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Daniel W. Lozier

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Center for Applied Mathematics
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Washington, D.C. 20234

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Final Report



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U.S. DEPARTMENT OF COMMERCE, Philip M. Klutznick, *Secretary*

Luther H. Hodges, Jr., *Deputy Secretary*

Jordan J. Baruch, *Assistant Secretary for Science and Technology*

NATIONAL BUREAU OF STANDARDS, Ernest Ambler, *Director*

ABSTRACT

Consider a given homogeneous or inhomogeneous linear difference equation $\sum_{s=0}^{\ell} d_s(r)y(r+s) = g(r)$ where $\ell \geq 2$ and $r = 0, 1, 2, \dots$.

Suppose y is a solution of this equation and u, v are solutions of the homogeneous form of this equation such that $u(r)/v(r) \rightarrow 0$, $y(r)/v(r) \rightarrow 0$, $u(r)/y(r) \rightarrow 0$. It is known that under these circumstances algorithms for the computation of y based on forward recurrence or backward recurrence, such as the Miller algorithm, are numerically unstable.

Stable algorithms, such as the method of Olver in the case $\ell = 2$, have been based on approximating $y(r)$ by the solutions $y_n(r)$ of a certain sequence of boundary value problems. More specifically, $y_n(r)$ is a solution of the difference equation that coincides with $y(r)$, over some fixed initial range of r , say $r = i, i+1, \dots, i+j-1$, and satisfies $y_n(r) = 0$ for $r = n, n+1, \dots, n+\ell-j-1$. Here j is an integer whose value depends on the asymptotic behavior of the chosen solution $y(r)$ and n is an arbitrary large integer. Boundary value problems of this type are shown to be equivalent to two initial value problems of order j and $\ell-j$ by factorization of the linear difference operator. The solution of the problem of order j is obtained by forward recurrence; the solution of the other problem is obtained by backward recurrence.

The algorithm is specified completely for a broad class of linear difference operators. This class includes, for example, every constant-coefficient operator. Convergence of $y_n(r)$ to $y(r)$ as $n \rightarrow \infty$ for fixed r is proved and an expansion of the truncation error is derived. Numerical stability is demonstrated under appropriate conditions. The method is tested by numerical examples involving fourth-order equations with variable coefficients.

PREFACE

This paper was submitted as a Ph.D. thesis in the Interdisciplinary Applied Mathematics Program of the University of Maryland at College Park. The Ph.D. degree was awarded on December 18, 1979.

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CHAPTER 1. INTRODUCTION

1.1 Miller's and Olver's Algorithms

Linear recurrence relations (linear difference equations) are satisfied by many of the higher transcendental functions of mathematics, and they arise quite naturally in applications of mathematics. Discretization of ordinary differential equations in numerical analysis is one important area in which they occur. Because of their special form, linear recurrence relations are especially well suited to the construction of computing algorithms. Stability requirements in the case of second-order homogeneous relations led to the well-known Miller backward recurrence algorithm [5], which was developed for the computation of modified Bessel functions. This reference was published in 1952. The review paper of Gautschi [9] provides a survey of algorithms and applications in the case of second-order homogeneous equations. Nowadays, the name of Miller often attends when a minimal solution (defined below) of a homogeneous equation of arbitrary order is computed using backward recurrence [28]. These algorithms always involve backward recurrence from assumed starting values (since these are not known in general) followed by a normalization procedure, such as matching one value of the computed solution with the desired solution in order to arrive at the proper scale factor which relates these two solutions.

A different class of algorithms becomes necessary when "intermediate" solutions of a linear recurrence relation are to be computed. Intuitively speaking, an intermediate solution is one for which two other solutions can be found such that one dominates, and the other is dominated by, the intermediate solution. We are speaking here of dominance in the following sense:

if $a = (a(0), a(1), \dots)$ and $b = (b(0), b(1), \dots)$ are two infinite sequences[†] then b dominates a whenever $a(r)/b(r) \rightarrow 0$ as $r \rightarrow \infty$.

Neither forward recurrence nor the Miller algorithm is numerically stable for an intermediate solution. This is because recurrence in either direction, when carried out in finite precision, will be contaminated by small components of the other solutions. An analysis of the propagation of these rounding errors appears in [18]. In the forward direction, as the recurrence proceeds through more and more steps, the dominant solutions grow and eventually overwhelm the desired intermediate solution. Since our concept of dominance is really dominance "at infinity", the situation for recurrence in the backward direction is not completely analogous. However, if we start at a sufficiently large value of r and recur backward, other solutions grow at a faster rate than the wanted intermediate solution as r decreases.

In addition to intermediate solutions, a difference equation has minimal (or "recessive") and maximal (or "dominant") solutions. Intuitively, a minimal solution does not dominate, and a maximal solution is not dominated by, any other solution of the difference equation as $r \rightarrow \infty$. Ultimately, forward recurrence is stable for maximal solutions since the error cannot grow more rapidly than the solution. Similarly, backward recurrence is stable for minimal solutions, at least in a region of sufficiently large r .

[†]We shall regard the solutions of a linear difference equation, as well as the coefficients of the equation, as complex-valued functions of a discrete real variable. We shall assume that this discrete real variable proceeds in unit increments; thus the solutions, as well as the coefficients, of linear difference equations are complex sequences. We shall be interested only in those linear difference equations whose coefficient sequences are of infinite length.

The lowest order of difference equation for which an intermediate solution can exist, and then only when the equation is inhomogeneous, is two. Such an equation is of the form

$$(1.1.1) \quad a(r)y(r-1) - b(r)y(r) + c(r)y(r+1) = d(r) \quad , \quad r = 1, 2, \dots \quad .$$

Since the general solution is

$$y(r) = \alpha u(r) + \beta v(r) + h(r) \quad , \quad r = 0, 1, 2, \dots \quad ,$$

where α and β are constants, $u = (u(0), u(1), \dots)$ and $v = (v(0), v(1), \dots)$ are linearly independent solutions of the homogeneous equation, and $h = (h(0), h(1), \dots)$ is a particular solution of (1.1.1), there can be at most three asymptotically distinct types of behavior of solutions as $r \rightarrow \infty$. Let us assume that v dominates u and that h is an intermediate solution. In addition, assume $u(0) \neq 0$ and $v(r) \neq 0$ for all sufficiently large r . Under these assumptions a stable algorithm for computing intermediate solutions of (1.1.1) was presented by Olver in [21]. If $y = (y(0), y(1), \dots)$ is the desired intermediate solution, Olver's algorithm approximates y by a sequence of solutions $y_n = (y_n(0), y_n(1), \dots)$ such that $y_n(0) = y(0)$ and $y_n(n) = 0$ where n is an appropriately chosen positive integer. Note that only one initial value of the wanted intermediate solution is required to determine the sequence of approximations. The sequence is shown to converge pointwise to y as $n \rightarrow \infty$.

Olver's algorithm efficiently produces the first $n-1$ unknown terms of y_n as the solution of a tridiagonal linear system of order $n-1$. The procedure used for solving these linear equations is equivalent to Gaussian elimination without pivoting. It consists, in effect, of a forward elimination stage followed by a back substitution stage. The algorithm is constructed so that the numerical stability of this procedure may be assessed. The

forward elimination stage is realized by two forward recurrences. One of these uses the homogeneous form of (1.1.1) to generate a dominant solution $p = (p(0), p(1), \dots)$. The initial values used for starting this recurrence are $p(0) = 0$ and $p(1) = 1$, and the solution p is asymptotically proportional to v . Therefore, the rounding errors ultimately propagate in proportion to p , although they could propagate faster than p at first (if p is essentially proportional to u for early values of r , corresponding to an excessively small value of $u(0)$). A combined algorithm that employs both Miller's algorithm and Olver's algorithm in adjacent ranges of r is proposed in [22].

The second forward recurrence in the forward elimination stage of Olver's algorithm uses the inhomogeneous first-order equation

$$a(r)e(r-1) - c(r)e(r) = d(r)p(r), \quad r = 1, 2, \dots,$$

where the solution $e = (e(0), e(1), \dots)$ satisfies $e(0) = y(0)$. Here two distinct asymptotic forms of behavior may exist, but there is no reason to expect that the calculation of e will be asymptotically unstable in general. However, as was the case for the solution p , there may be applications where rounding errors grow more rapidly than $e(r)$ over some finite initial range of values of r .

The back substitution consists of backward recurrence of the inhomogeneous first-order equation

$$p(r+1)y_n(r) - p(r)y_n(r+1) = e(r), \quad r = n-1, n-2, \dots, 1,$$

starting with $y_n(n) = 0$. The stability of this backward recurrence is guaranteed in the "asymptotic region" belonging to the chosen solution $y(r)$, that is, the infinite range of r which is such that the asymptotic

behavior of $y(r)$ relative to $u(r)$ and $v(r)$ is maintained.

Olver's method of solving the tridiagonal system leads to a very convenient representation of the truncation error. For each r and for all $n \geq r$, the expansion

$$(1.1.2) \quad y(r) = y_n(r) + p(r) \sum_{s=n}^{\infty} \frac{e(s)}{p(s)p(s+1)}$$

is valid. Consequently, for each r and $n \geq r$ and $v \geq 1$ we have

$$y_{n+v}(r) - y_n(r) = p(r) \sum_{s=n}^{n+v-1} \frac{e(s)}{p(s)p(s+1)} .$$

This expansion is useful as a basis for deciding on an appropriate value of n , i.e., for deciding when to stop the forward elimination and begin the back substitution. Suppose that for particular values of $\varepsilon > 0$ and r , say $r = m$, it is desired that

$$|y(m) - y_n(m)| < \varepsilon .$$

A value of $v(\geq 1)$ is selected and $y(m)$ is approximated by $y_{n+v}(m)$. The problem then is to determine the smallest value of n such that

$$|y_{n+v}(m) - y_n(m)| < \varepsilon .$$

This is done by continuing the forward elimination until $r = m$ at which point calculation of the numbers

$$\eta(r) = \left| p(m) \sum_{s=r}^{r+v-1} \frac{e(s)}{p(s)p(s+1)} \right| , \quad r = m, m+1, \dots$$

is begun. Forward elimination is continued further, together with the calculation of $\eta(r)$, until a value of r is reached such that $\eta(r) < \varepsilon$. This value of r is taken for n ; its adequacy may be checked by taking additional terms in the infinite expansion of $y^{(m)} - y_n^{(m)}$, given by (1.1.2) with $r = m$.

In addition to being appropriate for computing intermediate solutions of (1.1.1), Olver's algorithm is suited to computing the minimal solution of a homogeneous equation. It can therefore be compared to Miller's algorithm. The advantage of Olver's algorithm over Miller's algorithm is that the latter does not have a criterion for determining automatically an appropriate value for n . On the other hand, if an adequate value of n is known from other considerations (as, for example, from an asymptotic estimate), then Miller's algorithm may require less computational effort.

Zahar [27], Oliver [19,20], and Cash [6] have considered extensions of Olver's algorithm to linear difference equations of higher order. In the present work we shall construct and analyze an algorithm which has features in common with these extensions. We shall also illustrate the algorithm by means of numerical examples.

1.2 Factorization of Linear Difference Equations

Let F be the field of complex numbers and let S be the set of all complex-valued infinite sequences $x = (x(0), x(1), \dots)$. Clearly S is an infinite-dimensional linear space over F . Let \mathcal{D}_ℓ be the set of all infinite upper-triangular band matrices of the form

$$\begin{bmatrix} d_0(0) & \cdot & \cdot & \cdot & \cdot & d_\ell(0) & & & & & 0 \\ & d_0(1) & \cdot & \cdot & \cdot & \cdot & d_\ell(1) & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & & d_0(r) & \cdot & \cdot & \cdot & \cdot & d_\ell(r) & & \\ 0 & & & \cdot & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & & & & \cdot & \cdot & \cdot & \\ & & & & & & & & \cdot & \cdot & \\ & & & & & & & & & \cdot & \\ & & & & & & & & & & \cdot \end{bmatrix}$$

where $d_0, d_1, \dots, d_\ell \in S$ and $d_0(r) \neq 0$, $d_\ell(r) \neq 0$ for infinitely many values of r . We shall also use the concise notation $[d_0, d_1, \dots, d_\ell]$ to represent infinite matrices in the set \mathcal{D}_ℓ . If we regard the infinite sequences in S as infinite column vectors, then an infinite sequence $Dx \in S$ is defined for each $D \in \mathcal{D}_\ell$, $x \in S$ by ordinary matrix multiplication. In fact, the terms of Dx are given by

$$Dx(r) = \sum_{s=0}^{\ell} d_s(r)x(r+s), \quad r = 0, 1, 2, \dots$$

Clearly, each $D \in \mathcal{D}_\ell$ is a linear operator on S under this definition. We shall say that \mathcal{D}_ℓ is the class of linear difference operators of order ℓ on S .

Note that \mathcal{D}_ℓ is defined for each integer $\ell \geq 0$. We pointed out in §1.1 that intermediate solutions of linear difference equations

can exist only if the order of the equation is at least two. But we are going to factor linear difference operators into a product of two lower-order operators. Therefore, operators of order one will arise. Furthermore, the formalism to be introduced will allow even the degenerate case of operators of order zero.

If d_0 and d_ℓ have no terms equal to zero, we shall say the difference operator $[d_0, d_1, \dots, d_\ell] \in \mathcal{D}_\ell$ is nonsingular; compare [14, §12.0]. The terms of d_0 and d_ℓ will be called the leading and trailing coefficients of $[d_0, d_1, \dots, d_\ell]$, respectively.

We shall use the notation x^i to indicate a sequence from S whose first term is $x(i)$. Thus

$$x^i = (x(i), x(i+1), \dots) .$$

The point i will be called the initial point of the sequence. When we do not indicate a superscript it is to be understood that zero is the initial point of the sequence. Similarly, we shall use the notation $x^{i,n}$ to indicate the (finite) subsequence of x^i whose final term is $x(n)$. Thus

$$x^{i,n} = (x(i), x(i+1), \dots, x(n)) .$$

The point n will be called the terminal point of the subsequence.

Similarly, for linear difference operators from \mathcal{D}_ℓ , we shall use the notation D^i or $[d_0^i, d_1^i, \dots, d_\ell^i]$ to indicate that i is the initial point of each of the $\ell + 1$ sequences involved in the definition of the operator. See Figure 1. When the superscript is omitted, it is understood that these initial points are all zero unless it is clear from the context (as in equation (1.2.1) below) what these initial points are.

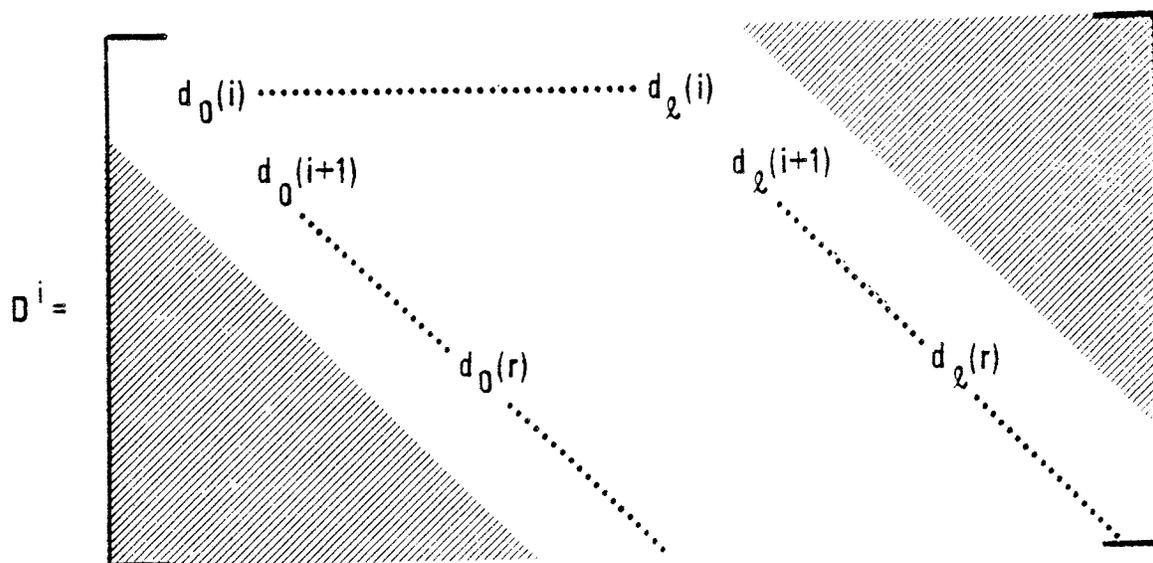


Figure 1. The operator D^i .

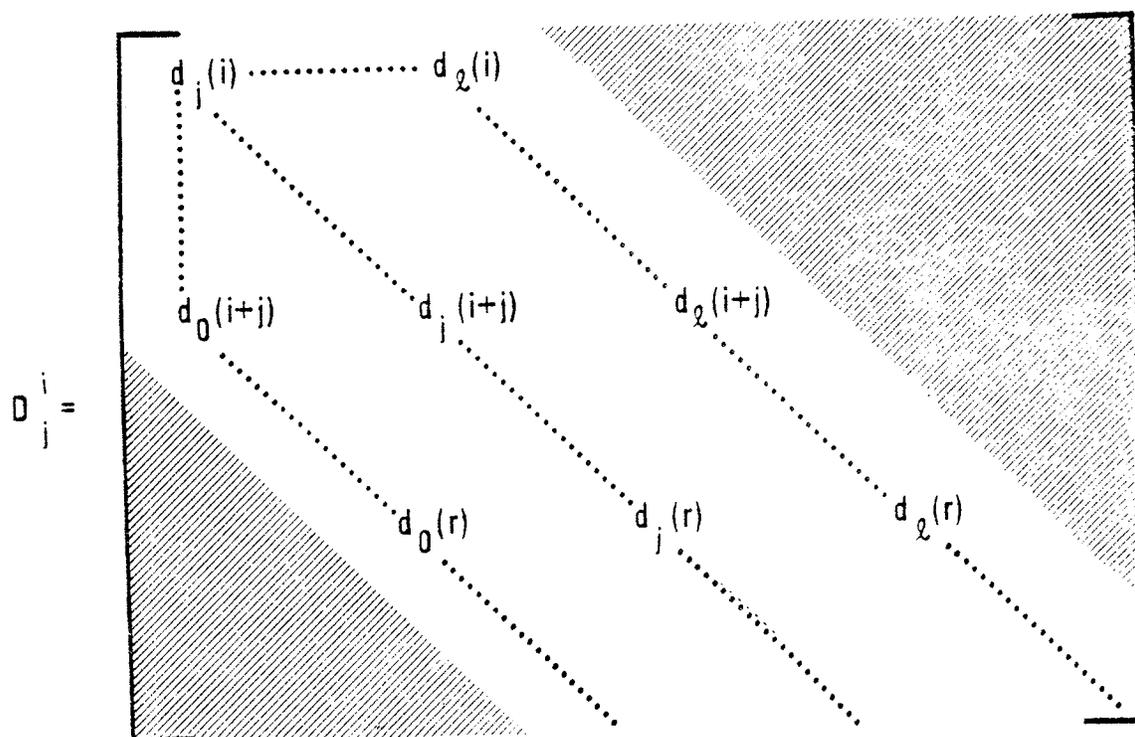


Figure 2. The matrix D_j^i .

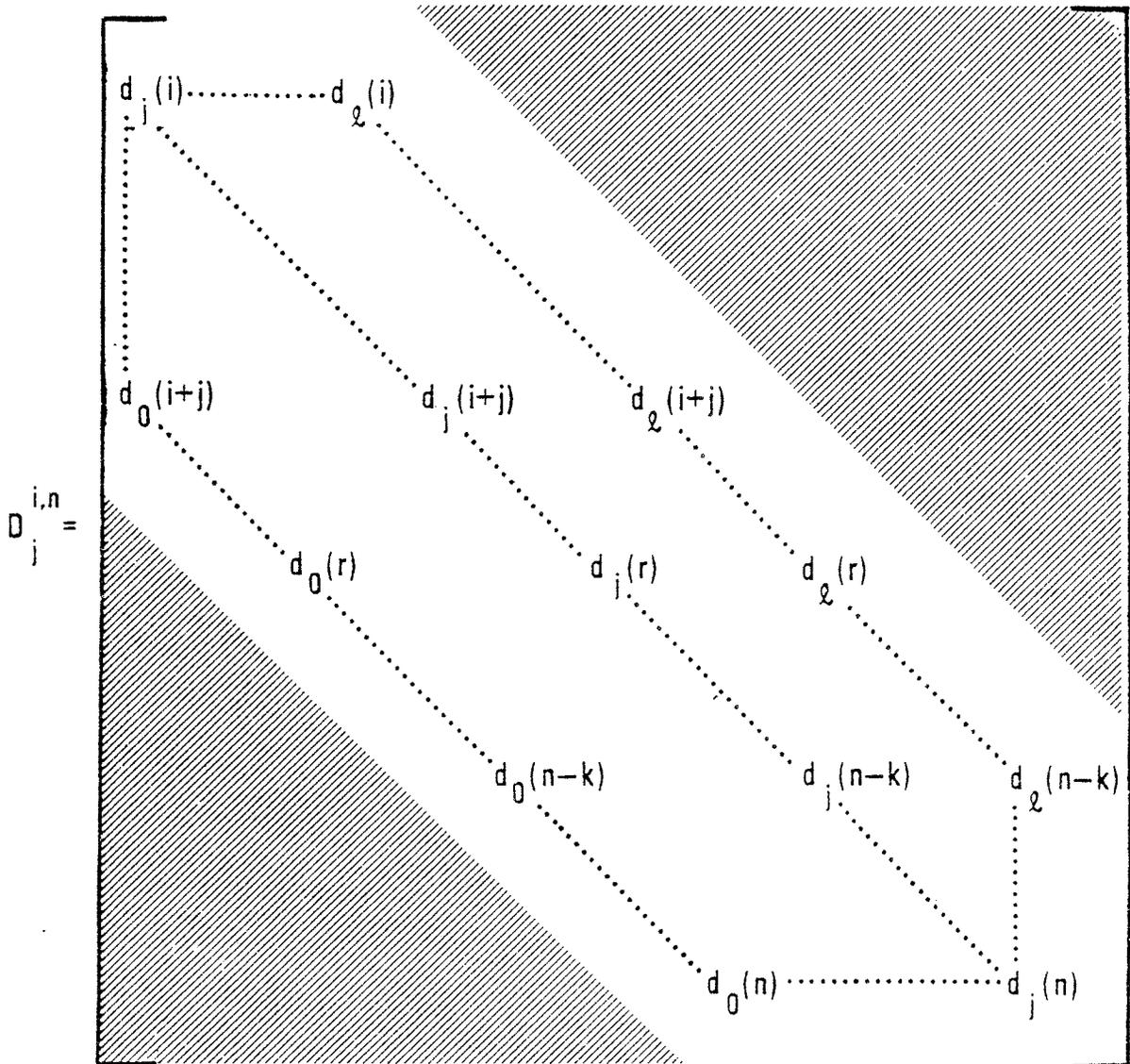


Figure 3. The matrix $D_j^{i,n}$.

