parameters) and other polynomials.

Acknowledgements

- Tom Koornwinder, Korteweg-de Vries Institute for Mathematics, The Netherlands
- George Gasper, USA
- Roberto S. Costas-Santos, Spain