Two further notational conventions to be used in this thesis are D_j^i , the infinite matrix obtained from D^i by deleting the first j columns ($0 \leq j \leq \ell$); and $D_j^{i,n}$, the leading principal submatrix of order $n - i + 1$ of D_j^i . See Figures 2 and 3. Note that each diagonal of D_j^i below the principal diagonal is a subsequence of one of the sequences $d_0^i, d_1^i, \dots, d_{j-1}^i$, the sequence of d_j^i is on the principal diagonal, and the sequences $d_{j+1}^i, \dots, d_\ell^i$ form the diagonals above the principal diagonal. The form of the finite matrix $D_j^{i,n}$ is similar, with the finite subsequence $d_j^{i,n}$ lying on the principal diagonal.

A linear difference equation is an equation of the form $Dx = g$, where $g \in S$ and $D \in \mathcal{D}_\ell$ are prescribed. Either $x \in S$ (or a subsequence of x of the form $x^{i,n}$) is to be determined. This equation is equivalent to the infinite linear algebraic system

$$Dx(r) = g(r), \quad r = 0, 1, \dots$$

Analogously to linear differential equations, auxiliary conditions are required in order to specify a unique solution. In our investigations these auxiliary conditions will consist of two finite subsequences $x^{i, i+j-1}$ and $x^{n, n+k-1}$ (or possibly only one of these) which are to be specified in advance, where $j + k = \ell$ and $j \geq 0, k \geq 0$. These are called the initial and terminal conditions, respectively, when $n \geq i + j$.

Let us consider the following finite boundary value problem: Determine $y^{i+j, n-1}$ such that

$$(1.2.1) \quad Dy(r) = g(r), \quad r = i, i+1, \dots, n-j-1$$

together with initial conditions

$$(1.2.2) \quad y(i+r) = \alpha_r, \quad r = 0, 1, \dots, j-1$$

and terminal conditions

$$(1.2.3) \quad y(n+r) = \beta_r, \quad r = 0, 1, \dots, k-1$$

are satisfied, where $D \in \mathcal{D}_\ell$, $\ell \geq 1$, $g \in S$, $j \in \{0, 1, \dots, \ell\}$, $k = \ell - j$, $i \geq 0$, $n \geq i + j$ and $\alpha_0, \alpha_1, \dots, \alpha_{j-1}, \beta_0, \beta_1, \dots, \beta_{k-1} \in F$. In view of (1.2.2) and (1.2.3), the finite subsequences $y^{i, i+j-1}$ and $y^{n, n+k-1}$ are known. Of course, if $j = 0$ then $y^{i, i+j-1}$ is null. Similarly if $k = 0$. These cases correspond to the absence of either (1.2.2) or (1.2.3). The unknown subsequence $y^{i+j, n-1}$ satisfies a linear system of algebraic equations which can be expressed in matrix terms using the notation we introduced above. The system is

$$(1.2.4) \quad D_j^{i, n-j-1} y^{i+j, n-1} = g^{i, n-j-1} - \begin{bmatrix} D^{i, i+j-1} y^{i, i+j-1} \\ 0 \\ \dots \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \dots \\ D_\ell^{n-\ell, n-j-1} y^{n, n+k-1} \end{bmatrix}$$

provided that $n - i - j \geq \max(j, k)$. The first partitioned column vector on the right side of (1.2.4) has j entries in the upper part, and the second partitioned column vector has k entries in the lower part. The length of the other column vectors appearing in (1.2.4) is $n - i - j$. This is what gives rise to the restriction $n - i - j \geq \max(j, k)$. This restriction is irrelevant for the applications we have in mind, since we shall be considering sequences of finite boundary value problems as $n \rightarrow \infty$. Note that the number of zeros represented by the symbol 0 in the partitioned column vectors is $n - i - 2j$ in the first vector and $n - i - \ell$ in the second vector. Obviously, the partitioned column vectors reflect the influence of the boundary conditions (1.2.2) and (1.2.3).

Solution of the boundary value problem (1.2.1) to (1.2.3) is equivalent to solution of the linear system (1.2.4). Equation (1.2.4) possesses

a unique solution if, and only if, the matrix $D_j^{i,n-j-1}$ is nonsingular. If in addition we assume that the difference operator D is nonsingular, then the solution $y^{i,n+k-1}$ of (1.2.1) to (1.2.3) extends uniquely to $y \in S$ by forward and backward recurrence. However, nonsingularity of D is not necessary for this unique extension to exist; for example, some of the leading and trailing coefficients of D which appear in the finite matrix $D_j^{i,n-j-1}$ could be zero.

Let us consider the nonsingular triangular cases of (1.2.4). The first is given by $j = \ell$. Then (1.2.4) assumes the form

$$(1.2.4a) \quad D_\ell^{i,n-\ell-1} y^{i+\ell,n-1} = g^{i,n-\ell-1} - \begin{bmatrix} D^{i,i+\ell-1} y^{i,i+\ell-1} \\ 0 \\ \dots\dots\dots \\ 0 \end{bmatrix}.$$

Writing out these equations in detail, we have

$$\begin{aligned} d_\ell(i)y(i+\ell) &= g(i) - \sum_{s=0}^{\ell-1} d_s(i)y(i+s) \\ d_{\ell-1}(i+1)y(i+\ell) + d_\ell(i+1)y(i+\ell+1) &= g(i+1) - \sum_{s=0}^{\ell-2} d_s(i+1)y(i+s+1) \\ &\vdots \\ &\vdots \\ &\vdots \\ \sum_{s=0}^{\ell} d_s(i+\ell)y(i+\ell+s) &= g(i+\ell) \\ &\vdots \\ &\vdots \\ &\vdots \\ \sum_{s=0}^{\ell} d_s(n-\ell-1)y(n-\ell-1+s) &= g(n-\ell-1) \end{aligned}$$

Since this system is lower triangular, it is solvable by determining $y(i+\ell)$ from the first equation, $y(i+\ell+1)$ from the second equation, and so on. This process is equivalent to developing $y(i+\ell), y(i+\ell+1), \dots, y(n-1)$ from (1.2.1) by forward recurrence starting with the given values (1.2.2).

The second triangular case is given by $j = 0$. Then (1.2.4) assumes the form

$$D_0^{i,n-1} y^{i,n-1} = g^{i,n-1} - \begin{bmatrix} \dots\dots\dots 0 \dots\dots\dots \\ D_\ell^{n-\ell,n-1} y^{n,n+\ell-1} \end{bmatrix}$$

Writing out these equations in detail, we have

$$\begin{aligned} \sum_{s=0}^{\ell} d_s(i)y(i+s) &= g(i) \\ &\vdots \\ &\vdots \\ \sum_{s=0}^{\ell} d_s(n-\ell-1)y(n-\ell-1+s) &= g(n-\ell-1) \\ &\vdots \\ &\vdots \\ d_0(n-2)y(n-2) + d_1(n-2)y(n-1) &= g(n-2) - \sum_{s=2}^{\ell} d_s(n-2)y(n-2+s) \\ d_0(n-1)y(n-1) &= g(n-1) - \sum_{s=1}^{\ell} d_s(n-1)y(n-1+s) \end{aligned}$$

Since this system is upper triangular, it is solvable by determining $y(n-1)$ from the last equation, $y(n-2)$ from the next-to-last equation, and so on. This process is equivalent to developing $y(n-1), y(n-2), \dots, y(i)$ from (1.2.1) by backward recurrence starting with the values (1.2.3).

The nontriangular cases of (1.2.4) in which $0 < j < \ell$ will be of greater interest to us. Our primary goal is to find a way to solve these problems by performing, in effect, a forward recurrence of order j followed by a backward recurrence of order $\ell - j$.

Definition 1.2.1 A nonsingular boundary value problem (1.2.4), or equivalently (1.2.1) to (1.2.3), having initial point i , terminal point n , number of initial conditions j and difference operator

$D = [d_0, d_1, \dots, d_\ell] \in \mathcal{D}_\ell$, will be called factorizable provided there

exist difference operators $A^i = [a_0^i, a_1^i, \dots, a_j^i]$ of order j and $B^{i+j} = [b_0^{i+j}, b_1^{i+j}, \dots, b_k^{i+j}]$ of order $k = \ell - j$ such that

$$(1.2.5) \quad D_j^{i, n-j-1} = A_j^{i, n-j-1} B_0^{i+j, n-1} .$$

If in addition the sequences a_j^i and b_0^{i+j} are free of zeros and the infinite matrix factorization

$$(1.2.6) \quad D_j^i = A_j^i B^{i+j}$$

is valid, then we shall say that the difference operator D is (i, j) - factorizable.

Several remarks may be made here. Note that $D_j^{i, n-j-1}$ and D_j^i are band matrices each having total bandwidth $\ell + 1$, lower bandwidth j , and upper bandwidth k . A nonsingular and factorizable boundary value problem is one for which a finite matrix factorization of $D_j^{i, n-j-1}$ exists such that the left factor is lower triangular with total bandwidth $j + 1$ and the right factor is upper triangular with total bandwidth $k + 1$. The difference operators A^i and B^{i+j} may be arbitrary as long as they satisfy the condition (1.2.5), since (1.2.5) involves only a finite submatrix of each of A^i and B^{i+j} . Equation (1.2.5) is illustrated in Figure 4. Since $D_j^{i, n-j-1}$ is nonsingular, both $A_j^{i, n-j-1}$ and $B_0^{i+j, n-1}$ are nonsingular. Thus the entries on the principal diagonals of $A_j^{i, n-j-1}$ and $B_0^{i+j, n-1}$ are all nonzero.

Now suppose the difference operator $D \in \mathcal{D}_\ell$ is (i, j) - factorizable; it need not be a nonsingular operator. Because of the triangular forms of A_j^i and B^{i+j} , equation (1.2.6) implies equation (1.2.5) for every $n \geq i + j + 1$. In words, every leading principal submatrix of D_j^i is equal to the product of the corresponding leading principal submatrices of A_j^i and B^{i+j} . The assumption that the subsequences a_j^i and b_0^{i+j} are free of zeros is equivalent to assuming that the entries on the

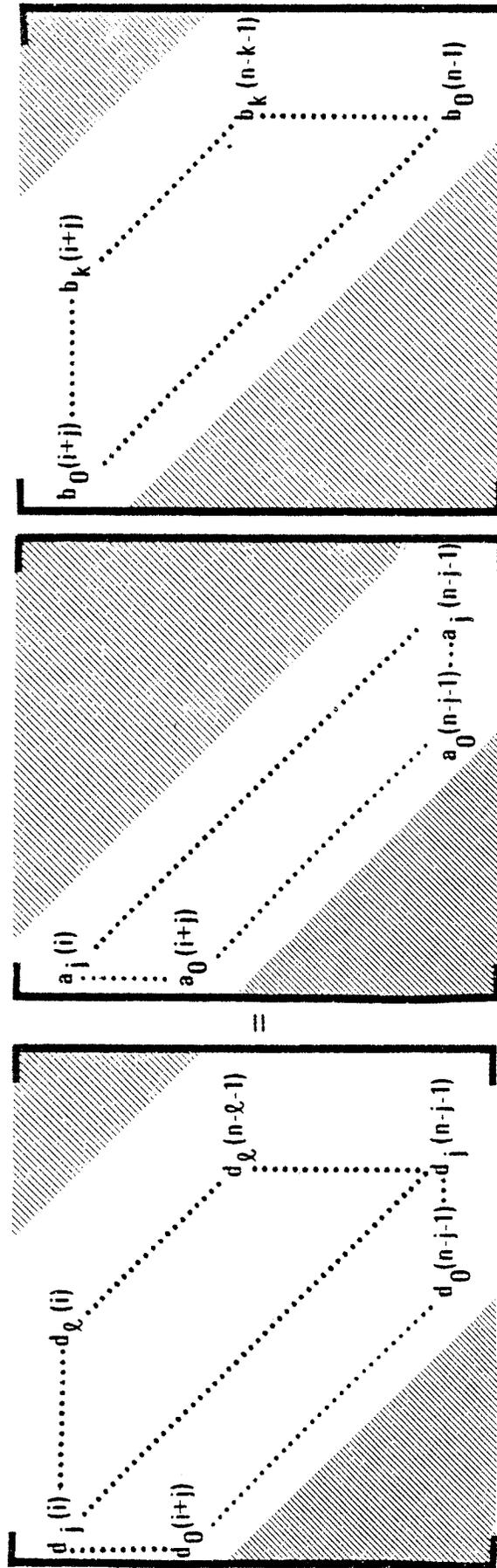


Figure 4. The factorization (1.2.5).

principal diagonals of the infinite matrices A_j^i and B^{i+j} are all non-zero. It follows that every leading principal submatrix of D_j^i is nonsingular. Thus we have proved the following theorem:

Theorem 1.2.1 Let $D \in \mathcal{D}_\ell$ be (i,j) - factorizable. Then, for every $n \geq i + j + \max(j, \ell - j)$, the boundary value problem (1.2.4) having difference operator D , initial point i , number of initial conditions j and terminal point n is nonsingular and factorizable. \blacksquare

The next theorem shows it is possible to obtain a solution to a factorizable boundary value problem, in theory at least, by performing an appropriate forward recurrence followed by an appropriate backward recurrence.

Theorem 1.2.2 Let (1.2.4) be a nonsingular and factorizable boundary value problem with initial point i , number of initial conditions j , terminal point n and difference operator $D \in \mathcal{D}_\ell$. If A^i and B^{i+j} are difference operators of order j and $k = \ell - j$, respectively, such that (1.2.5) is valid, then the solution of (1.2.4) is identical to the solution of

(1.2.7)

$$B_0^{i+j, n-1} y^{i+j, n-1} = z^{i+j, n-1} - \left[\begin{array}{c} \dots\dots\dots 0 \dots\dots\dots \\ (A_j^{n-\ell, n-j-1})^{-1} D_\ell^{n-\ell, n-j-1} y^{n, n+k-1} \end{array} \right]$$

where $z^{i+j, n-1}$ is the solution of

(1.2.8)

$$A_j^{i, n-j-1} z^{i+j, n-1} = g^{i, n-j-1} - \left[\begin{array}{c} D^{i, i+j-1} y^{i, i+j-1} \\ 0 \\ \dots\dots\dots \\ 0 \end{array} \right].$$

Substitution into the equation and comparison with (1.2.4) completes the proof. \square

An important remark is that if (1.2.6) is valid, i.e. if D is (i,j) -factorizable with left factor A^i and right factor B^{i+j} , then

$$D_{\ell}^{n-\ell, n-j-1} = A_j^{n-\ell, n-j-1} B_k^{n-k, n-1}.$$

In this case $B_k^{n-k, n-1}$ may be written in place of $(A^{n-\ell, n-j-1})^{-1} D^{n-\ell, n-j-1}$ in (1.2.7). The truth of this remark is apparent from Figure 4.

The next theorem gives necessary and sufficient conditions for a difference operator to be (i,j) - factorizable. The proof also shows that (i,j) - factorizations may be constructed by Gaussian elimination in natural order, i.e., without any row or column interchanges.

Theorem 1.2.3 Let $D \in \mathcal{D}_{\ell}$. Then

i) A nonsingular boundary value problem having initial point i , number of initial conditions j and terminal point n is factorizable if every leading principal submatrix of $D_j^{i, n-j-1}$ is nonsingular.

ii) D is (i,j) - factorizable if every leading principal submatrix of D_j^i is nonsingular. Furthermore, this condition is necessary if D is nonsingular.

Proof Our approach is to apply Gaussian elimination [10, §2.1] without any row or column interchanges. Because Gaussian elimination is so familiar, we merely sketch the proof. The proof of (i) is contained in the first $n - i - j$ stages of the proof of (ii); therefore we supply only the proof of (ii). We begin by proving sufficiency of the condition.

The first row of D_j^i is used to annihilate every nonzero element below the first element in the first column. This is possible because the pivot element, which is the leading principal minor of order one, is nonzero. Because of the band structure, only the j rows immediately follow-

ing the first row are affected. Also (again because of the band structure) the trailing coefficient in each of these rows is not changed. The multipliers used in these annihilations go into positions 2 through $j + 1$ of the first column of the lower triangular factor A_j^i . The first element is unity and every other element of the first column of A_j^i is zero. This completes the first stage.

Now suppose the $(r-1)$ st stage has been completed. Let the notation \bar{D}_j^i denote the transformation of the original D_j^i resulting from the first $r - 1$ stages. The leading principal submatrix of \bar{D}_j^i of order r is upper triangular, having resulted in effect from the corresponding principal submatrix of the original D_j^i by Gaussian elimination in natural order. Its diagonal elements are all nonzero, by hypothesis. Therefore the pivotal element is nonzero and the r -th row of \bar{D}_j^i may be used to annihilate every nonzero element below the r -th element in the r -th column. Only the j rows immediately following the r -th row are affected, and the trailing coefficient in each of these rows is not changed. The multipliers used in this annihilation go into positions $r + 1$ through $r + j$ of the r -th column of A_j^i ; unity goes into the r -th position, and zeros go into every other position. This completes the r -th stage.

Let B^{i+j} be the infinite upper triangular matrix which results from the full sequence of transformations on D_j^i . By our construction, both A_j^i and B^{i+j} have the bandwidths required by (i,j) - factorizability. Also by our construction, formally multiplying A_j^i on the right by B^{i+j} results in D_j^i ; compare (1.2.6).

In order to prove necessity of the condition in part (ii), note that the factorization produced by the foregoing process is such that leading and trailing coefficients of both A_j^i and B^{i+j} are nonzero if D is

nonsingular. Therefore, no row or column interchanges could be made without widening the bandwidth. Therefore, Gaussian elimination in natural order is the only possibility for arriving at the factorization. But Gaussian elimination in natural order is possible only if every leading principal minor is nonzero. This completes the proof. \square

Examination of the proof of Theorem 1.2.3 shows that a unique (i,j) -factorization does not exist, since a different scaling of the rows is possible. This would result in the appearance of entries on the principal diagonal of A_j^i that are other than unity. In fact, the scaling used in the $(0,1)$ -factorization in Olver's second-order algorithm put the sequence $(p(2), p(3), \dots)$ on the principal diagonal of the right factor; compare §1.1. Thus the principal diagonal of the left factor will contain entries other than unity, in general.

1.3 A Preliminary Development of the Algorithm

Let $D \in \mathcal{D}_\ell$. It is clear that an extension of Olver's algorithm for computing a particular solution y of a given difference equation $Dy = g$ should involve the posing of a boundary value problem whose solution y_n is an approximation of y over some finite subsequence $y^{i,m}$. The boundary value problem will have some number $j \geq 0$ of initial conditions (depending on the particular solution being sought). These will consist of a subsequence of j consecutive values of y . The terminal conditions of the boundary value problem will be $y_n(n) = y_n(n+1) = \dots = y_n(n+k-1) = 0$, where $k = \ell - j$.

The first task is to decide how to choose j . A necessary prerequisite is a classification of possible solution types when the operator is of order higher than two. Oliver [19] provided an early extension of Olver's algorithm. He also provided the following definition of dominance. Suppose we are given two sequences $a, b \in S$. Then b dominates a if there exists some $i \geq 0$ such that

$$(1.3.1) \quad |b(r+1)/b(r)| > |a(r+1)/a(r)|$$

for all $r \geq i$. Oliver considered only those difference operators for which a system of complementary solutions could be found such that the complementary solutions are linearly ordered under this definition of dominance. Furthermore, he considered only those particular solutions of the given linear difference equation which could be compared to each of the complementary solutions. But not all possible particular solutions can be so compared; for example, no particular solution which has an infinite subsequence of zeros can play the role of either a or b in (1.3.1).

In §1.1, we adopted a different definition of dominance: b dominates a if

$$(1.3.2) \quad \lim_{r \rightarrow \infty} \frac{a(r)}{b(r)} = 0 .$$

This differs from Oliver's definition (which is equivalent to requiring that ultimately $|a(r)/b(r)|$ be monotonically decreasing). Another difference is that the limit of $|a(r)/b(r)|$ under Oliver's definition may be greater than zero, as the example $a(r) = 1 + 1/r$ and $b(r) = 1$ immediately shows.

Let $D \in \mathcal{D}_\ell$. The set $K(D) = \{x \in S \mid Dx=0\}$ will be called the kernel of D . Obviously, $K(D)$ is a linear subspace of S . Let us suppose that $K(D)$ has dimension ℓ , so that there exist ℓ linearly independent sequences $x_1, x_2, \dots, x_\ell \in K(D)$ which span $K(D)$. A sufficient condition for this to be the case is that D be nonsingular; compare [14, §12.1]. We shall say that the set x_1, x_2, \dots, x_ℓ forms a basis for D .

Definition 1.3.1 We shall say that $D \in \mathcal{D}_\ell$ is totally separable (as $r \rightarrow \infty$) provided there exists a basis y_1, y_2, \dots, y_ℓ for D such that y_{s+1} dominates y_s in the sense of (1.3.2) for each s , $1 \leq s < \ell$.

Any basis which satisfies the conditions of Definition 1.3.1 will be called totally ranked. Such a basis is not unique because y_1 , for example, could be added to each of y_2, \dots, y_ℓ and the resulting basis would also be totally ranked.

For each s , $1 \leq s \leq \ell$, let K_s denote the linear subspace of S which is spanned by y_1, y_2, \dots, y_s . In particular, $K_\ell = K(D)$. It seems obvious, and it will be proved in a more general setting in Chapter 2, that every totally ranked basis generates the same subspaces. The subspaces satisfy

$$K_1 \subset K_2 \subset \dots \subset K_\ell = K(D) .$$

Each set inclusion is proper, since the dimension of K_s is s . This ascending chain of subspaces induces an obvious classification of the solutions of $Dy = 0$ into ℓ distinct types. However, we are interested in classifying the solutions of $Dy = g$ for nonzero, and indeed, arbitrary $g \in S$. Therefore, we frame the definition so as to cover all cases:

Definition 1.3.2 Let $D \in \mathcal{D}_\ell$ be totally separable with totally ranked basis y_1, y_2, \dots, y_ℓ . Then, for each $y \in S$, we shall say that the type of y (with respect to D) is s , where $0 < s < \ell$, and we shall write $\text{type}(y) = s$, provided that

$$(1.3.3) \quad \lim_{r \rightarrow \infty} \frac{y(r)}{y_{s+1}(r)} = 0$$

and

$$(1.3.4) \quad \infty \geq \limsup_{r \rightarrow \infty} \left| \frac{y(r)}{y_s(r)} \right| > 0 .$$

Furthermore, if (1.3.3) is true for $s = 0$, we define $\text{type}(y) = 0$, and if (1.3.4) is true for $s = \ell$, we define $\text{type}(y) = \ell$.

The type of each $y \in S$ is the same regardless of which totally ranked basis is used. This will be proved for a more general classification of sequences in Chapter 2. Thus we are justified in defining the type with respect to D rather than with respect to a particular totally ranked basis.

The conditions for Olver's algorithm [21] suggest that Definitions 1.3.1 and 1.3.2 are appropriate for an extension of the algorithm to arbitrary order; see also Cash [6] . But it would be desirable to have a classification scheme which is applicable, for example, to every difference

equation with constant coefficients. Not every constant-coefficient operator is totally separable, however. In the next chapter we shall introduce a more general class of difference operators (separable operators) which does include every constant-coefficient operator. Incidentally, the class of difference equations considered by Oliver [19] includes every constant-coefficient equation, since the standard basis (see §2.1 below) is linearly ordered under his definition of dominance.

Before stating the next theorem, we explain what we mean by matrices and determinants associated with the name Casorati. If $x_1, x_2, \dots, x_s \in S$ are arbitrary sequences, then for each $r \geq 0$ the $s \times s$ matrix

$$\begin{bmatrix} x_1(r) & x_2(r) & \dots & x_s(r) \\ x_1(r+1) & x_2(r+1) & \dots & x_s(r+1) \\ \vdots & \vdots & & \vdots \\ x_1(r+s-1) & x_2(r+s-1) & \dots & x_s(r+s-1) \end{bmatrix}$$

will be called a Casorati matrix. Let $X = \{x_1, x_2, \dots, x_s\}$. We shall use the concise notation $[X](r)$ to denote the Casorati matrix above. Furthermore, $|X|(r)$ will denote the Casoratian (of X at r), defined by

$$|X|(r) = \det [X](r), \quad r = 0, 1, \dots$$

If $D \in \mathcal{D}_\ell$ is nonsingular, Casorati's theorem [14, §12.11] states that x_1, x_2, \dots, x_ℓ forms a basis for D if, and only if, $|X|(r) \neq 0$ for all r .

Theorem 1.3.1 Let y be a solution of the difference equation $Dy = g$, where $D \in \mathcal{D}_\ell$ has a totally ranked basis $Y = \{y_1, y_2, \dots, y_\ell\}$. Let

$j = \text{type}(y)$. Then y is uniquely determined by its values $y(i), y(i+1), \dots, y(i+j-1)$ provided that $i \geq 0$ is a point at which the leading principal minor of order j of $|Y|(i)$ is nonzero.

Proof Assume $j = 0$; then the condition on the Casoratian is vacuous and we must show y is uniquely determined without knowledge of any of its values. Suppose $z \neq y$ is another solution of type zero; then there exist unique scalars $\alpha_1, \alpha_2, \dots, \alpha_\ell$ (not all zero) such that

$$y(r) = z(r) + \alpha_1 y_1(r) + \dots + \alpha_\ell y_\ell(r) , \quad r = 0, 1, \dots .$$

Let α_t be the nonzero scalar of highest subscript. Dividing through by $y_t(r)$ and allowing $r \rightarrow \infty$, we see that $\alpha_t = 0$, which is a contradiction.

Next, assume $j > 0$. Suppose $z \neq y$ is another solution of type j such that

$$z(i+r) = y(i+r) , \quad r = 0, 1, \dots, j-1 .$$

There exist unique scalars $\alpha_1, \alpha_2, \dots, \alpha_\ell \in F$ (not all zero) such that

$$y(r) = z(r) + \alpha_1 y_1(r) + \dots + \alpha_\ell y_\ell(r) , \quad r = 0, 1, \dots .$$

It is proved easily that each of $\alpha_{j+1}, \dots, \alpha_\ell$ is zero by a method similar to that of the previous paragraph. The remaining scalars are determined by the linear system

$$\alpha_1 y_1^{i, i+j-1} + \dots + \alpha_j y_j^{i, i+j-1} = 0$$

(where 0 stands for a column vector of j zeros and we have used the notation developed in §1.2). This system is nonsingular by the assumption on the Casoratian; therefore, $\alpha_t = 0$, $t = 1, 2, \dots, j$, which is a contradiction. \blacksquare

Theorem 1.3.1 confirms the adequacy of the selection of j initial conditions in the approximating boundary value problem, where $j = \text{type}(y)$. In §3.3 we will demonstrate that stability considerations demand exactly j initial conditions, in general, for a solution of type j .

Now let us suppose that a value of j has been selected, where $j \geq \text{type}(y)$. Since $y_n^{i,i+j-1} = y^{i,i+j-1}$ and $y_n^{n,n+k-1} = 0$, the linear system to be solved is

$$(1.3.5) \quad D_j^{i,n-j-1} y_n^{i+j,n-1} = g^{i,n-j-1} - \begin{bmatrix} D^{i,i+j-1} y^{i,i+j-1} \\ 0 \\ \dots\dots\dots \\ 0 \end{bmatrix};$$

compare (1.2.4). The value of n is to be found in the course of the computation. In view of Theorem 1.3.1, y is the unique solution of the infinite linear system

$$(1.3.6) \quad D_j^i y^{i+j} = g^i - \begin{bmatrix} D^{i,i+j-1} y^{i,i+j-1} \\ 0 \\ \dots\dots\dots \\ 0 \end{bmatrix}.$$

We assume that D is (i,j) -factorizable, i.e., that linear difference operators $A^i \in \mathcal{D}_j$ and $B^{i+j} \in \mathcal{D}_{\ell-j}$ exist such that $D_j^i = A_j^i B^{i+j}$. As shown in the proof of Theorem 1.2.3, successive rows of this factorization, and indeed of the triangularization of the infinite system (1.3.6), may be generated indefinitely by the forward stage of Gaussian elimination without pivoting. The stability of this process, also, will be discussed in §3.3.

In consequence of Theorems 1.2.1 and 1.2.2 and the terminal conditions $y_n^{n,n+k-1} = 0$, the solution of (1.3.5) may be obtained as the solution of

$$(1.3.7) \quad B_0^{i+j,n-1} y_n^{i+j,n-1} = z^{i+j,n-1}$$

where $z^{i+j, n-1}$ is a finite subsequence of the solution of

$$(1.3.8) \quad A_j^i z^{i+j} = g^i - \begin{bmatrix} D_{i, i+j-1} & y_{i, i+j-1} \\ 0 & \dots \\ \dots & \dots \\ 0 & \dots \end{bmatrix} .$$

Comparison of (1.3.8) and (1.3.6) and reference to the identity

$$D_j^i = A_j^i B^{i+j} \quad \text{show that}$$

$$(1.3.9) \quad B^{i+j} y^{i+j} = z^{i+j} .$$

Thus (1.3.7) represents the finite linear system that results from (1.3.9) by truncation at the $(n-i-j)$ th row and column. Since B^{i+j} and z^{i+j} are generated by Gaussian elimination without pivoting, the solution of (1.3.5) for any n is easily programmed for an automatic digital computer. The algorithm comprises two stages: (i) forward elimination (solution of (1.3.8)); (ii) back substitution (solution of (1.3.7)). The forward elimination can be continued arbitrarily far. The back substitution is numerically equivalent to backward recurrence, and proceeds until we return to the initial point i .

The question of the stability of the back substitution is easily settled. For totally separable operators we have the following theorem:

Theorem 1.3.2 Let $D \in \mathcal{D}_\ell$ have a totally ranked basis $Y = \{y_1, y_2, \dots, y_\ell\}$.

Suppose $i \geq 0$ and $j \in \{0, 1, \dots, \ell-1\}$ are such that D is (i, j) -factorizable and the leading principal minor of order j of $|Y|(i)$ is nonzero. If $D_j^i = A_j^i B^{i+j}$ is an (i, j) -factorization of D , then B^{i+j} is totally separable. Furthermore, if $z \in S$ has type j or less with respect to D , then z^{i+j} has type zero with respect to B^{i+j} .

Proof Suppose $j = 0$. Then A_j^i is a diagonal matrix with nonzero elements on the diagonal; see Definition 1.2.1. For each $y \in Y$ we have

$$D^i y^i = A^i B^i y^i = 0, \text{ i.e.}$$

$$\sum_{s=0}^{\ell} a_0(r) b_s(r) y(r+s) = 0, \quad r = i, i+1, \dots$$

Therefore,

$$\sum_{s=0}^{\ell} b_s(r) y(r+s) = 0, \quad r = i, i+1, \dots$$

which proves $\{y^i | y \in Y\}$ is a totally ranked basis for B^i . It follows that z^i has type zero with respect to B^i whenever z has type zero with respect to D .

Suppose $0 < j < \ell$. Let $U = \{y_1, y_2, \dots, y_j\}$. Since $|U|(i) \neq 0$ it is possible to find a linear combination x of y_1, y_2, \dots, y_j such that $x(i), x(i+1), \dots, x(i+j-1)$ take on any prescribed set of values. Therefore, appropriate linear combinations x_s , $s = j+1, j+2, \dots, \ell$, can be found such that $y_s(r) + x_s(r)$ vanish for $r = i, i+1, \dots, i+j-1$. Let $v_{s-j} = y_s + x_s$, $s = j+1, j+2, \dots, \ell$. It is easy to verify that $\{y_1, \dots, y_j, v_1, \dots, v_k\}$ is a totally ranked basis for D . Now let $v \in \{v_1, v_2, \dots, v_k\}$. In view of Theorems 1.2.1 and 1.2.2, the remark following Theorem 1.2.2, and the facts that $Dv = 0$ and $v(i+r) = 0$ for $r = 0, 1, \dots, j-1$, we conclude that $v^{i+j, n-1}$ satisfies

$$B_0^{i+j, n-1} v^{i+j, n-1} = - \begin{bmatrix} \dots\dots\dots 0 \dots\dots\dots \\ B_k^{n-k, n-1} v^{n, n+k-1} \end{bmatrix}$$

for every $n \geq i + j + \max(j, k)$; compare equation (1.2.7). On letting $n \rightarrow \infty$ this proves $B^{i+j} v^{i+j} = 0$. Thus $\{v_1^{i+j}, v_2^{i+j}, \dots, v_k^{i+j}\}$ is a totally ranked basis for B^{i+j} . Finally, let z be of type j or less with respect to D . Let $s \in \{1, 2, \dots, k\}$. Then

$$\frac{z(r)}{v_s(r)} = \frac{z(r)}{y_{j+s}(r)\{1+o(1)\}} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

which shows that z^{i+j} has type zero with respect to B^{i+j} . \square

In consequence of Theorem 1.3.2 and our remarks in §1.1, when $j \geq \text{type}(y)$ the back substitution stage of the algorithm is stable, at least for all r in a sufficiently large range.

It remains to determine the appropriate value of the truncation parameter n . It is necessary to introduce Green's formula for linear difference operators; see, e.g., Miller [13]. We apply Green's formula not to the operator D but to the operator B^{i+j} . Our development is essentially the same as that of Cash [6].

Let $B = [b_0, b_1, \dots, b_k] \in \mathcal{D}_k$. The infinite upper triangular matrix $\hat{B} = [\hat{b}_0, \hat{b}_1, \dots, \hat{b}_n] \in \mathcal{D}_k$ which is obtained from B by deleting the first k columns and transposing the result, i.e. whose entries are defined by

$$(1.3.10) \quad \hat{b}_s(r) = b_{k-s}(r+s),$$

is a linear difference operator of order k . We shall call \hat{B} the operator which is adjoint to B , or more briefly, the adjoint operator of B . An operator and its adjoint satisfy a number of elementary properties which are easily verified from the definition. Among these are

$$(1.3.11) \quad \hat{B}^p = (B_k^p)^{\text{Tr}}$$

$$(1.3.12) \quad \hat{B}_0^{p,q} = (B_k^{p,q})^{\text{Tr}}$$

$$(1.3.13) \quad \hat{B}_k^p = (B^{p+k})^{\text{Tr}}$$

$$(1.3.14) \quad \hat{B}_k^{p,q} = (B_0^{p+k,q+k}) \text{Tr} ,$$

valid for $q \geq p$ and $p = 0, 1, 2, \dots$.

Now take any $u, v \in S$ and $B \in \mathcal{D}_k$, and assume $k \geq 1$. Green's formula is the identity

$$(1.3.15) \quad \sum_{r=p}^{q-1} \{v(r)Bu(r) - u(r+k)\hat{B}v(r)\} \\ = (v^{p,p+k-1}) \text{Tr}_{B^{p,p+k-1}} u^{p,p+k-1} \\ - (v^{q,q+k-1}) \text{Tr}_{B^{q,q+k-1}} u^{q,q+k-1}$$

valid for $q \geq p + k$, $p \geq 0$. Green's formula may be verified by expressing the left side of (1.3.15) in the form

$$(v^{p,q-1}) \text{Tr}_{(Bu)^{p,q-1}} - (u^{p+k,q+k-1}) \text{Tr}_{(\hat{B}v)^{p,q-1}}$$

and reducing this expression with the aid of the definition (1.3.10) and the identities (1.3.11) - (1.3.14).

Theorem 1.3.3 Let $D_j^i = A_j^i B^{i+j}$ be an (i,j) -factorization of $D \in \mathcal{D}_\ell$, where D is totally separable and $j < \ell$. Let y be a solution of $Dy = g$, where $\text{type}(y) \leq j$ and $g \in S$. Let $y_n^{i+j, n-1}$ and z^{i+j} be the corresponding solutions of (1.3.5) and (1.3.8). Also, let s be a fixed integer satisfying $i + j + k - 1 \leq s < n$ and $w_s^{s+1} = (w_s(s+1), w_s(s+2), \dots)$ be the solution of

$$(1.3.16) \quad \hat{B}_k^{s-k+1, s} w_s^{s+1} = - \begin{bmatrix} \hat{B}_0^{s-k+1, s} w_s^{s-k+1, s} \\ 0 \dots \dots \dots 0 \end{bmatrix}$$

where $k = \ell - j$ and

$$(1.3.17) \quad w_s^{s-k+1,s} = (0,0,\dots,0,1) .$$

Then

$$(1.3.18) \quad y_n(s) = \sum_{r=s}^{n-1} w_s(r)z(r)/b_0(s) ,$$

where $b_0(s)$ is the $(s-i-j+1)$ st leading coefficient of B^{i+j} .

Remark: Equations (1.3.16), (1.3.17) are equivalent to the initial value problem

$$\sum_{t=0}^k \hat{b}_t(r)w_s(r+t) = 0 , \quad r = s-k+1, s-k+2, \dots$$

with initial conditions $w_s(s-k+1) = \dots = w_s(s-1) = 0$, $w_s(s) = 1$;

compare (1.2.1)-(1.2.3) and (1.2.4a).

Proof: Let $p = s - k + 1$, $q = n$, $v = w_s$ and $u = y_n$ in (1.3.15). By virtue of (1.3.7), (1.3.16) and (1.3.17), the left side of (1.3.15)

becomes

$$\sum_{r=s}^{n-1} w_s(r)z(r) .$$

Since $u^{q,q+k-1} = y_n^{n,n+k-1} = 0$, the second term on the right side of (1.3.15) is zero. Thus, expanding the right side and using (1.3.17), we derive

The difference operator which is implicit in (1.1.1) is $D = [d_0, d_1, d_2]$ where

$$d_0(r) = a(r+1) , d_1(r) = - b(r+1) , d_2(r) = c(r+1)$$

for $r = 0, 1, 2, \dots$. Similarly, the right side of (1.1.1) is, in our present notation,

$$g(r) = d(r+1) , r = 0, 1, 2, \dots .$$

The rows of the (0,1)-factorization $D_1^0 = A_1^0 B_1^1$ are scaled so that

$$b_0(r) = p(r+1) , b_1(r) = - p(r)$$

for $r = 1, 2, \dots$, where $B_1^1 = [b_0^1, b_1^1]$ and $p = (p(0), p(1), \dots)$ is the sequence defined in §1.1. The solution z^1 of the forward elimination stage of the algorithm is given by

$$z(r) = e(r) , r = 1, 2, \dots ;$$

compare (1.3.8). Finally, it may be verified readily that the solution of the adjoint equation $\hat{B}w_s(r) = 0$, $r = s, s+1, \dots$ such that $w_s(s) = 1$ is given by

$$w_s(r) = \frac{p(s)p(s+1)}{p(r)p(r+1)} , r = s, s+1, \dots .$$

Making these substitutions into (1.3.20) and exchanging the roles of r and s , we obtain (1.1.2).

The simple form of the equation for $w_s(r)$ depends solely on the particular scaling of the (0,1)-factorization used by Olver and the fact that $k = 1$. A similar simplification can be obtained for difference equations of arbitrary order provided only that $k = 1$. It stems from

the fact that the order of the adjoint equation is one, hence all solutions can be expressed as multiples of a single particular solution. These conclusions were also arrived at in Cash's paper [6].

The expansions (1.3.19) and (1.3.20) may be employed in the estimation of the optimal value of n such that a given termination criterion is met. Typical termination criteria would be to require one or more values of $y_n(r)$ in the range $i + j \leq r \leq m$ to approximate $y(r)$ to a specified absolute or relative precision. Since the back substitution stage is stable, controlling the absolute or relative error only at the point $r = m$ is often sufficient. In practice equations (1.3.19) and (1.3.20) are employed by replacing the upper limit by $n + \nu$, where $\nu \geq 1$ is a parameter chosen to suit the problem at hand.

The solution of (1.3.16), (1.3.17) is obtained by forward recurrence. The stability of this process will depend on the behavior of $w_s(r)$ for large r relative to the other solutions of the adjoint equation. In general there is no reason to expect that $w_s(r)$ will not contain a component of a maximal solution. Thus the solution of (1.3.16) will normally be stable, although there could be an initial range where some unstable rounding error propagation takes place.

CHAPTER 2. CLASSIFICATION OF SEQUENCES

2.1 Constant-Coefficient Operators

Let $D = [d_0, d_1, \dots, d_\ell] \in \mathcal{D}_\ell$ be a constant-coefficient operator. By this we mean that there exist $\delta_0, \delta_1, \dots, \delta_\ell \in F$ such that $\delta_0 \neq 0$, $\delta_\ell \neq 0$ and $d_s(r) = \delta_s$ for all r . We assume without loss of generality that $\delta_\ell = 1$ in every constant-coefficient operator. Since every constant-coefficient operator is nonsingular, by our definitions, the kernel $K(D)$ has dimension ℓ ; see discussion preceding Definition 1.3.1.

A particular basis, which we shall call the standard basis, exists for each constant-coefficient operator; see [14, §§13.0-13.1]. The polynomial

$$P(\xi) = \xi^\ell + \delta_{\ell-1}\xi^{\ell-1} + \dots + \delta_0$$

is known as the characteristic polynomial for D . The standard basis is the set of sequences

$$\Lambda = \{(r^m \lambda^r) \in S \mid P(\lambda) = 0 \text{ and } m \in \{0, 1, \dots, \mu(\lambda)-1\}\}$$

where $\mu(\lambda)$ denotes the multiplicity of the root λ . We are using the notation $(r^m \lambda^r)$ here as an alternative way of denoting the sequence

$$(\rho_m, \lambda, 2^m \lambda^2, 3^m \lambda^3, \dots)$$

where $\rho_0 = 1$ and $\rho_m = 0$ when $m > 0$.

The standard basis of a constant-coefficient operator is not necessarily totally ranked; recall Definition 1.3.1. For example, if λ_1 and λ_2 are two distinct zeros of the characteristic polynomial such that

$|\lambda_1| = |\lambda_2|$ then $|\lambda_1^r| = |\lambda_2^r|$ for all r . Neither solution dominates the other.

Let us write $a < b$ whenever $a, b \in S$ are such that a is dominated by b . We shall say that a and b are separated (at infinity) if $a < b$ or $b < a$. A totally ranked basis is linearly ordered by $<$ but in general the standard basis is not, as we have just seen. However, we can introduce a linearly ordered partition of the standard basis. We shall proceed to do this.

It is apparent that $(r^{m_1 \lambda_1^r}) < (r^{m_2 \lambda_2^r})$ if, and only if, either (i) $|\lambda_1| < |\lambda_2|$ or (ii) $|\lambda_1| = |\lambda_2|$ and $m_1 < m_2$. Let σ denote the number of roots of $P(\xi) = 0$ of distinct absolute value, and let $\lambda_1, \lambda_2, \dots, \lambda_\sigma$ be representative roots of $P(\xi) = 0$ such that $|\lambda_s| < |\lambda_{s+1}|$ for each s . Define $\mu_s = \max\{\mu(\lambda) \mid |\lambda| = |\lambda_s|\}$. We define the sets

$$\Lambda_s = \{(r^m \lambda^r) \in \Lambda \mid |\lambda| = |\lambda_s|\}, \quad s = 1, 2, \dots, \sigma$$

and, for each s , the sets

$$\Lambda_{s,m} = \{(r^m \lambda^r) \in \Lambda_s \mid \mu(\lambda) \geq m\}, \quad m = 0, 1, \dots, \mu_s - 1.$$

It may be verified readily that the sets $\Lambda_{s,m}$ are disjoint and nonempty and

$$(2.1.1) \quad \Lambda = \bigcup_{s=1}^{\sigma} \bigcup_{m=0}^{\mu_s - 1} \Lambda_{s,m}$$

The sets $\Lambda_{s,m}$ are linearly ordered according to the definition:

$\Lambda_{s,m} < \Lambda_{s',m'}$ if, and only if, either (i) $s < s'$ or (ii) $s = s'$ and $m < m'$.

An illustrative example is afforded by the constant-coefficient equation

$$\begin{aligned}
& y(r+8) - (4+4\sqrt{2})y(r+7) + (15+16\sqrt{2})y(r+6) - (48+12\sqrt{2})y(r+5) \\
& - 12y(r+4) + (192+64\sqrt{2})y(r+3) - (208+256\sqrt{2})y(r+2) + (256+192\sqrt{2})y(r+1) \\
& - 192y(r) = 0 .
\end{aligned}$$

Its characteristic polynomial, in real factored form, is

$$(2.1.2) \quad (\xi-1)(\xi-2)(\xi+2)(\xi^2-2\sqrt{2}\xi+4)^2(\xi-3) = 0 .$$

The roots of (2.1.2) include the complex numbers $\sqrt{2}(1+i)$ and $\sqrt{2}(1-i)$, each of multiplicity two. The subsets of the linearly ordered partition in this example are

$$\begin{aligned}
\Lambda_{1,0} &= \{1^r\} \\
\Lambda_{2,0} &= \{2^r, (-2)^r, \sqrt{2}^r(1-i)^r, \sqrt{2}^r(1+i)^r\} \\
\Lambda_{2,1} &= \{r\sqrt{2}^r(1-i)^r, r\sqrt{2}^r(1+i)^r\} \\
\Lambda_{3,0} &= \{3^r\} .
\end{aligned}$$

Note that if $a \in \Lambda_{2,0}$ and $b \in \Lambda_{2,1}$ then $|a(r)/b(r)| = r^{-1}$ when $r \geq 1$. In contrast, if $a \in \Lambda_s$ and $b \in \Lambda_{s'}$, where $s < s'$ then $|a(r)/b(r)|$ contains a decreasing exponential factor.

In general, if $a, b \in \Lambda$ are such that $a < b$, then the separation ratios $|a(r)/b(r)|$ decrease as the product of a power of r^{-1} and an exponential function in r . If the exponential function is identically equal to one, as will be the case whenever a and b are standard basis solutions which correspond to characteristic roots of equal magnitude, then we shall say that a and b are algebraically separated. Otherwise, we shall say that a and b are exponentially separated.

Figure 5 is obtained by plotting the roots of (2.1.2) in the complex plane and drawing circles through them, centered at the origin. There are three circles, corresponding to the distinct absolute values of the roots.

The double roots are indicated on the circle of radius two; it is helpful to imagine them duplicated on a second circle of radius two that is raised slightly out of the plane. Let us number the four circles, thus arrived at, by increasing three-dimensional distance from the origin. This determines geometrically the proper linear ordering of the four subsets $\Lambda_{s,m}$ of the partition. The ordering of elements within any one of the subsets is unimportant for our purposes.

Clearly, this geometric indication of the linear ordering of the subsets of the partition can be extended to any constant-coefficient operator.

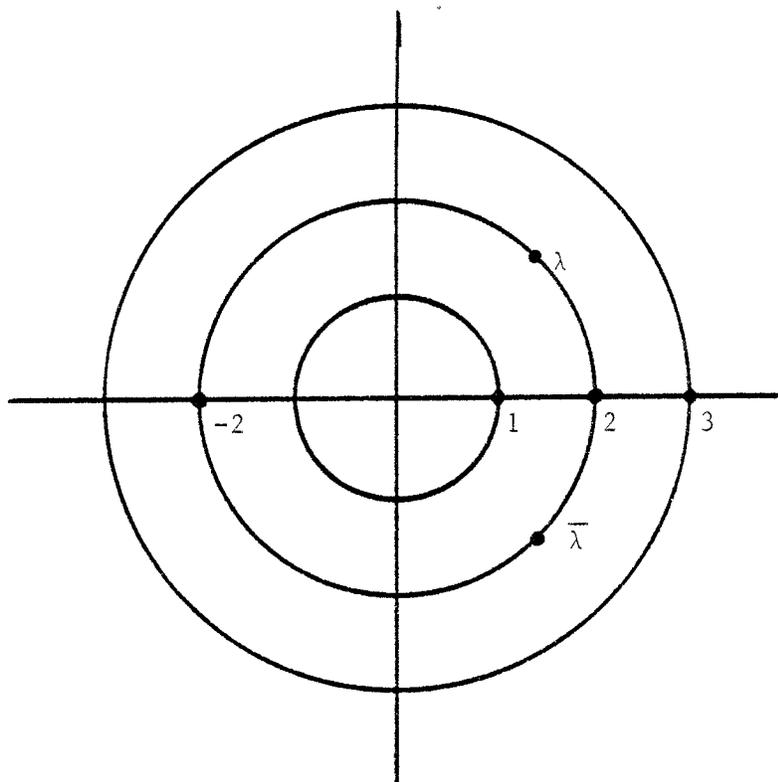


Figure 5. The roots of (2.1.2). The double roots $\sqrt{2}(1 \pm i)$ are denoted $\lambda, \bar{\lambda}$.

Theorem 2.1.1 Let (2.1.1) denote the linearly ordered partition of the standard basis for a given constant-coefficient operator. Then:

- i) If $a \in \wedge_{s,m}$ and $b \in \wedge_{s',m'}$, then $a < b$ if, and only if, $\wedge_{s,m} < \wedge_{s',m'}$.
- ii) If $a, b \in \wedge_{s,m}$, then $|a(r)| = |b(r)|$ for all r .
- iii) If a_1, a_2, \dots, a_q , $b \in \wedge_{s,m}$ where the a 's are distinct, and $\alpha_1, \alpha_2, \dots, \alpha_q \in F$ are all nonzero, then

$$(2.1.3) \quad \limsup_{r \rightarrow \infty} \left| \sum_{p=1}^q \alpha_p a_p(r)/b(r) \right| > 0 .$$

Proof: Parts (i) and (ii) are trivial consequences of our definitions. Turning to part (iii), we note that the sequence $\sum_{p=1}^q \alpha_p a_p(r)/b(r)$ is not identically zero (since the a 's are linearly independent). Let $\lambda_1, \lambda_2, \dots, \lambda_q, \lambda$ be the characteristic roots associated with the set $\wedge_{s,m}$ such that

$$a_p(r) = r^m \lambda_p^r, \quad p = 1, 2, \dots, q$$

and $b(r) = r^m \lambda^r$. Since the λ 's lie on the same circle in the complex plane, there exist real constants $\beta_1, \beta_2, \dots, \beta_q, \beta$ all lying in the half-open real interval $[-\pi, \pi)$ such that

$$a_p(r) = r^m |\lambda|^r e^{ir\beta_p}, \quad p = 1, 2, \dots, q$$

and $b(r) = r^m |\lambda|^r \exp(ir\beta)$, $i = \sqrt{-1}$. Thus

$$\sum_{p=1}^q \alpha_p a_p(r)/b(r) = \sum_{p=1}^q \alpha_p e^{ir\gamma_p}$$

where $\gamma_p = \beta_p - \beta$. The γ 's are distinct real numbers (modulo 2π)

because the sequences a_1, a_2, \dots, a_q are assumed distinct.

The right member of the last equation represents a so-called almost periodic function of r when r is regarded as a continuous real variable. To complete the proof we appeal to the following lemma, the proof of which is given in the Appendix.[†]

Lemma 2.1.1 Let $\alpha_1, \alpha_2, \dots, \alpha_q$ be nonzero complex numbers and $\gamma_1, \gamma_2, \dots, \gamma_q$ distinct real numbers (modulo 2π). Define the infinite sequence

$$(2.1.4) \quad x(r) = \sum_{p=1}^q \alpha_p e^{ir\gamma_p}, \quad r = 0, 1, 2, \dots$$

Then $\limsup_{r \rightarrow \infty} |x(r)| > 0$. \square

An equivalent statement to inequality (2.1.3) is that there exists an infinite subsequence of the sequence

$$\sum_{p=1}^q \alpha_p a_p(r)/b(r), \quad r = 0, 1, 2, \dots$$

which is bounded away from zero. Intuitively, this means that it is impossible to create a new sequence with qualitatively different asymptotic behavior merely by forming a linear combination of standard basis solutions of similar asymptotic behavior. Note, however, that unlike the original sequences infinitely many zeros may occur in the new sequence. For example, if 2^r and $(-2)^r$ are sequences in a standard basis, then every other term of the sequence $2^r + (-2)^r$ is zero.

[†]We relegate the proof of Lemma 2.1.1 to the Appendix because the theory of almost periodic functions is not germane to the rest of this thesis.

2.2 Separable Operators

In this section we introduce a subclass of the general set of infinite upper-triangular band matrices \mathcal{D}_ℓ introduced in §1.2, and examine in detail the general structure of the kernel of an operator in this subclass. This will prepare the way for the presentation, in the next section, of a general classification of all sequences in S with respect to a given linear difference operator. In Theorem 2.1.1 we presented three properties of the standard basis of a constant-coefficient operator. These properties suggest the following extension of Definition 1.3.1:

Definition 2.2.1 Let $D \in \mathcal{D}_\ell$ be such that $K(D)$ has dimension ℓ .

Then D will be said to be separable if there exists a basis X for D which can be partitioned into nonempty disjoint subsets $X_1, X_2, \dots, X_\sigma$ in such a way that the following three conditions are all satisfied:

i) If $x \in X_s$ and $y \in X_t$, where $s < t$, then $\lim_{r \rightarrow \infty} x(r)/y(r) = 0$;

ii) If $x, y \in X_s$ for some s , then $0 < \liminf_{r \rightarrow \infty} |x(r)/y(r)|$

and $\limsup_{r \rightarrow \infty} |x(r)/y(r)| < \infty$;

iii) If x is any linear combination of sequences from X_s , other than $x = 0$, and $y \in X_s$, then $\limsup_{r \rightarrow \infty} |x(r)/y(r)| > 0$.

Let D be a separable operator. A basis X for D which is such that

$$(2.2.1) \quad X = \bigcup_{s=1}^{\sigma} X_s,$$

where the X_s satisfy the conditions of Definition 2.2.1 will be said to be optimally ranked. Let

$$(2.2.2) \quad X_s = \{x_{s,1}, x_{s,2}, \dots, x_{s,n_s}\},$$

where the subscript n_s is positive (because X_s is nonempty).

Obviously, we have

$$(2.2.3) \quad \ell = \sum_{s=1}^{\sigma} n_s .$$

In Lemma 2.2.1 below we will show that two distinct optimally ranked bases for a given separable operator have the same number (σ) of disjoint subsets and the same number (n_s) of sequences in corresponding subsets.

An optimally ranked basis exhibits the full range of possible asymptotic behavior of solutions to the corresponding homogeneous difference equation. Conditions (i) and (ii) of Definition 2.2.1 establish the ranking of sequences within the basis. If (2.2.1) is a particular optimally ranked basis, we shall say that $x \in X$ is a sequence of type s provided that $x \in X_s$. More generally, we shall say that $x \in K(D)$ is a sequence of type s provided that $x \neq 0$ and there exist scalars $\alpha_{p,q} \in F$ such that

$$(2.2.4) \quad x = \sum_{p=1}^s \sum_{q=1}^{n_p} \alpha_{p,q} x_{p,q} ,$$

where $\alpha_{s,q} \neq 0$ for some q , $1 \leq q \leq n_s$. Such a representation of x exists and is unique for every nonzero $x \in K(D)$. Again, in consequence of Lemma 2.2.1 below, if we change to a different optimally ranked basis, the type of every sequence in $K(D)$ remains invariant. Condition (iii) of Definition 2.2.1 presages this; it requires that no nontrivial linear combination of basis sequences of the same type may be dominated by a basis sequence of that type.

Let $L(Y)$ denote the linear subspace of S which is spanned by Y where $Y \subset S$ may be any finite subset of sequences. If (2.2.1) is an optimally ranked basis for a separable operator D , then we shall say that the subspace $L(\bigcup_{p=1}^s X_p)$ is the subdominant subspace of type s for D .

These subspaces are independent of the basis; again, see Lemma 2.2.1 below. If j_s denotes the dimension of $L(\bigcup_{p=1}^s X_p)$, then

$$(2.2.5) \quad j_s = \sum_{p=1}^s n_p ;$$

compare (2.2.2) and (2.2.3). In addition, we shall say that $\{0\}$, the linear subspace of dimension zero (whose only sequence is the zero sequence) is the subdominant subspace of type 0.

Let U be one of the subdominant subspaces for D . The complementary subspace of U in $K(D)$, i.e., the subspace V of $K(D)$ such that

$$(2.2.6) \quad K(D) = U \oplus V ,$$

will be called the corresponding dominant subspace. The symbol \oplus on the right side of (2.2.6) means that $K(D)$ is the direct sum of the subspaces U and V ; that is, every nonzero sequence $x \in K(D)$ is uniquely expressible in one of the forms $x = u$, or $x = v$, or $x = u+v$, where $u \in U$ and $v \in V$. Clearly, if U is the subdominant subspace of type s , $0 \leq s < \sigma$, then $V = L(\bigcup_{p=s+1}^{\sigma} X_p)$; and if U is the subdominant subspace of type σ , then $V = \{0\}$.

Before stating and proving Lemma 2.2.1, we note that

$$L(\bigcup_{p=1}^s X_p) = L(X_1) \oplus L(X_2) \oplus \cdots \oplus L(X_s)$$

Thus, an alternative characterization of a sequence of type s in $K(D)$ is the following: $x \in K(D)$ is of type s , $1 \leq s \leq \sigma$, provided that

$$x = \sum_{p=1}^s x_p , \quad x_p \in L(X_p) , \quad x_s \neq 0 ;$$

compare (2.2.4).

Lemma 2.2.1 Let

$$X = \bigcup_{s=1}^{\sigma} X_s = \bigcup_{s=1}^{\sigma} \{x_{s,1}, x_{s,2}, \dots, x_{s,m_s}\}$$

and

$$Y = \bigcup_{s=1}^{\tau} Y_s = \bigcup_{s=1}^{\tau} \{y_{s,1}, y_{s,2}, \dots, y_{s,n_s}\}$$

be distinct optimally ranked bases for a separable operator $D \in \mathcal{D}_\ell$.

Then $\sigma = \tau$ and $m_s = n_s$ for all s . Furthermore,

$$L\left(\bigcup_{p=1}^s X_p\right) = L\left(\bigcup_{p=1}^s Y_p\right), \quad s = 1, 2, \dots, \sigma,$$

and the type of every nonzero sequence $z \in K(D)$ is the same with respect to X as it is with respect to Y .

Proof Let $u \in X_s$ and $v \in X_t$, so that relative to X the types of u and v are s and t . Let the types of u and v relative to Y be s' and t' . Then there exist unique representations

$$u = \sum_{j=1}^{s'} u_j, \quad u_j \in L(Y_j), \quad u_{s'} \neq 0$$

and

$$v = \sum_{j=1}^{t'} v_j, \quad v_j \in L(Y_j), \quad v_{t'} \neq 0.$$

Furthermore, by condition (i) of Definition 2.2.1, we have, as $r \rightarrow \infty$,

$$(2.2.7) \quad u(r) = u_{s'}(r) + o\{y(r)\}, \quad y \in Y_{s'},$$

and

$$(2.2.8) \quad v(r) = v_{t'}(r) + o\{y(r)\} \quad , \quad y \in Y_{t'} \quad .$$

i) First we prove that $s' = t'$ if $s = t$, i.e., if u and v are of the same type with respect to X , then they are also of the same type with respect to Y .

Since $s = t$, the sequence $u(r)/v(r)$ is asymptotically bounded away from zero; compare part (ii) of Definition 2.2.1. Suppose that $s' < t'$. We shall obtain a contradiction by showing that, under this assumption, $u(r)/v(r)$ has an infinite subsequence which converges to zero. Choose any $y \in Y_{t'}$. Then we have

$$\lim_{r \rightarrow \infty} \frac{u(r)}{y(r)} = 0 \quad .$$

From (2.2.8), we have

$$\frac{v(r)}{y(r)} = \frac{v_{t'}(r)}{y(r)} + o(1)$$

and, using the triangle inequality, an elementary property of the limit superior, and condition (iii) of Definition 2.2.1, we obtain

$$\limsup_{r \rightarrow \infty} \left| \frac{v(r)}{y(r)} \right| = \limsup_{r \rightarrow \infty} \left| \frac{v_{t'}(r)}{y(r)} \right| > 0 \quad .$$

Thus there exists an infinite subsequence of $v(r)/y(r)$ that is bounded away from zero, and therefore also a subsequence of

$$\frac{u(r)}{v(r)} = \frac{u(r)/y(r)}{v(r)/y(r)}$$

that converges to zero. This supplies the needed contradiction.

A similar contradiction is obtained if we suppose that $s' > t'$. Hence $s' = t'$.

ii) Next we prove that $s' < t'$ if $s < t$.

Since $s < t$, the sequence $u(r)/v(r)$ converges to zero. Suppose that $s' \geq t'$. We shall derive a contradiction by showing that, under this assumption, $u(r)/v(r)$ has a subsequence which does not converge to zero. Choose any $y \in Y_s$. First, by using (2.2.7) and an argument similar to that used in part (i) of this proof, we have

$$\limsup_{r \rightarrow \infty} \left| \frac{u(r)}{y(r)} \right| > 0.$$

Thus there exists an infinite subsequence of $u(r)/y(r)$ that is bounded away from zero.

Assume now that $s' > t'$. Then

$$\lim_{r \rightarrow \infty} \frac{v(r)}{y(r)} = 0.$$

In this case it follows that there exists a subsequence of

$$\frac{u(r)}{v(r)} = \frac{u(r)/y(r)}{v(r)/y(r)}$$

that diverges to ∞ , which supplies the needed contradiction.

Alternatively, assume that $s' = t'$. Using (2.2.8), we have for sufficiently large r

$$\left| \frac{v(r)}{y(r)} \right| \leq \left| \frac{v_{t'}(r)}{y(r)} \right| + 1.$$

Furthermore, using condition (ii) of Definition 2.2.1, we see that there exist $r_0 \geq 0$ and $\beta > 0$ such that

$$\left| \frac{v_{t'}(r)}{y(r)} \right| \leq \beta, \quad r \geq r_0.$$

Therefore, for sufficiently large r , we have

$$\left| \frac{v(r)}{y(r)} \right| \leq \beta + 1$$

and

$$\left| \frac{u(r)}{v(r)} \right| \geq (1+\beta)^{-1} \left| \frac{u(r)}{y(r)} \right|.$$

It follows that $u(r)/v(r)$ has a subsequence that is bounded away from zero, which again supplies the needed contradiction.

iii) Next, we prove that $\sigma = \tau$ and $m_s = n_s$ for all s , and also that $L(\bigcup_{p=1}^s X_p) = L(\bigcup_{p=1}^s Y_p)$ for all s .

Part (i) of this proof implies that for each s , $1 \leq s \leq \sigma$, the type relative to Y of every sequence in X_s is the same, say t_s . Using part (ii), we have

$$(2.2.9) \quad 1 \leq t_1 < t_2 < \cdots < t_\sigma \leq \tau.$$

Furthermore, that $s \leq t_s$ for all s , $1 \leq s \leq \sigma$, is easily proved by induction. In particular, $\sigma \leq t_\sigma$ and, using (2.2.9), we see that $\sigma \leq \tau$.

Inverting the roles of X and Y and repeating the argument, we see that $\tau \leq \sigma$. Thus $\sigma = \tau$, and (2.2.9) implies $t_s = s$ for all s .

For each s , it follows that

$$L(X_1 U \cdots U X_s) \subset L(Y_1 U \cdots U Y_s),$$

and by symmetry,

$$L(Y_1 U \cdots U Y_s) \subset L(X_1 U \cdots U X_s) .$$

Thus

$$L(X_1 U \cdots U X_s) = L(Y_1 U \cdots U Y_s)$$

for each s . In particular, $L(X_1) = L(Y_1)$. Since X_1 and Y_1 are bases of the same linear space, we have $m_1 = n_1$. An easy induction completes the proof of this part.

iv) To complete the proof, let $z \in K(D)$ be a nonzero sequence such that

$$z = \sum_{p=1}^s u_p , \quad u_s \neq 0 , \quad u_p \in L(X_p) ,$$

and

$$z = \sum_{p=1}^t v_p , \quad v_t \neq 0 , \quad v_p \in L(Y_p) .$$

Assume $s < t$, and let $y \in Y_t$. Then

$$\lim_{r \rightarrow \infty} \frac{v_t(r)}{y(r)} = \lim_{r \rightarrow \infty} \left[\sum_{p=1}^s \frac{u_p(r) - v_p(r)}{y(r)} - \sum_{p=s+1}^{t-1} \frac{v_p(r)}{y(r)} \right] = 0 .$$

But this result contradicts condition (iii) of Definition 2.2.1. Hence we must have $s \geq t$. A similar contradiction is obtained if we assume $s > t$. Therefore $s = t$ and the proof of Lemma 2.2.1 is complete. \square

In general, the actual determination of optimally ranked bases for variable coefficient difference operators is a difficult problem. Some results have been given by Wimp [26]. This reference extends earlier work by Birkhoff and Trjitzinsky [2, 3] on the analytic theory of singular difference equations whose coefficients possess asymptotic expansions

of a prescribed form. All equations with coefficients rational in r are included, for example. Wimp's analysis provides a means of constructing a basis, which he calls a canonical set; see [26, Theorem 3.3 and Definition 3.6]. The canonical set is analogous to our optimally ranked basis. More specifically, there exists a basis $\{y_1, y_2, \dots, y_\ell\}$ such that, for each $s = 1, 2, \dots, \ell$,

$$y_s(r) = c_s M_s(r) \{1 + o(1)\} \quad \text{as } r \rightarrow \infty$$

where $c_s \neq 0$ and

$$M_s(r) = e^{Q_s(r)} r^{\theta_s} (\ln r)^{p_s}.$$

Here θ_s is a complex number and p_s is a positive integer. Q_s is of the form

$$Q_s(r) = \mu_0 r \ln r + \mu_1 r + \mu_2 r^{1-1/\rho} + \dots + \mu_\rho r^{1/\rho}$$

where ρ is an integer, $\rho \geq 1$, and $\mu_0, \mu_1, \dots, \mu_\rho$ are complex numbers. The rich variety of possible separation ratios of solutions as $r \rightarrow \infty$, compared with the algebraic or exponential separation of solutions in the constant-coefficient case, described in §2.1 above, is evident.

2.3 The General Classification

In §2.2 we showed that every nonzero sequence in the kernel of a separable operator has a unique type with respect to that particular operator. In this section we extend this classification to every nonzero sequence in S . The following definition should be compared with Definition 1.3.2.

Definition 2.3.1 Let $D \in \mathcal{D}_\ell$ be separable with optimally ranked basis $X = \bigcup_{s=1}^{\sigma} X_s$. Then, for each $y \in S$, we say that $\text{type}(y) = s$, or more fully, the type of y with respect to D is s , provided that $0 < s < \sigma$ and the following two conditions are satisfied:

- i) If $x \in X_{s+1}$ then $\lim_{r \rightarrow \infty} \frac{y(r)}{x(r)} = 0$;
- ii) If $x \in X_s$ then $\infty \geq \limsup_{r \rightarrow \infty} \left| \frac{y(r)}{x(r)} \right| > 0$.

If (i) is satisfied for $s = 0$, then (ii) does not apply, and we define $\text{type}(y) = 0$. Similarly, if (ii) is satisfied for $s = \sigma$, (i) does not apply, and we define $\text{type}(y) = \sigma$.

The classification of sequences introduced in Definition 2.3.1 allows us to treat both homogeneous and inhomogeneous difference equations in a uniform manner when the difference operator is separable. The kernel of the operator contains every solution of the homogeneous equation, by definition. But solutions of the inhomogeneous equation with arbitrary right-hand side may lie anywhere in S . This is why a classification of all sequences in S is desirable.

Theorem 2.3.1 Every sequence in S possesses an unambiguous type with respect to each separable operator in \mathcal{D}_ℓ .

Proof (i) First we show that the type with respect to a fixed optimally

ranked basis is determined uniquely by Definition 2.3.1 for every nonzero sequence in S .

Let $z \in S$. Assuming that

$$(2.3.1) \quad \limsup_{r \rightarrow \infty} \left| \frac{z(r)}{x(r)} \right| > 0$$

for at least one sequence $x \in X$, let $\hat{x} \in X$ be a sequence of largest type such that (2.3.1) is satisfied. Let $s = \text{type}(\hat{x})$. Then (2.3.1) is satisfied for every $x \in X_s$, since

$$\left| \frac{z(r)}{x(r)} \right| = \left| \frac{z(r)}{\hat{x}(r)} \right| \left| \frac{\hat{x}(r)}{x(r)} \right|$$

and $\liminf_{r \rightarrow \infty} |\hat{x}(r)/x(r)| > 0$ as a consequence of condition (ii) of Definition 2.2.1.

If $s = \sigma$, then $\text{type}(z) = \sigma$ according to Definition 2.3.1. Therefore suppose that $s < \sigma$. By the maximality of \hat{x} we see that $\limsup_{r \rightarrow \infty} |z(r)/x(r)| = 0$ whenever $\text{type}(x) > s$ or, equivalently,

$$(2.3.2) \quad \lim_{r \rightarrow \infty} \frac{z(r)}{x(r)} = 0 \quad \text{whenever} \quad \text{type}(x) > s.$$

Accordingly, $\text{type}(z) = s$.

To conclude the proof of part (i), let us suppose that (2.3.1) is not satisfied for any $x \in X$. Then (2.3.2) is satisfied for every $x \in X$, and in particular for every $x \in X_1$. Then $\text{type}(z) = 0$.

(ii) In this part we prove that if $X = \bigcup_{s=1}^{\sigma} X_s$ and $X' = \bigcup_{s=1}^{\sigma} X'_s$ are two distinct optimally ranked bases for D then the type of every sequence in S is the same with respect to X and X' .

Choose any s , $1 \leq s \leq \sigma$, and choose $x \in X_s$, $x' \in X'_s$. There

exist $x_p \in X_p$, $1 \leq p \leq s$, such that $x' = x_1 + \dots + x_s$; see Lemma 2.2.1.

Furthermore, we have

$$\frac{x'(r)}{x(r)} = o(1) + \frac{x_s(r)}{x(r)} \quad \text{as } r \rightarrow \infty .$$

Since x_s is a linear combination of sequences of type s , a brief computation and application of condition (ii) of Definition 2.2.1 yields

$$\limsup_{r \rightarrow \infty} \left| \frac{x_s(r)}{x(r)} \right| < \infty .$$

It follows that

$$(2.3.3) \quad \limsup_{r \rightarrow \infty} \left| \frac{x'(r)}{x(r)} \right| < \infty$$

$$\text{and } \liminf_{r \rightarrow \infty} \left| \frac{x(r)}{x'(r)} \right| > 0 \text{ whenever } \text{type}(x') = \text{type}(x) .$$

Since the roles of x and x' are interchangeable, we also have

$$(2.3.4) \quad \limsup_{r \rightarrow \infty} \left| \frac{x(r)}{x'(r)} \right| < \infty$$

$$\text{and } \liminf_{r \rightarrow \infty} \left| \frac{x'(r)}{x(r)} \right| > 0 \text{ whenever } \text{type}(x') = \text{type}(x) .$$

Let $z \in S$ and suppose $\text{type}(z) = 0$ with respect to X . Thus $z(r)/x(r) \rightarrow 0$ for every $x \in X$. Choose $x' \in X'$. Let $s = \text{type}(x')$ and choose $x \in X_s$. Then

$$\left| \frac{z(r)}{x'(r)} \right| = \left| \frac{z(r)}{x(r)} \right| \left| \frac{x(r)}{x'(r)} \right| \rightarrow 0 \text{ as } r \rightarrow \infty$$

by virtue of (2.3.4). This proves $\text{type}(z) = 0$ with respect to X' .

Let $z \in S$ and suppose $\text{type}(z) = \sigma$ with respect to X . Thus $\limsup_{r \rightarrow \infty} |z(r)/x(r)| > 0$ for every $x \in X$. Choose $x' \in X'$. Let $s = \text{type}(x')$ and choose $x \in X_s$. Then the sequence

$$\left| \frac{z(r)}{x'(r)} \right| = \left| \frac{z(r)}{x(r)} \right| \left| \frac{x(r)}{x'(r)} \right|$$

satisfies $\limsup_{r \rightarrow \infty} |z(r)/x'(r)| > 0$ by virtue of (2.3.3). This proves $\text{type}(z) = \sigma$ with respect to X' .

Finally, let $z \in S$ and suppose $\text{type}(z) = s$, $0 < s < \sigma$. Thus $z(r)/x(r) \rightarrow 0$ for every $x \in X_{s+1} \cup \dots \cup X_\sigma$ and $\limsup_{r \rightarrow \infty} |z(r)/x(r)| > 0$ for every $x \in X_s$. It can be verified that $z(r)/x'(r) \rightarrow 0$ for every $x' \in X'_{s+1} \cup \dots \cup X'_\sigma$ and that $\limsup_{r \rightarrow \infty} |z(r)/x'(r)| > 0$ for every $x' \in X'_s$ using arguments similar to the ones used in the preceding two paragraphs. \square

CHAPTER 3. THE GENERAL ALGORITHM

3.0 Preliminaries and Overview

In this chapter we fix our attention on a particular linear difference equation

$$(3.0.1) \quad Dy(r) = g(r) , \quad r = 0,1,2\dots$$

assuming that $D \in \mathcal{D}_\ell$ is separable and $g \in S$. More specifically, we are interested in computing an approximation y_n of a particular solution y of (3.0.1) which is valid over some finite subsequence $y^{i,m}$, i and m given.

Let σ be the number of distinct types in any optimally ranked basis for D . We assume that y is known to be a sequence of type t with respect to D , where $0 \leq t \leq \sigma$. We also assume that initial values $y(i), y(i+1), \dots, y(i+j-1)$ are known, where j is the dimension of the subdominant subspace of type t for D . The extension of Theorem 1.3.1, from totally separable to separable operators, is given in §3.1. Accordingly, these initial values (in the presence of one additional condition) suffice to determine the solution uniquely.

Let $k = \ell - j$. The approximating sequence y_n is defined as the solution of (3.0.1) which satisfies the conditions

$$(3.0.2) \quad y_n(i+r) = y(i+r) , \quad r = 0,1,\dots,j-1$$

and

$$(3.0.3) \quad y_n(n+r) = 0 , \quad r = 0,1,\dots,k-1 .$$

Since $y^{i,m}$ is to be approximated by $y_n^{i,m}$, it is clear that the value

of n must exceed m . The existence and uniqueness of the approximating sequences y_n , for all sufficiently large n , and the convergence of the sequence of approximate values $y_n(r)$ to $y(r)$ for each value of r in the range $i + j \leq r < \infty$, are proved under appropriate conditions in §3.2.

The general algorithm, which is valid for any separable operator, reduces to the algorithm presented in §1.3 for totally separable operators. We again assume that the separable operator is (i,j) -factorizable, just as we did for totally separable operators. The linear system (1.3.5) is solved by the factorization method by means of a forward elimination stage followed by a back substitution stage; see §1.3. The expansions (1.3.19) and (1.3.20), which are used to estimate the optimal value of n , are unchanged for separable operators. In §3.3 we discuss the stability of the forward elimination stage, which was omitted in §1.3. We also extend Theorem 1.3.2 to separable operators, showing that the back substitution is stable. Finally, we note that our remarks on the stability of the forward recurrence solution of the adjoint equation, whose solution enters in the truncation error expansion of §1.3, apply to separable operators as well as totally separable operators.

We conclude this introductory section by fixing some notation for later use in this chapter. Let

$$(3.0.4) \quad X = \bigcup_{s=1}^{\sigma} X_s$$

be an optimally ranked basis for D , where

$$(3.0.5) \quad X_s = \{x_{s,1}, x_{s,2}, \dots, x_{s,n_s}\}, \quad s = 1, 2, \dots, \sigma.$$

Then the dimension of the subdominant subspace of type t for D is

$$(3.0.6) \quad j = \sum_{s=1}^t n_s .$$

Define the set

$$(3.0.7) \quad U = \{u_1, u_2, \dots, u_j\} = \bigcup_{s=1}^t X_s$$

by means of the correspondence

$$x_{p,q} = u_{n_1+n_2+\dots+n_{p-1}+q} .$$

When $t = 0$ it is understood that $j = 0$ and U is empty. Similarly, define the set

$$(3.0.8) \quad V = \{v_1, v_2, \dots, v_k\} = \bigcup_{s=t+1}^{\sigma} X_s ;$$

when $t = \sigma$ it is understood that V is empty.

3.1 Existence and Uniqueness

Let y be a solution of (3.0.1) of type t such that

$$(3.1.1) \quad y(i+r) = \gamma_r, \quad r = 0, 1, \dots, j-1$$

for some $i \geq 0$, where j is given by (3.0.6). Using the notation $|U|(i)$ for the Casoratian of U at i , we also suppose that

$$(3.1.2) \quad |U|(i) \neq 0;$$

see (3.0.7). Note that for every i (3.1.2) is satisfied vacuously if $t = 0$, and also (by virtue of Casorati's theorem [14, §12.11]) if $t = \sigma$ and D is nonsingular. The following theorem generalizes Theorem 1.3.1.

Theorem 3.1.1 If (3.1.2) is valid then there exists a unique solution of (3.0.1) of type t such that (3.1.1) is satisfied.

Proof: Suppose $t = 0$. Then $j = 0$ and we must prove that without any specified initial values y is uniquely determined. Let $v \in S$ be a solution of (3.0.1), other than y , such that $\text{type}(v) = 0$. Since $y - v \in K(D)$, there exist scalars $\alpha_{p,q} \in F$ such that

$$y - v = \sum_{p=1}^{\sigma} \sum_{q=1}^{n_p} \alpha_{p,q} x_{p,q};$$

see (3.0.5). At least one of the $\alpha_{p,q}$ is nonzero; let the largest value of p such that $\alpha_{p,q} \neq 0$ for some q , $1 \leq q \leq n_p$, be $p = s$.

Then

$$\lim_{r \rightarrow \infty} \frac{y(r) - v(r)}{x_{s,1}(r)} = 0$$

because $\text{type}(y) = \text{type}(v) = 0$. By part (i) of Definition 2.2.1, we

have

$$\lim_{r \rightarrow \infty} \sum_{p=1}^{s-1} \sum_{q=1}^n \frac{\alpha_{p,q} x_{p,q}(r)}{x_{s,1}(r)} = 0 .$$

It follows from the above three equations that

$$\lim_{r \rightarrow \infty} \sum_{q=1}^n \frac{\alpha_{s,q} x_{s,q}(r)}{x_{s,1}(r)} = 0 .$$

But this result contradicts part (iii) of Definition 2.2.1; therefore we conclude that $y = v$.

Alternatively suppose $t > 0$. Let $v \in S$ be a sequence of type t which satisfies (3.0.1) and is coincident with $y(i+r)$ for $r = 0, 1, \dots, j-1$. Then there exist scalars $\alpha_{p,q} \in F$ such that

$$v = y + \sum_{p=1}^{\sigma} \sum_{q=1}^n \alpha_{p,q} x_{p,q} .$$

Let the largest value of p in this sum such that at least one of the scalars $\alpha_{p,q}$ is nonzero be $p = s$. Then $s \leq t$, because $s > t$ would imply

$$\lim_{r \rightarrow \infty} \frac{v(r) - y(r)}{x_{s,1}(r)} = 0$$

(since $\text{type}(v) = \text{type}(y) = t$) and

$$\lim_{r \rightarrow \infty} \sum_{p=1}^{s-1} \sum_{q=1}^n \frac{\alpha_{p,q} x_{p,q}(r)}{x_{s,1}(r)} = 0$$

(by part (i) of Definition 2.2.1). This gives

$$\lim_{r \rightarrow \infty} \sum_{q=1}^n \frac{\alpha_{s,q} x_{s,q}^{(r)}}{x_{s,1}^{(r)}} = 0 ,$$

which is a contradiction.

Thus, $\alpha_{p,q} = 0$ whenever $p > s$. The $\alpha_{p,q}$ having $p \leq s$ satisfy the system of linear equations

$$\sum_{p=1}^s \sum_{q=1}^n \alpha_{p,q} x_{p,q}^{(i+r)} = 0 , \quad r = 0, 1, \dots, j-1 .$$

This system is nonsingular by the condition $|U|(i) \neq 0$, so we have

$\alpha_{p,q} = 0$ for all p and q . Thus $v(r) = y(r)$ for all r and the theorem is proved. \square

Theorem 3.1.2 If D is nonsingular then, for each $i \geq 0$, $|U|(i+r) \neq 0$ for at least one value of r in the range $0 \leq r \leq \ell - j$.

Proof: Since D is nonsingular, Casorati's theorem is valid, i.e.,

$|X|(i) \neq 0$ for all i . Theorem 3.1.2 is then an immediate consequence

of Laplace's general theorem on the expansion of determinants [15, §93]. \square

Theorem 3.1.2 implies there is no shortage of points i which satisfy condition (3.1.2) when the difference operator is nonsingular. Our next goal is to show that any optimally ranked basis may be used to locate an admissible value of i .

Theorem 3.1.3 If $|U|(i) \neq 0$ for some fixed $i \geq 0$, then the leading principal minor of order j of the Casoratian evaluated at i of every optimally ranked basis for D is nonzero.

Proof: If $j = 0$ then there is nothing to prove, so let us assume $j > 0$.

Also, let us assume temporarily that $j < \ell$. Let X' be an optimally ranked basis distinct from X , where

$$X' = \bigcup_{s=1}^{\sigma} X'_s$$

and

$$X'_s = \{x'_{s,1}, x'_{s,2}, \dots, x'_{s,n_s}\}, \quad s = 1, 2, \dots, \sigma;$$

compare (3.0.4), (3.0.5). Define the sets

$$U' = \{u'_1, u'_2, \dots, u'_j\} = \bigcup_{s=1}^t X'_s$$

and

$$V' = \{v'_1, v'_2, \dots, v'_k\} = \bigcup_{s=t+1}^{\sigma} X'_s.$$

There exist scalars $\alpha_{p,q} \in F$ such that

$$u'_p = \sum_{q=1}^j \alpha_{p,q} u'_q + \sum_{q=j+1}^{\ell} \alpha_{p,q} v'_{q-j}, \quad p = 1, 2, \dots, j,$$

and

$$v'_{p-j} = \sum_{q=1}^j \alpha_{p,q} u'_q + \sum_{q=j+1}^{\ell} \alpha_{p,q} v'_{q-j}, \quad p = j+1, j+2, \dots, \ell.$$

The matrix of coefficients $(\alpha_{p,q})$ is nonsingular because both X and X' are bases of the finite-dimensional linear space $K(D)$. Furthermore, from the definition of an optimally ranked basis it is easy to show that $(\alpha_{p,q})$ is block lower triangular with the s -th diagonal block having order n_s . Because of the block triangular structure, the determinant of $(\alpha_{p,q})$ is equal to the product of the determinants of the diagonal blocks. And because the determinant of $(\alpha_{p,q})$ is nonzero, so is the determinant of every diagonal block. Therefore, the leading principal minor of order j of the determinant of $(\alpha_{p,q})$ is nonzero. Let Δ_j denote this minor.

Next, consider the Casorati matrix $[X](i)$. It can be written in block form as

$$[X](i) = \begin{bmatrix} [U](i) & \vdots & * \\ \cdots & \cdots & \cdots \\ * & \vdots & [V](i+j) \end{bmatrix}$$

where the actual form of the blocks indicated by asterisks is unimportant.

Similarly, we write

$$[X'](i) = \begin{bmatrix} [U'](i) & \vdots & * \\ \cdots & \cdots & \cdots \\ * & \vdots & [V'](i+j) \end{bmatrix} .$$

If

$$(\alpha_{p,q}) = \begin{bmatrix} A_{11} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix}$$

is the corresponding block form of $(\alpha_{p,q})$, we verify readily that

$$[X'](i) = \begin{bmatrix} [U](i) & \vdots & * \\ \cdots & \cdots & \cdots \\ * & \vdots & [V](i+j) \end{bmatrix} \begin{bmatrix} A_{11}^{Tr} & \vdots & A_{21}^{Tr} \\ \cdots & \cdots & \cdots \\ 0 & \vdots & A_{22}^{Tr} \end{bmatrix} .$$

The leading principal minor of order j of $|X'|(i)$ is evidently given by

$$|U'|(i) = |U|(i) \cdot \det(A_{11}) .$$

Since $|U|(i) \neq 0$, by hypothesis, and $\det(A_{11}) = \Delta_j \neq 0$ we have

$|U'|(i) \neq 0$. This completes the proof for $j < \ell$.

Finally, we note that the proof in the case $j = \ell$ follows by an argument analogous to that just used for $j < \ell$, except that there is no partitioning of matrices since V and V' are empty. \square

3.2 Convergence

Assume[†]

$$(3.2.1) \quad |U|(i) \neq 0$$

and let y be a solution of type t of (3.0.1), where $0 \leq t \leq \sigma$. For each $n \geq m+1$ such that a solution y_n of (3.0.1) exists satisfying both (3.0.2) and (3.0.3), there exist scalars $\alpha_1(n), \alpha_2(n), \dots, \alpha_j(n)$ and $\beta_1(n), \beta_2(n), \dots, \beta_k(n)$ such that

$$(3.2.2) \quad y_n = y + \sum_{s=1}^j \alpha_s(n) u_s + \sum_{s=1}^k \beta_s(n) v_s.$$

In the case $t = \sigma$, we have

$$y_n = y + \sum_{s=1}^{\ell} \alpha_s(n) u_s.$$

Then we prove readily, using (3.0.2) and (3.2.1), that $\alpha_s(n) = 0$ for $s = 1, 2, \dots, \ell$. Thus for each n , $y_n(r) = y(r)$ for $r = i, i+1, \dots$.

Therefore we restrict the ensuing discussion to the cases in which $0 \leq t < \sigma$.

Consider the set C of all subsets of k distinct sequences from X . We have $k > 0$ by our assumption that $t < \sigma$. Obviously, $V = \{v_1, v_2, \dots, v_k\} \in C$. We shall say that the optimally ranked basis X is j -normal provided that $|V|(r) \neq 0$ for all sufficiently large r , and

$$(3.2.3) \quad |\tilde{V}|(r) = o\{|V|(r)\} \text{ as } r \rightarrow \infty$$

for every $\tilde{V} \in C - \{V\}$. When $t = 0$, $C = \{V\}$ and X is automatically

[†]The reader is referred to §3.0 for notation and underlying assumptions.

j -normal (0-normal). When $t > 0$ the sets \tilde{V} contain at least one sequence from $U = \{u_1, u_2, \dots, u_j\}$. In this case (3.2.3) expresses the quite reasonable condition that whenever one or more of the "dominant" solutions v_1, v_2, \dots, v_k is replaced in the Casoratian $|V|$ by a "subdominant" solution, then the resulting Casoratian is dominated by $|V|(r)$ as $r \rightarrow \infty$.

We now state and prove the convergence theorem. A similar result, but only for homogeneous linear difference equations, is given by Zahar [27, Th. 5.1].

Theorem 3.2.1 Assume X is a j -normal optimally ranked basis such that (3.2.1) is satisfied and $y(r)$ is a solution of (3.0.1) of type t . Also assume

$$(3.2.4) \quad \lim_{r \rightarrow \infty} \frac{|V_s(y)|(r)}{|V|(r)} = 0, \quad s = 1, 2, \dots, k$$

where $V_s(y) = \{v_1, \dots, v_{s-1}, y, v_{s+1}, \dots, v_k\}$. Then for sufficiently large n there exists a unique solution y_n of (3.0.1) which satisfies (3.0.2) and (3.0.3). Furthermore,

$$(3.2.5) \quad \lim_{n \rightarrow \infty} y_n(r) = y(r)$$

for each fixed value of $r \geq i + j$.

Proof: From (3.2.2), (3.0.2) and (3.0.3) we derive the linear system

$$\begin{bmatrix} u_1(i) & \dots & u_j(i) & \vdots & v_1(i) & \dots & v_k(i) \\ \vdots & & \vdots & & \vdots & & \vdots \\ u_1(i+j-1) & \dots & u_j(i+j-1) & \vdots & v_1(i+j-1) & \dots & v_k(i+j-1) \\ \vdots & & \vdots & & \vdots & & \vdots \\ u_1(n) & \dots & u_j(n) & \vdots & v_1(n) & \dots & v_k(n) \\ \vdots & & \vdots & & \vdots & & \vdots \\ u_1(n+k-1) & \dots & u_j(n+k-1) & \vdots & v_1(n+k-1) & \dots & v_k(n+k-1) \end{bmatrix} \begin{bmatrix} \alpha_1(n) \\ \vdots \\ \alpha_j(n) \\ \beta_1(n) \\ \vdots \\ \beta_k(n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dots \\ -y(n) \\ \vdots \\ -y(n+k-1) \end{bmatrix}.$$

Although the block structure indicated in this system disappears when $t = 0$, this modification[†] causes no difficulty in the following proof.

Suppose V were such that the upper right block of the matrix is zero (again, this is satisfied vacuously if $t = 0$). Then the linear system becomes

$$\begin{bmatrix} u_1(i) & \dots & u_j(i) & \vdots & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_1(i+j-1) & \dots & u_j(i+j-1) & \vdots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ u_1(n) & \dots & u_j(n) & \vdots & v_1(n) & \dots & v_k(n) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_1(n+k-1) & \dots & u_j(n+k-1) & \vdots & v_1(n+k-1) & \dots & v_k(n+k-1) \end{bmatrix} \begin{bmatrix} \alpha_1(n) \\ \vdots \\ \alpha_j(n) \\ \beta_1(n) \\ \vdots \\ \beta_k(n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dots \\ -y(n) \\ \vdots \\ -y(n+k-1) \end{bmatrix} .$$

The determinant of this system is $|U|(i) \cdot |V|(n)$, and it is nonzero for all sufficiently large n , in view of our assumptions. Using Cramer's rule and Laplace's general theorem on the expansion of determinants [15, §93], we then find that

$$\alpha_1(n) = \alpha_2(n) = \dots = \alpha_j(n) = 0$$

and

$$\beta_s(n) = - \frac{|V_s(y)|(n)}{|V|(n)}, \quad s = 1, 2, \dots, k .$$

Therefore, for all r and for all sufficiently large n , we have

$$(3.2.6) \quad y_n(r) = y(r) - \sum_{s=1}^k \frac{|V_s(y)|(n)}{|V|(n)} v_s(r) .$$

[†]The other case in which the block structure disappears, which is $t = \sigma$, was disposed of earlier.

Then (3.2.5) follows from (3.2.4).

In order to complete the proof, let us introduce the sequences

$$(3.2.7) \quad v'_p = v_p + \gamma_{p,1}u_1 + \cdots + \gamma_{p,j}u_j, \quad p = 1, 2, \dots, k,$$

where the $\gamma_{p,q}$ are to be chosen in such a way that

$$v'_p(i+r) = 0, \quad r = 0, 1, \dots, j-1.$$

Thus the $\gamma_{p,q}$ have to satisfy the linear systems

$$\begin{bmatrix} u_1(i) & \cdots & u_j(i) \\ \vdots & & \vdots \\ u_1(i+j-1) & \cdots & u_j(i+j-1) \end{bmatrix} \begin{bmatrix} \gamma_{p,1} \\ \vdots \\ \gamma_{p,j} \end{bmatrix} = \begin{bmatrix} -v_p(i) \\ \vdots \\ -v_p(i+j-1) \end{bmatrix}, \quad p = 1, 2, \dots, k,$$

each of which is nonsingular because $|U|(i) \neq 0$, by hypothesis. Let $V' = \{v'_1, v'_2, \dots, v'_k\}$. Clearly it suffices to show that $U \cup V'$ is a j -normal optimally ranked basis for D such that

$$(3.2.8) \quad \lim_{r \rightarrow \infty} \frac{|v'_s(y)|(r)}{|V'|}(r) = 0, \quad s = 1, 2, \dots, k.$$

Since v_p dominates every sequence in U , we have from (3.2.7)

$$\frac{v'_p(r)}{v_p(r)} \rightarrow 1 \quad \text{as } r \rightarrow \infty$$

for each p , $1 \leq p \leq k$. Therefore v'_p has the same type as v_p . Furthermore, we may verify the three conditions of Definition 2.2.1, as follows:

$$i) \quad \frac{u(r)}{v'_p(r)} = \frac{u(r)}{v_p(r)} \cdot \frac{v_p(r)}{v'_p(r)} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

for every $u \in U$, and

$$\frac{v'_p(r)}{v'_q(r)} = \frac{v'_p(r)}{v_p(r)} \cdot \frac{v_p(r)}{v_q(r)} \cdot \frac{v_q(r)}{v'_q(r)} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

whenever $\text{type}(v'_p) < \text{type}(v'_q)$;

$$\text{ii) } \frac{v'_p(r)}{v'_q(r)} = \frac{v'_p(r)}{v_p(r)} \cdot \frac{v_p(r)}{v_q(r)} \cdot \frac{v_q(r)}{v'_q(r)} \sim \frac{v_p(r)}{v_q(r)} \quad \text{as } r \rightarrow \infty$$

whenever $\text{type}(v'_p) = \text{type}(v'_q)$;

(iii) Let $v'_p, v'_{p+1}, \dots, v'_q \in V'$ be all the sequences of one type, and let v' be any fixed one of these. Let the corresponding unprimed sequences be $v, v_p, v_{p+1}, \dots, v_q \in V$. Take $\delta_p, \delta_{p+1}, \dots, \delta_q \in F$, not all zero. For each $s = p, p+1, \dots, q$ we have

$$\frac{\delta_s v'_s}{v'} = \frac{\delta_s (v_s + \gamma_{s,1} u_1 + \dots + \gamma_{s,j} u_j)}{v'} = \frac{\delta_s v_s}{v'} + o(1)$$

(since u_1, u_2, \dots, u_j are of lower type than $v' = v\{1+o(1)\}$). Hence

$$\frac{\delta_p v'_p + \dots + \delta_q v'_q}{v'} = \frac{\delta_p v_p + \dots + \delta_q v_q}{v'} + o(1) .$$

Furthermore

$$\frac{\delta_p v'_p + \dots + \delta_q v'_q}{v'} = \frac{\delta_p v_p + \dots + \delta_q v_q}{v} \{1+o(1)\} + o(1)$$

from which it is clear that

$$\limsup_{r \rightarrow \infty} \left| \sum_{s=p}^q \delta_s v'_s(r)/v'(r) \right| > 0 .$$

Thus, $U \cup V'$ is an optimally ranked basis.

Next we prove that $|V'|(\mathbf{r}) \neq 0$ for all sufficiently large \mathbf{r} . We recall that a determinant is a multilinear function of its columns. If we express

$$V'(\mathbf{r}) = |v'_1, v'_2, \dots, v'_k|(\mathbf{r})$$

where v'_1, v'_2, \dots, v'_k temporarily denote the successive columns of $|V'|(\mathbf{r})$, then we derive

$$\begin{aligned} |V'|(\mathbf{r}) &= |v_1, v'_2, \dots, v'_k|(\mathbf{r}) + \sum_{q=1}^j \gamma_{1,q} |u_q, v'_2, \dots, v'_k|(\mathbf{r}) \\ &= |v_1, v_2, v'_3, \dots, v'_k|(\mathbf{r}) + \sum_{q=1}^j \gamma_{1,q} |u_q, v'_2, \dots, v'_k|(\mathbf{r}) \\ &\quad + \sum_{q=1}^j \gamma_{2,q} |v_1, u_q, v'_3, \dots, v'_k|(\mathbf{r}) \\ &= \dots, \end{aligned}$$

and hence

$$\begin{aligned} |V'|(\mathbf{r}) &= |V|(\mathbf{r}) + \sum_{q=1}^j \gamma_{1,q} |u_q, v'_2, \dots, v'_k|(\mathbf{r}) \\ &\quad + \sum_{q=1}^j \gamma_{2,q} |v_1, u_q, v'_3, \dots, v'_k|(\mathbf{r}) \\ &\quad + \dots \\ &\quad + \sum_{q=1}^j \gamma_{j,q} |v_1, v_2, \dots, v_{k-1}, u_q|(\mathbf{r}) . \end{aligned}$$

If we continue this construction until every reference to a primed v is removed, then it is clear that $|V'|(\mathbf{r})$ is equal to the sum of $|V|(\mathbf{r})$ plus a linear combination of determinants formed from $|V|(\mathbf{r})$ by replace-

ment of at least one v by a u . By our assumption that X is j -normal, we conclude

$$(3.2.9) \quad |V'| (r) = |V| (r) \{1+o(1)\} \quad \text{as } r \rightarrow \infty .$$

Since $|V| (r) \neq 0$ for sufficiently large r , we also have $|V'| (r) \neq 0$ for sufficiently large r .

If one or more of the solutions v_1', v_2', \dots, v_k' are replaced in $|V'| (r)$ by subdominant solutions from U , then a construction like the one preceding shows that the modified Casoratian $|\tilde{V}'| (r)$ is dominated by $|V| (r)$ as $r \rightarrow \infty$, since the original optimally ranked basis is j -normal. Consequently, using (3.2.9), we conclude $U \in V'$ is j -normal. A similar argument shows that (3.2.8) is satisfied, and the theorem is proved. \square

3.3 Stability

Let us consider solutions of (3.0.1) of type t , where $0 \leq t < \sigma$.[†] Let j be the dimension of the subdominant subspace of type t for D . Let i be a point at which $|U|(i) \neq 0$; see (3.0.7). Suppose D is (i,j) -factorizable; see Definition 1.2.1. Let $A^i = [a_0^i, a_1^i, \dots, a_j^i]$ be a left factor and $B^{i+j} = [b_0^{i+j}, b_1^{i+j}, \dots, b_k^{i+j}]$ the corresponding right factor of D_j^i , so that $D_j^i = A_j^i B^{i+j}$ is an (i,j) -factorization of D .

The next theorem provides a generalization of Theorem 1.3.2, from totally separable linear difference equations to equations that are merely separable. The remark following Theorem 1.3.2 applies here as well. That is, the back substitution stage of the algorithm is stable, at least for sufficiently large r .

Theorem 3.3.1 If $|U|(i) \neq 0$ and $D_j^i = A_j^i B^{i+j}$, then the difference operator B^{i+j} is separable and any solution y of (3.0.1) of type t or less is a sequence of type zero with respect to B^{i+j} .

Proof: Suppose $j = 0$, i.e., type $(y) = 0$ with respect to D . Then $A^i = [a_0^i]$ and $B^i = [b_0^i, b_1^i, \dots, b_\ell^i]$ and we have $D^i = A^i B^i$.

Now let $x \in X$, where X is the optimally ranked basis (3.0.4). Then $Dx = 0$; hence $D^i x^i = A^i B^i x^i = 0$. This last equation is equivalent to

$$a_0^i(r) \sum_{s=0}^{\ell} b_s^i(r) x(r+s) = 0, \quad r = i, i+1, \dots$$

Since a_0^i is free of zeros by the definition of (i,j) -factorizability, we have $B^i x^i = 0$. Thus $\{x^i | x \in X\}$ is an optimally ranked basis for B^i and it follows immediately that type $(y) = 0$ with respect to B^i .

[†]The case $t = \sigma$ corresponds to a maximal solution of (3.0.1). Since the general algorithm reduces to pure forward recurrence in this case, it is stable; see §1.1.

Next suppose j satisfies $0 < j < \ell$. Let $v \in V$, where V is given by (3.0.8). In view of our assumption that $|U|(i) \neq 0$, we may assume that

$$v(i+r) = 0, \quad r = 0, 1, \dots, j-1;$$

compare the proof of Theorem 3.2.1. Since $v \in K(D)$ we have $Dv = 0$. It follows that for each $n \geq i + j + \max(j, k)$ the subsequence $v^{i, n+k-1}$ satisfies the finite boundary value problem

$$D_j^{i, n-j-1} v^{i+j, n-1} = \begin{bmatrix} \dots\dots\dots 0 \dots\dots\dots \\ -D^{n-\ell, n-j-1} v^{n, n+k-1} \end{bmatrix};$$

compare (1.2.4). For each n , these boundary value problems are nonsingular and factorizable as shown by Theorem 1.2.1.

Therefore, Theorem 1.2.2 is applicable and we see that $v^{i+j, n-1}$ satisfies

$$B_0^{i+j, n-1} v^{i+j, n-1} = - \begin{bmatrix} \dots\dots\dots 0 \dots\dots\dots \\ B_k^{n-k, n-1} v^{n, n+k-1} \end{bmatrix}$$

for each $n \geq i + j + \max(j, k)$; compare (1.2.7) and the remark following the proof of Theorem 1.2.2. The last equation is equivalent to

$$\sum_{s=0}^k b_s(r) v(r+s) = 0, \quad r = i+j, i+j+1, \dots, n-1.$$

Since n may be arbitrarily large, we have

$$\sum_{s=0}^k b_s(r) v(r+s) = 0, \quad r = i+j, i+j+1, \dots$$

or, equivalently, $B^{i+j} v^{i+j} = 0$. Therefore, $\{v^{i+j} | v \in V\}$ is an optimally ranked basis for B^{i+j} . Since y is dominated by every sequence in V ,

Then the arguments of the preceding paragraph again apply.

Since the wanted solution y is of type t , we conclude that the process of forward elimination is stable in the sense that each perturbation subsequently grows at a rate that does not exceed that of the wanted solution. Of course, there will be a loss of accuracy when an element of E_j^i (or e^i) is large compared with the rounding error in the stored value of the corresponding element of D_j^i (or of the right side of (3.3.1)). This may occur, for example, when there is heavy cancellation in the formation of an element that is subsequently used as a pivot. However, the important point is that the effect of each such loss is not magnified in subsequent steps of the algorithm.

An extension of the foregoing discussion shows that if we select a value of j (that is, the number of prescribed initial conditions) that is less than the dimension of the subdominant subspace of the wanted solution, then the forward elimination remains stable. However, the back-substitution is now unstable; compare the proof of Theorem 3.3.1. Similarly if j is too large then the forward elimination is unstable, and the back-substitution is stable. Nevertheless if, in practice, the actual instabilities are weak owing to weak separation of solutions of consecutive types as $r \rightarrow \infty$, then it may be advantageous to select a value of j that differs from the dimension of the subdominant subspace of the wanted solution. This is because the loss of accuracy caused by the instability is offset by a substantial reduction in the value of the terminal point n owing to increased convergence. This modification is illustrated by numerical examples in Chapter 4.

CHAPTER 4. NUMERICAL EXAMPLES

4.0 Introduction

Several numerical examples will be given to illustrate the general algorithm. The examples all involve fourth-order linear difference operators with nonconstant coefficients. Both homogeneous and inhomogeneous difference equations are treated.

The fourth-order operators were produced by using the method described in [14, §12.22], starting with two second-order recurrence relations which have as solutions Bessel functions, modified Bessel functions or associated Legendre functions. The inhomogeneous equations have as particular solutions Anger-Weber functions or Struve functions. The right sides of these equations were produced as a by-product of the method used to produce the fourth-order operators.

The production of the fourth-order difference operators from the corresponding pairs of second-order operators requires only elementary algebraic manipulations. These calculations are rather lengthy, however, and would be extremely difficult (as well as tedious) to complete accurately by hand. Instead, the MACSYMA symbol manipulation system, described in [12], was employed[†]. This procedure had the added advantage of storing the required formulas for the coefficients directly in the computer, ready for numerical evaluation, thereby further reducing the possibility of human error. Similarly, MACSYMA was used to advantage in the production of the right sides for the inhomogeneous examples.

[†]Development of which is currently supported, in part, by the United States Department of Energy under Contract Number DE-AC02-79ER10357 and by the National Aeronautics and Space Administration under Contract Number NSG1321.

In the case of the Bessel functions $J_r(x)$, $Y_r(x)$ and modified Bessel functions $I_r(x)$, $K_r(x)$ we restrict ourselves to integer order $r \geq 0$ and real argument $x > 0$. Relevant properties of these functions are given in [16, Chapter 9]. For example, for fixed x the functions $J_r(x)$ and $Y_r(x)$ satisfy the linear recurrence relation

$$(4.0.1) \quad y(r-1) - \frac{2r}{x} y(r) + y(r+1) = 0 .$$

Similarly, $I_r(x)$ and $(-)^r K_r(x)$ satisfy

$$(4.0.2) \quad y(r-1) - \frac{2r}{x} y(r) - y(r+1) = 0 .$$

The Anger-Weber functions $E_r(x)$ satisfy

$$(4.0.3) \quad y(r-1) - \frac{2r}{x} y(r) + y(r+1) = - \frac{2\{1-(-1)^r\}}{\pi x} ;$$

and the Struve functions $H_r(x)$ satisfy

$$(4.0.4) \quad y(r-1) - \frac{2r}{x} y(r) + y(r+1) = \frac{\left(\frac{1}{2} x\right)^r}{\sqrt{\pi} \Gamma\left(r + \frac{3}{2}\right)}$$

compare (4.0.1) in both cases. Relevant properties of the $E_r(x)$ and $H_r(x)$ appear in Olver's paper [21] where they were used as examples for second-order inhomogeneous equations.

The coefficients of the fourth-order homogeneous recurrence relation obtained from (4.0.1) with $x = x_1$ and (4.0.2) with $x = x_2$ are shown in Figure 6 in the form of FORTRAN statements produced by MACSYMA. In addition, Figure 6 shows the right-hand sides corresponding to $E_r(x_1)$ and $H_r(x_1)$. We designate the difference operator D defined by these

```

DCOEFF(1) = -A0*X1*X2*(X1**2*X2**2-R**2*X2**2-R**2*X1**2+R
1 *X1**2)
DCOEFF(2) = 2*A0*R*(Y2+X1)*(X1**2*X2**2-R**2*X2**2+Y2**2+R**2*X1**
1 2-X1**2)
DCOEFF(3) = -2*A0*R*(X1*Y2**3+2*R*X1**2*Y2**2-2*R**3*Y2**2+2*R**2*Y2*
1 **2+Y1**3*Y2+2*R**3*X1**2-2*R*X1**2)
DCOEFF(4) = -2*A0*R*(Y2-X1)*(X1**2*X2**2-R**2*X2**2+Y2**2+R**2*X1**
1 **2-Y1**2)
DCOEFF(5) = A0*X1*Y2*(X1**2*X2**2-R**2*X2**2+R**2*X1**2-R*
1 X1**2)
CCOEFF=0.0D0
IF (ISUR.EQ.0) RETURN
ACOEFF(1) = -A0*X1*Y2*(X1**2*X2**2-R**2*X2**2+R**2*X1**2+R
1 *X1**2)
ACOEFF(2) = 2*A0*X1*(X1*Y2**3+R*X1**2*Y2**2-R**3*Y2**2+R**2*Y2**2+R**
1 3*X1**2-R**3*X1**2)
ACOEFF(3) = A0*X1*Y2*(X1**2*X2**2-R**2*X2**2+R**2*X1**2-1R
1 X1**2)
CO TO (101,102), ISUR
CCOEFF=-((1+(-1)**R)*(ACOEFF(1)+ACOEFF(3)))+
101 (1-(-1)**R)*ACOEFF(2))/(1.57079632679489662D0*X1)
RETURN
CCOEFF=1.0D0/1.57079632679489662D0
102 BD JOB I=1,R
CCOEFF=XI*CCOEFF/(2*I+1)
CONTINUE
103 CCOEFF=(C2*B-1)*ACOEFF(1)/X1+ACOEFF(2)+X1*ACOEFF(3)/(2*R*3)
1 CCOEFF
RETURN

```

Figure 6. Fragment of FORTRAN subroutine for computing coefficients of recurrence relations $\sum_{s=-2}^2 d_s(r)y(r+s) = g(r)$ involving the JYIK operator. A0 is an arbitrary scale factor. X1, X2 and R are input parameters. ISWR (=0,1 or 2) selects the homogeneous equation, the inhomogeneous equation having as a particular solution the Anger-Weber function $E_r(x_1)$, or the inhomogeneous equation having as a particular solution the Struve function $H_r(x_1)$.

coefficients the JYIK operator. Since the functions $J_r(x_1)$, $Y_r(x_1)$, $I_r(x_2)$, $(-)^r K_r(x_2)$ are linearly independent, they form a fundamental set of solutions of $Dy = 0$.

Similarly in the case of the associated Legendre functions $P_r^\mu(x)$, $Q_r^\mu(x)$ we restrict consideration to integer degree $r \geq 0$, integer order $\mu \geq 0$, and real argument $x > 0$. Relevant properties of these functions may be found in [16, Chapter 8]. We use the r -wise linear recurrence relation

$$(4.0.5) \quad (r+\mu)y(r-1) - (2r+1)xy(r) + (r-\mu+1)y(r+1) = 0 ,$$

valid for fixed x and μ . The coefficients of the fourth-order difference operator D , shown in Figure 7, were produced (using MACSYMA) from (4.0.1) with $x = x_1$ and (4.0.5) with $x = x_2$. Figure 7 also shows the right sides corresponding to $E_r(x_1)$ and $H_r(x_1)$. In this case the functions $J_r(x_1)$, $Y_r(x_1)$, $P_r^\mu(x_2)$, $Q_r^\mu(x_2)$ form a fundamental set of solutions of $Dy = 0$. Accordingly, we designate D the JYPQ operator.

A FORTRAN program has been written to implement the general algorithm. The user may select either the JYIK operator or the JYPQ operator. He may also elect to solve the homogeneous equation, the inhomogeneous equation having the Anger-Weber function as a solution, or the inhomogeneous equation having the Struve function as a solution. Alternatively, he may provide a subroutine which generates the coefficients of an arbitrary linear difference equation of any order.

The user supplies the program with the starting point i and the **terminal** point m of the desired subsequence of the solution to be computed, the number j of initial values, the subsequence $y^{i,i+j-1}$ of initial values, and the parameters ν and ε for use in the convergence test (to be described

```

DOOFF(1) = (-4*R**3+(4*M-4)*R**2+(5-S*D)*R-3*M+3)*X1**2*Y2**2+(8
1 *R**4+(8*M+12)*R**3+(20*M-6)*R**2+(12*M-10)*R-2*M**2+6*M-4)*X1*
2 Y2+((-4*M**3+4*M-1)*R-4*M**3+8*M**2-5*M+1)*X1**2-4*R**5+(-4*M-8
3 )*R**4+(4*M**3-20*M+4)*R**3+(4*M**3-8*M**2-6*M+8)*R**2+(4*M**3-
4 12*M**2+8*M)*R
DOOFF(2) = ((8*R**3+12*M**2-2*M-8)*X1**3*Y2**3+(-8*R**4+(8*M-24)*
1 R**3+(8*M-14)*R**2+(6-2*M)*R-2*M+4)*X1**2*Y2**2+((8*M**2-2)*R+
2 4*M**2-1)*X1**3+(-8*M**5+(4-16*M)*R**4+(16-8*M**2)*R**3+(-4*M**
3 2+20*M-12)*R**2+(12*M**2-12*M)*R)*X1**2+((8*M**3-24*M+10)*R**2
4 +(8*M**3-24*M**2+6*M+2)*R-4*M**2+2*M)*X1**2+8*R**6+(8*M+8)*R**5
5 +(-8*M**3+32*M-24)*R**4+(-8*M**3+24*M**2-24*M-8)*R**3+(8*M**2-8
6 2*M+16)*R**2+(8*M**3-24*M**2+16*M)*R/X1
DOOFF(3) = -((16*R**4+24*R**3-4*R**2-6*R)*X1**2*Y2**3+((8*M**3+12
1 *R**2+(8*M-6)*R+4*M-5)*X1**3+(-32*R**5-40*R**4+8*R**3+(16*M-4)*
2 R**2+(8*M-4)*R)*X1)*Y2**2+((-16*R**4-24*R**3+(16*M**2-24*M+12)*
3 R**2+(8*M**3-12*M+10)*R)*X1**2+16*R**6+24*R**5+(-16*M**2+40*M-4
4 0)*R**4+(-8*M**2+24*M-40)*R**3+(16*M**2-40*M+24)*R**2+(8*M**2-2
5 4*M+16)*R)*Y2+((8*M**2-8*M+2)*R+4*M**2-4*M+1)*X1**3+(8*R**5+12*
6 R**4-8*R**2*R**3+(-4*M**2-4*M)*R**2+(8*M**3-12*M**2-4*M+4)*R)*Y
7 1/X1
DOOFF(4) = ((8*R**3+12*R**2-2*R-8)*X1**3*Y2**3+(-8*R**4-8*M**R**3+
1 (18-8*M)*R**2+(10*M-4)*R+6*M-6)*X1**2*Y2**2+((8*M**2-2)*R+4*M
2 *R-1)*X1**3+(16*M-20)*R**4+(-8*M**2+32*M-24)*R**3+(-4*
3 R**2+4*M+4)*R**2+(4*M**2-12*M+8)*R)*X1**2+((8*M**2-24*M+10)*R*
4 *Y2+(-8*M**3+32*M**2-36*M+12)*R+4*M**2-6*M+2)*X1**2+8*R**6+(16-8
5 *R)*R**5+(-8*M**2+16*M-16)*R**4+(8*M**3-32*M**2+40*M-32)*R**3+(
6 8*M**2-16*M+8)*R**2+(-8*M**3+32*M**2-40*M+16)*R/X1
DOOFF(5) = (-4*M**3+(4*M-10)*R**2-2*M+2*M**2-6*M+4)*X1*Y2+((-4*M**2+4*M-1)
1 *R+4*M**3-12*M**2+9*M-2)*X1**2-4*R**5+(4*M-4)*R**4+(4*M**2-12*M
2 +12)*R**3+(-4*M**3+12*M**2-12*M+4)*R**2+(4*M**3-16*M**2+20*M-8)
3 *R
4
CMMPT=0
IF (ISV,EC,0) RETURN

```

Figure 7: Fragment of FORTRAN subroutine for computing coefficients of recurrence relations $\sum_{s=0}^2 d_s(r)y(r+s) = g(r)$ involving the JYPQ operator. $M(=μ)$, $X1$, $X2$ and R are input parameters. ISWR ($=0, 1$ or 2) selects the homogeneous equation, the inhomogeneous equation having as a particular solution the Anger-Weber function $E_r(x_1)$, or the inhomogeneous equation having as a particular solution the Struve function $H_r(x_1)$.

```

ACOFF(1) = -(R-I-1)*(4*R**2*X1**2+X2**2+C*R*X1**2+X2**2+O*X
1 1**2+2**2-2*C*R**2*X1**2-20*R**2*X1**2-14*R*X1**2+2*R*X1**2-4*X1
2  **2+4*I**2*X1**2-4*I**2*X1**2+X1**2+4*I**2+12*I**2-4*I**2*I**2+O*I
3  **2+O*I**2-4*I**2*I**2+O*I**2)
ACOFF(2) = (8*I**2*X1**2+X2**2+12*I**2*X1**2+X2**2-2*I**2*X1**
1  2**2+O*I**2+O*I**2*X1**2+2**2-10*I**2*X1**2+2**2-24*I**2*X1**2+4*I**2*
2  *X1**2+O*I**2*X1**2+2**2+2*I**2+4*I**2*X1**2+2**2-2*X1**2+O*I
3  **2+O*I**2*X1**2+2**2-2*I**2*X1**2+4*I**2*X1**2+2**2-4*I**2*X1**2
4  +12*I**2+O*I**2-4*I**2*X1**2+24*I**2*X1**2-20*I**2*X1**2+4*I**2*X1**2
5  +12*I**2+O*I**2-20*I**2*X1**2+O*I**2*X1**2-24*I**2*X1**2+12*I**2+4*I**2
6  **2-12*I**2+O*I**2-16*I**2*X1**2+O*I**2*X1**2-12*I**2*X1**2+O*I**2
7  **2-20*I**2*X1**2+12*I**2*X1**2)
ACOFF(3) = -(R-I+2)*(4*R**2*X1**2+X2**2-4*I**2*X1**2-20*R**2*X1**2-14*R**2*X1
1  **2+4*I**2*X1**2+2*R**2*X1**2+2*I**2+O*I**2*X1**2+2**2-20*I**2*X1**2-14*I**2*X1
2  **2+4*I**2*X1**2+2*I**2+O*I**2*X1**2+2**2-4*I**2*X1**2-4*I**2*X1**2+O*I
3  **2+O*I**2*X1**2+2**2-4*I**2*X1**2-4*I**2*X1**2+O*I**2*X1**2+2**2)
CO TO (101,102),ISUR
RETURN
CCOFF=1.000/1.5707963267948966200
D0 102 I=1,R
CCOFF=X1*CCOFF/(2*I+1)
CGTIN0E
CCOFF=(2*I+1)*ACOFF(1)/X1+ACOFF(2)+X1*ACOFF(3)/(2*I+3)
1  *CCOFF
RETURN
101
102
103

```

Figure 7 (Concluded).

below). The program solves the proposed problem by executing three phases. In the initial phase the first $m + l + v - i - j$ rows of the infinite linear system (1.3.6) are stored. Forward elimination without pivoting is performed on these rows. This results in the storing of the first $m + l + v - i - j$ rows of the left and right factors of the (i,j) -factorization $D_j^i = A_j^i B^{i+j}$, and also the subsequence $z^{i+j, m+v}$ of the solution of (1.3.8). The elimination procedure is carried out by modifications, described below, of the subroutines DGBFA and DGBSL which are available in the mathematical software package known as LINPACK; see [8].

The unmodified DGBFA performs forward elimination with partial pivoting on a finite band matrix, ignoring the right side. It produces an upper triangular matrix, a lower triangular matrix and a vector of pivoting information. These results may then be used by the unmodified DGBSL to solve the corresponding linear system with specified right side. Alternatively, the user can direct that DGBSL solve a linear system having as its matrix of coefficients the transpose of the matrix input to DGBFA. Provision has been made in DGBFA to suppress the partial pivoting. When this is done the vector of pivoting information is superfluous and (except for rounding errors) the matrix obtained by premultiplying the upper triangular matrix by the lower triangular matrix is the original band matrix. This factorization is therefore of exactly the kind required by Definition 1.2.1; see equation (1.2.5).

When partial pivoting has been suppressed in DGBFA, DGBSL proceeds, in effect, by forward elimination without pivoting followed by back substitution. Provision has been made in DGBSL to exit after the forward elimination stage and to re-enter at the start of the back substitution stage. Thus the user has the option of examining the result of the

forward elimination stage and perhaps electing to continue it without having to do any back substitution. All of the above modifications of DGBFA and DGBSL, to provide just the tools that are needed in our algorithm, were easily achieved because of the clarity and flexibility of the coding and documentation of LINPACK.

We return to the description of the initial phase. The next computation is the subsequence $w_m^{m+1, m+v}$; compare equations (1.3.16) and (1.3.17) with $s = m$. Since w_m^{m-k+1} satisfies the adjoint equation, it is produced using the forward elimination stage of the (modified) subroutine DGBSL in which the input matrix to DGBFA is automatically transposed. Finally, the initial phase computes

$$(4.0.6) \quad y_r(m) = \sum_{s=m}^{r-1} w_m(s) z(s) / b_o(m)$$

for $r = m+1, m+2, \dots, m+v, m+v+1$ and

$$(4.0.7) \quad \eta_{r,v}(m) = \sum_{s=r}^{r+v-1} w_m(s) z(s) / b_o(m)$$

for $r = m+1$. Equation (4.0.6) is the same as (1.3.18) apart from a change of symbols. Equation (4.0.7) is the approximation to the truncation error

$$\eta_r(m) = y(m) - y_r(m)$$

which is obtained when $y(m)$ is replaced by $y_{r+v}(m)$: compare (1.3.20).

After completion of the initial phase, the program proceeds to the iteration phase. One additional row of the infinite linear system (1.3.6) is stored. Since the previously determined part of the (i,j) -factorization has replaced the corresponding rows of the original matrix in the computer

storage, these rows are no longer available to extend the forward elimination stage. Therefore, extension of the forward elimination is accomplished by an alternative procedure that requires the solutions of two triangular linear systems of orders j and k . This is achieved using DTRSL, a LINPACK subroutine that solves triangular systems. Next, the formerly obtained subsequences of the solutions z^{i+j} and w_m^{m+1} are extended to include the next higher term. Then the series (4.0.6) and (4.0.7) are summed with the new value of r , and the following convergence test is applied: If

$$(4.0.8) \quad |\eta_{r,v}(m)| \leq \varepsilon |y_{r+v}(m)|$$

then the program proceeds to the third and final phase; otherwise, the iteration phase is repeated and the convergence test re-applied.

The final phase of the program is the back substitution stage of the algorithm. Let n denote the value of r which satisfies (4.0.8). The back substitution is begun at this point, using the (modified) subroutine DGBSL. This yields the computed solution $y_n^{i+j,n-1}$ of (1.3.7) and also the desired approximation of $y(r)$ over the range $r = i+j, i+j+1, \dots, m$.

In all of the examples presented in this thesis the chosen termination criterion corresponds to an estimated relative error of 10^{-10} in the final term $y_n(m)$ of the requested sequence of values of the function. Thus $\varepsilon = 10^{-10}$ in equation (4.0.8). For the values of x_1 and x_2 selected in the examples the solutions are reasonably well-separated as $r \rightarrow \infty$, hence we take the "testing parameter" v in equation (4.0.7) to be 1 throughout. Initial values were entered to 10 significant figures in every case. All computations were performed internally to 18 figures

and the output printed to 10 figures. For each example we present (i) the solution $z(i+j), z(i+j+1), \dots, z(n)$ of equation (1.3.8) (the forward elimination stage); (ii) the solution $w_m^{(m+1)}, w_m^{(m+2)}, \dots, w_m^{(n)}$ of (1.3.16), (1.3.17) (the forward recurrence of the adjoint equation for use in the termination criterion); (iii) the sequence $y_{m+1}^{(m)}, y_{m+2}^{(m)}, \dots, y_{n+v}^{(m)}$ of approximants to the value $y(m)$ (see equation (4.0.6)); (iv) the sequence $\eta_{m+1,1}^{(m)}, \eta_{m+2,1}^{(m)}, \dots, \eta_{n,1}^{(m)}$ of estimates of the truncation error incurred in accepting the terminal points of $m+1, m+2, \dots, n$ respectively (see equation (4.0.7)); (v) the sequence $y_n^{(n-1)}, y_n^{(n-2)}, \dots, y_n^{(i+j)}$ derived from equation (1.3.7) (the back substitution stage).

The initial values that were used in the boundary value problem actually solved in each example are shown in parentheses in the tables; see, for example, Table 5. The terminal values (which are all zero) are also shown. Forward recurrence of the j -th order difference equation $A z^i = g^i$ starting from the given initial values produces a solution of (1.3.8) (the algorithm produces this solution by forward elimination). Similarly, backward recurrence of the k -th order difference equation $B y_n^{i+j} = z^{i+j}$ starting from the terminal values (which are all zero) produces the corresponding solution of the boundary value problem; this is the back substitution stage. Data is given in the tables only for selected values of r . All of the values of r that were used beyond $r = m$ are included, however. These values illustrate the termination procedure.

It should be noted that when $j = 4$ the method as implemented by the FORTRAN program is equivalent to, but not identical with, forward recurrence of the given difference equation. See, for example, Table 8. This is because DGBFA, the LINPACK matrix factorization subroutine, scales the linear system in such a way

that the principal diagonal of the lower triangular factor is a sequence of ones. Therefore, the result of the forward elimination stage differs from the result of the back substitution stage (unless the trailing coefficients of the original linear difference operator are all equal to one). Since there is no question of truncation error when $j = 4$, the sequences $w_m(r)$, $y_r(m)$ and $\eta_{r,1}(m)$, $r = m+1, m+2, \dots$, are not defined.

It has been observed that Olver's original second-order method is likely to cause overflow in most computers. This same likelihood of overflow exists for the extended algorithm. The examples to be presented in this chapter were computed using double precision on a Univac 1108. Double precision was used not for higher precision but for wider exponent range. The double precision range extends to 10^{308} whereas the largest number generated in the examples was somewhat greater than 10^{200} . Van der Cruyssen [25] gives a rescaled version of the second-order algorithm, based on LU decomposition, that greatly reduces the incidence of overflow or underflow. An equivalent procedure for overcoming this difficulty, based directly on the use of ratios of consecutive terms in each solution, is described by Aggarwal and Burgmeier [1]. In the present writer's view, however, it is preferable (if possible) to avoid mathematical modifications of algorithms simply to suit arbitrary limitations of existing computers. A software alternative to rescaling is proposed in [24], in which a full integer storage location is allocated to the exponent of each floating-point number. A FORTRAN implementation of this proposal is available; see [11].

4.1 Examples Involving the Homogeneous JYIK Equation

From the asymptotic forms

$$(4.1.1) \quad J_r(x) \sim \frac{1}{(2\pi r)^{1/2}} \left(\frac{ex}{2r}\right)^r, \quad Y_r(x) \sim -\left(\frac{2}{\pi r}\right)^{1/2} \left(\frac{ex}{2r}\right)^{-r}$$

and

$$(4.1.2) \quad I_r(x) \sim \frac{1}{(2\pi r)^{1/2}} \left(\frac{ex}{2r}\right)^r, \quad (-)^r K_r(x) \sim (-)^r \left(\frac{\pi}{2r}\right)^{1/2} \left(\frac{ex}{2r}\right)^{-r}$$

which are valid for fixed x and large r , we obtain

$$(4.1.3) \quad J_r(x_1) < I_r(x_2) < (-)^r K_r(x_2) < Y_r(x_1)$$

when $x_1 < x_2$ and

$$(4.1.4) \quad I_r(x_2) < J_r(x_1) < Y_r(x_1) < (-)^r K_r(x_2)$$

when $x_2 < x_1$, where in (4.1.3) and (4.1.4) the symbol $<$ is used to indicate the order of dominance of the functions as $r \rightarrow \infty$. Thus when $x_1 \neq x_2$ the JYIK operator is totally separable. As with any totally separable operator of order 4, there are four distinct types of solution and the dimension of the subdominant subspace of type s is s . This means that $j = s$ is the proper number of initial values to specify for a solution of type s .

When all four Bessel functions have the same argument, from (4.1.1) and (4.1.2) we have

$$(4.1.5) \quad J_r(x) \sim I_r(x), \quad (-)^r K_r(x) \sim (-)^{r+1} \frac{1}{2\pi} Y_r(x), \quad r \rightarrow \infty.$$

The first of these relations suggests that we define the further solution

$$(4.1.6) \quad A_r(x) = J_r(x) - I_r(x) .$$

Then it may be verified that

$$(4.1.7) \quad A_r(x) \sim -\frac{1}{2} \frac{x^2}{r+1} \left\{ \begin{array}{c} I_r(x) \\ J_r(x) \end{array} \right\}, \quad r \rightarrow \infty$$

where either function in the braces on the right may be chosen. Furthermore, since there is no nonzero linear combination of $(-)^r K_r(x)$ and $Y_r(x)$ that reduces to a solution of lower type, we conclude that there are exactly three distinct types of solution. Symbolically, we have

$$(4.1.8) \quad A_r(x) < \left\{ \begin{array}{c} I_r(x) \\ J_r(x) \end{array} \right\} < \left\{ \begin{array}{c} (-)^r K_r(x) \\ Y_r(x) \end{array} \right\}$$

where in each pair of braces either function may be chosen. Thus when $x_1 = x_2$ the JYIK operator is separable but not totally separable. The dimensions of the subdominant subspaces of types 1,2 and 3 are 1,2 and 4 respectively.

Although $A_r(x)$ has type 1, it is separated only by a relative factor of $1/r$ from solutions of type 2. Let $y_n(r)$ denote the approximation to $A_r(x)$ that is obtained from the algorithm with terminal point n . Then, suppressing the arguments of $A_r(x)$ and all the Bessel functions and using equation (3.2.6), we have[†]

$$y_n(r) \sim A_r - \beta_1(n)I_r - \beta_2(n)Y_r - \beta_3(n)(-)^r K_r, \quad n \rightarrow \infty$$

where

[†]Since $I_r(x)$, $Y_r(x)$ and $K_r(x)$ are not normalized as required for the validity of (3.2.6), we have only the weaker statement given here.

$$\beta_1(n) = \left| \begin{array}{ccc|ccc} A_n & Y_n & (-)^n K_n & I_n & Y_n & (-)^n K_n \\ A_{n+1} & Y_{n+1} & (-)^{n+1} K_{n+1} & I_{n+1} & Y_{n+1} & (-)^{n+1} K_{n+1} \\ A_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} & I_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} \end{array} \right| ,$$

$$\beta_2(n) = \left| \begin{array}{ccc|ccc} I_n & A_n & (-)^n K_n & I_n & Y_n & (-)^n K_n \\ I_{n+1} & A_{n+1} & (-)^{n+1} K_{n+1} & I_{n+2} & Y_{n+1} & (-)^{n+1} K_{n+1} \\ I_{n+2} & A_{n+2} & (-)^{n+2} K_{n+2} & I_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} \end{array} \right| ,$$

and

$$\beta_3(n) = \left| \begin{array}{ccc|ccc} I_n & Y_n & A_n & I_n & Y_n & (-)^n K_n \\ I_{n+1} & Y_{n+1} & A_{n+1} & I_{n+1} & Y_{n+1} & (-)^{n+1} K_{n+1} \\ I_{n+2} & Y_{n+2} & A_{n+2} & I_{n+2} & Y_{n+2} & (-)^{n+2} K_{n+2} \end{array} \right| .$$

In consequence of (4.1.1), (4.1.2), and (4.1.7) we see that $\beta_1(n) \sim -x^2(2n)^{-1}$ for large n , whereas $\beta_2(n)$ and $\beta_3(n)$ approach zero much more rapidly. Thus we expect that the rate of convergence of the algorithm will be approximately as $1/n$. This has been confirmed in numerical tests. Since the separation of $A_r(x)$ from solutions of type two is so weak, it is usually advantageous to regard $A_r(x)$ as belonging to the same type as $J_r(x)$ and $I_r(x)$; this means that we specify two initial conditions rather than one. This modification improves convergence; in consequence the value of the terminal point n is reduced very considerably. This gain is offset to a minor extent by some loss in accuracy in the values of $A_r(x)$ for higher values of r ; the extent of this loss is no more than the cancellation that takes place between the corresponding values of $I_r(x)$ and $J_r(x)$ in Eq. (4.1.6).

Example 4.1.1 In this example we compute the solutions $J_r(1)$, $I_r(10)$, $(-)^r K_r(10)$ and $Y_r(1)$ of the homogeneous equation defined in Figure 6, with $x_1 = 1$, $x_2 = 10$, for $r = 0, 1, 2, \dots, 100$. Since each of these solutions is monotonic (in magnitude), the algorithm will be numerically stable provided that the proper number j of initial values is used in each case. This number is $j = 1$ for $J_r(1)$, $j = 2$ for $I_r(10)$, $j = 3$ for $(-)^r K_r(10)$ and $j = 4$ for $Y_r(1)$; compare (4.1.3).

Table 1 gives the numerical coefficients of the JYIK operator with $x_1 = 1$ and $x_2 = 10$ for $r = 0, 1, 2, \dots, 109$. Tables 2, 3 and 4 give the numerical coefficients of the $(0, j)$ -factorizations of this JYIK operator for $j = 1, 2$, and 3. (We do not present the $(0, 4)$ -factorization.) Since the (i, j) -factorization $D_j^i = A_j^i B^{i+j}$ produced by the test program has the principal diagonal of A_j^i equal to $(1, 1, 1, \dots)$, we do not include values of $a_j(r)$.

Tables 5 through 8 give the results of the computation of the four desired solutions $J_r(1)$, $I_r(10)$, $(-)^r K_r(10)$ and $Y_r(1)$ for selected values of r ; see §4.0 for a full description of the tables. Initial values were taken from Tables 9.4 and 9.11 in [16]; they are given in parentheses. These tables also supply values of the four Bessel functions for certain other values of r in the range $0 \leq r \leq 100$. These entries all agree with our computed results, except for the tenth significant figure in some instances. This slight discrepancy is explained by the fact that the starting values that were used inevitably have an error of several units in the eleventh significant figure. Incidentally, because in this (and other examples) we have not attempted rigorous control of individual rounding errors, a guaranteed error bound cannot be given.

Nevertheless, the stability of the algorithm for all solutions in this example is firmly demonstrated.

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
0	.4940000000+004	-.8668000000+004	-.8880000000+003	.7092000000+004	-.9500000000+003
1	.1080000000+005	-.4567200000+005	.1885200000+005	.8736800000+005	-.4940000000+004
2	.1880000000+005	-.1218300000+006	.8086000000+005	.9972000000+005	-.1080000000+005
3	.2870000000+005	-.2508600000+006	.2175000000+006	.2040400000+006	-.1880000000+005
4	.4050000000+005	-.4441200000+006	.4724400000+006	.3684200000+006	-.2870000000+005
5	.5444000000+005	-.7164000000+006	.8976520000+006	.5886152000+006	-.4050000000+005
6	.7025000000+005	-.1080112000+007	.1554912000+007	.8887280000+006	-.5444000000+005
7	.8810000000+005	-.1548360000+007	.2515500000+007	.1266840000+007	-.7025000000+005
8	.1079000000+006	-.2134200000+007	.3860200000+007	.1746120000+007	-.8810000000+005
9	.1296000000+006	-.2889760000+007	.5679800000+007	.2382440000+007	-.1079000000+006
10	.1534000000+006	-.3711040000+007	.8072592000+007	.3086312000+007	-.1296000000+006
11	.1791800000+006	-.4728152000+007	.1114987200+008	.3868488000+007	-.1534000000+006
12	.2069000000+006	-.5915140000+007	.1592844000+008	.4839660000+007	-.1791800000+006
13	.2366000000+006	-.7285080000+007	.1983810000+008	.5960820000+007	-.2069000000+006
14	.2682000000+006	-.8851040000+007	.2571616000+008	.7241760000+007	-.2366000000+006
15	.3019400000+006	-.1063600000+008	.3200998200+008	.8694072000+007	-.2682000000+006
16	.3375800000+006	-.1262329200+008	.4127628200+008	.1082814800+008	-.3019400000+006
17	.3752000000+006	-.1495972000+008	.5178188000+008	.1215468000+008	-.3375800000+006
18	.4148000000+006	-.1788644000+008	.6500120000+008	.1418486000+008	-.3752000000+006
19	.4563800000+006	-.2007852000+008	.7662102000+008	.1642782000+008	-.4148000000+006
20	.4999400000+006	-.2259950200+008	.9283567200+008	.1899598200+008	-.4563800000+006
21	.5454800000+006	-.2638998200+008	.1108049200+009	.2159928000+008	-.4999400000+006
22	.5930000000+006	-.3090360000+008	.1208768200+009	.2454840000+008	-.5454800000+006
23	.6425000000+006	-.3623150000+008	.1541895000+009	.2775420000+008	-.5930000000+006
24	.6939000000+006	-.4216670000+008	.1808718000+009	.3123780000+008	-.6425000000+006
25	.7474000000+006	-.4878186000+008	.2096158120+009	.3497889200+008	-.6939000000+006
26	.8028000000+006	-.5614000000+008	.2437881820+009	.3901816000+008	-.7474000000+006
27	.8608000000+006	-.6427900000+008	.2798552600+009	.4335732000+008	-.8028000000+006
28	.9197000000+006	-.7326466000+008	.3198800000+009	.4800654000+008	-.8608000000+006
29	.9810000000+006	-.8300000000+008	.3640867400+009	.5297652000+008	-.9197000000+006
30	.1044400000+007	-.7128280000+008	.4148688520+009	.5827795200+008	-.9810000000+006
31	.1097000000+007	-.7012681200+008	.4685918120+009	.6392152800+008	-.1044400000+007
32	.1177000000+007	-.6854585800+008	.5282002000+009	.6991794000+008	-.1097000000+007
33	.1246400000+007	-.6628352000+008	.5932017000+009	.7627788000+008	-.1177000000+007
34	.1317000000+007	-.6349916000+009	.6640286000+009	.8301204000+008	-.1246400000+007
35	.1399040000+007	-.6011602400+009	.7410082920+009	.9018112000+008	-.1317000000+007
36	.1466180000+007	-.5624488200+009	.8244876720+009	.9764878800+008	-.1399040000+007

Table 1. Numerical coefficients of the JYIK operator with $x_1 = 1$ and $x_2 = 10$.

r	d ₀ (r)	d ₁ (r)	d ₂ (r)	d ₃ (r)	d ₄ (r)
37	1.154840000000+007	-.12902660400+009	.9148881400+009	.1055667600+009	-.14661600000+007
38	1.62260000000+007	-.13921600000+009	.1012405600+010	.1139047200+009	-.15484000000+007
39	1.70378000000+007	-.1493204400+009	.11175804600+010	.1237036600+009	-.16226000000+007
40	1.73694000000+007	-.1611797800+009	.1280742682+010	.1318743720+009	-.17037800000+007
41	1.87208000000+007	-.1729779920+009	.1352286582+010	.1415274400+009	-.17869400000+007
42	1.95920000000+007	-.18533361200+009	.14822616000+010	.1516402000+009	-.18720800000+007
43	2.04880000000+007	-.1983732400+009	.1622144700+010	.1622235600+009	-.19592000000+007
44	2.13938000000+007	-.2117964200+009	.1771295820+010	.1732879800+009	-.20488000000+007
45	2.23224000000+007	-.2259207200+009	.1900500872+010	.1848442320+009	-.21393800000+007
46	2.32748000000+007	-.2406592320+009	.2100201792+010	.1969030010+009	-.22322400000+007
47	2.42435000000+007	-.25600250000+009	.2280851020+010	.2094750000+009	-.23274800000+007
48	2.52335000000+007	-.27220310000+009	.2472909000+010	.2225700000+009	-.24243500000+007
49	2.62424000000+007	-.28869060000+009	.2676046180+010	.2362014000+009	-.25233500000+007
50	2.72744000000+007	-.30660165680+009	.2893142512+010	.2503771920+009	-.26242400000+007
51	2.83280000000+007	-.3242020720+009	.3122287452+010	.2651039600+009	-.27274400000+007
52	2.93960000000+007	-.34272018000+009	.3364779960+010	.2804074200+009	-.28328000000+007
53	3.04830000000+007	-.3621239600+009	.3621128800+010	.2962882400+009	-.29396000000+007
54	3.15930000000+007	-.38222464800+009	.3891851040+010	.3127471200+009	-.30483000000+007
55	3.27194000000+007	-.4031000000+009	.4177475052+010	.3298097520+009	-.31593000000+007
56	3.38650000000+007	-.42470001200+009	.4478537512+010	.3474618280+009	-.32719400000+007
57	3.50330000000+007	-.44720571600+009	.4795584900+010	.3657740400+009	-.33867800000+007
58	3.62240000000+007	-.4701882000+009	.5129178200+010	.3846970800+009	-.35036000000+007
59	3.74383000000+007	-.4940975600+009	.5479867900+010	.4042616400+009	-.36224000000+007
60	3.86730000000+007	-.51880069400+009	.5848244992+010	.4244784120+009	-.37448180000+007
61	3.99290000000+007	-.5444545400+009	.6234085972+010	.4453350800+009	-.38659400000+007
62	4.12111000000+007	-.57070069400+009	.663897940+010	.4669113600+009	-.39906000000+007
63	4.25244000000+007	-.5978806800+009	.705353100+010	.4891489200+009	-.41174000000+007
64	4.38770000000+007	-.6255587200+009	.747613832+010	.5120814600+009	-.42461000000+007
65	4.52694000000+007	-.6547468800+009	.7916135800+010	.5357196720+009	-.43767000000+007
66	4.66990000000+007	-.6845458800+009	.8372300832+010	.5600742480+009	-.45094400000+007
67	4.81670000000+007	-.7151905200+009	.8847235700+010	.5851550800+009	-.46440000000+007
68	4.96730000000+007	-.7472916820+009	.9339168800+010	.61097752600+009	-.47807000000+007
69	5.12180000000+007	-.7819198200+009	.9858866600+010	.6375480000+009	-.49193000000+007
70	5.28024000000+007	-.8182198200+009	1.0405800000+010	.6649480000+009	-.50590000000+007
71	5.44250000000+007	-.8566965820+009	1.112413400+010	.6929668800+009	-.52024400000+007
72	5.60880000000+007	-.89732358000+009	1.187017252+010	.7218441000+009	-.53466000000+007
73	5.66420000000+007	-.94183151500+009	1.2642565550+010	.7515126600+009	-.54938000000+007

Table 1 (Continued)

r	a ₀ (r)	b ₀ (r+1)	b ₁ (r+1)	b ₂ (r+1)	b ₃ (r+1)
0	.4940000000+004	- .5668000000+004	- .8280000000+003	.7092000000+004	- .9500000000+003
1	-.1255191509+001	- .2678666100+005	.2775381818+005	.3633791282+005	- .4940000000+004
2	-.032486646+000	- .2072127111+006	.9666109877+005	.977498762+005	- .1683000000+005
3	-.2591940754+000	- .2255127155+006	.2433828298+006	.20220199685+006	- .1820000000+005
4	-.1797627691+000	- .9005222222+006	.5087159919+006	.3600400644+006	- .2870000000+005
5	-.1359228905+000	- .6472356185+006	.946589562+006	.5322510274+006	- .4050000000+005
6	-.100103350+000	- .2299658127+006	.1618183865+007	.8798217654+006	- .4444000000+005
7	-.9014315871-001	- .402495674+007	.2594765193+007	.1261932585+007	- .7028000000+005
8	-.7663232655-001	- .2585495454+007	.3957285879+007	.1740773089+007	- .8810000000+005
9	-.6788216689-001	- .2585495454+007	.5795987789+007	.2326584466+007	- .1079000000+006
10	-.5934643266-001	- .2626262626+007	.6210668689+007	.3029908515+007	- .1296200000+006
11	-.5321580869-001	- .2626262626+007	.131060950+008	.3861587089+007	- .1534400000+006
12	-.4821215959-001	- .2626262626+007	.1521462544+008	.4882361931+007	- .1791600000+006
13	-.4469273756-001	- .2626262626+007	.208510588+008	.5952625111+007	- .2069000000+006
14	-.4266969696-001	- .2626262626+007	.255658225+008	.7283368525+007	- .2366000000+006
15	-.4199696969-001	- .2626262626+007	.3888168822+008	.8685184129+007	- .2682000000+006
16	-.4246969696-001	- .2626262626+007	.4158002921+008	.1081276889+008	- .3019400000+006
17	-.4383838383-001	- .2626262626+007	.5161908883+008	.1214449978+008	- .3375300000+006
18	-.4627383829-001	- .2626262626+007	.6387449516+008	.1417398379+008	- .3752000000+006
19	-.4918188922-001	- .2626262626+007	.8004021254831	.1641700789+008	- .4145000000+006
20	-.5270576405-001	- .2626262626+007	.9964081208+008	.1888456408+008	- .4568000000+006
21	-.5702222222-001	- .2626262626+007	.1200+008	.22788559814+008	- .5099940000+006
22	-.6222222222-001	- .2626262626+007	.200+008	.2788559811	- .5745000000+006
23	-.6825185885-001	- .2626262626+007	.300+008	.3345959599+008	- .6425000000+006
24	-.7522222222-001	- .2626262626+007	.400+008	.3996969696+008	- .7199900000+006
25	-.8325185885-001	- .2626262626+007	.500+008	.4804457191+008	- .8020000000+006
26	-.9206666666-001	- .2626262626+007	.600+008	.5782444444+008	- .8900000000+006
27	-.1019066666-001	- .2626262626+007	.700+008	.6945555555+008	- .9900000000+006
28	-.1141906666-001	- .2626262626+007	.800+008	.8300000000+008	- .1000000000+007
29	-.1282222222-001	- .2626262626+007	.900+008	.9888000000+008	- .1100000000+007
30	-.1441906666-001	- .2626262626+007	1.000+008	1.1777121212+008	- .1200000000+007
31	-.1619066666-001	- .2626262626+007	1.100+008	1.3999999999+008	- .1300000000+007
32	-.1819066666-001	- .2626262626+007	1.200+008	1.6666666666+008	- .1400000000+007
33	-.2044444444-001	- .2626262626+007	1.300+008	1.9999999999+008	- .1500000000+007
34	-.2299999999-001	- .2626262626+007	1.400+008	2.4000000000+008	- .1600000000+007
35	-.2583333333-001	- .2626262626+007	1.500+008	2.8888888888+008	- .1700000000+007
36	-.2899999999-001	- .2626262626+007	1.600+008	3.4666666666+008	- .1800000000+007

Table 2. Numerical coefficients of the (0,1)-factorization of the JYK operator with x₁ = 1 and x₂ = 10.

r	a ₀ (r)	b ₀ (r+1)	b ₁ (r+1)	b ₂ (r+1)	b ₃ (r+1)
37	1.422805490-001	-1.172764096+009	.9162221751+009	1.055469696+009	-1.14661000000+007
38	1.885560959-001	-1.286408144+009	.1018865915+010	1.133844344+009	-1.5434000000+007
39	1.846482564-001	-1.191113387+010	.1119113387+010	1.226495792+009	-1.6226000000+007
40	1.811111111-001	-1.465056089+009	.1242850858+010	1.318530959+009	-1.7087300000+007
41	1.277821428-001	-1.872307487+009	.1353971879+010	1.415056767+009	-1.7869500000+007
42	1.246006669-001	-1.684667385+009	.1484379885+010	1.516100135+009	-1.8720000000+007
43	1.215048350-001	-1.802254583+009	.1623988145+010	1.622007983+009	-1.9592000000+007
44	1.027057231-001	-1.19251827482+009	.1773220737+010	1.732647232+009	-2.0488000000+007
45	1.159396104-001	-2.053585230+009	.1932509543+010	1.848204600+009	-2.1393000000+007
46	1.1188373921-001	-2.187566729+009	.2102296499+010	1.962707603+009	-2.2324400000+007
47	1.1086309049-001	-2.327250577+009	.2283088045+010	2.094502577+009	-2.3274000000+007
48	1.086326277-001	-2.472755625+009	.2475169125+010	2.235456625+009	-2.4245700000+007
49	1.063841247-001	-2.624200674+009	.2679208187+010	2.361756674+009	-2.5235000000+007
50	1.041558249-001	-2.781704522+009	.2896597183+010	2.508509642+009	-2.6244000000+007
51	1.0199979219-001	-2.945338971+009	.3124886569+010	2.650822451+009	-2.7274100000+007
52	1.000000000-002	-3.121111111+009	.3367425305+010	2.808302020+009	-2.8323000000+007
53	1.000000000-002	-3.291756868+009	.3623871853+010	2.969300000+007	-2.9393000000+007
54	1.000000000-002	-3.474668396+009	.3894694137+010	3.127189117+009	-3.0482000000+007
55	1.000000000-002	-3.664268766+009	.4180419773+010	3.297810406+009	-3.1590000000+007
56	1.000000000-002	-3.859091521+009	.4481585509+010	3.474526295+009	-3.2719000000+007
57	1.000000000-002	-4.068885706+009	.4793738115+010	3.657744433+009	-3.3857000000+007
58	1.000000000-002	-4.274406310+009	.5132433335+010	3.846666689+009	-3.5000000000+007
59	1.000000000-002	-4.49487162+009	.5483236737+010	4.042309562+009	-3.6224000000+007
60	1.000000000-002	-4.716113011+009	.5851722313+010	4.244747444+009	-3.7431000000+007
61	1.000000000-002	-4.948104460+009	.6238477559+010	4.454502664+009	-3.8699000000+007
62	1.000000000-002	-5.181580309+009	.6644093375+010	4.666879109+009	-3.9990000000+007
63	1.000000000-002	-5.424665988+009	.7069172458+010	4.886879109+009	-4.1174000000+007
64	1.000000000-002	-5.679946040+009	.7514303337+010	5.120560660+009	-4.2440000000+007
65	1.000000000-002	-5.948210225+009	.7979393939+010	5.366040000+009	-4.3740000000+007
66	1.000000000-002	-6.228203705+009	.846496479+010	5.624000000+009	-4.5000000000+007
67	1.000000000-002	-6.519300604+009	.8970303030+010	5.895606600+009	-4.6340000000+007
68	1.000000000-002	-6.822000000+009	.9500000000+010	6.180000000+009	-4.7700000000+007
69	1.000000000-002	-7.138000000+009	1.0000000000+010	6.478000000+009	-4.9000000000+007
70	1.000000000-002	-7.460000000+009	1.0500000000+010	6.780000000+009	-5.0300000000+007
71	1.000000000-002	-7.780000000+009	1.1000000000+010	7.090000000+009	-5.1600000000+007
72	1.000000000-002	-8.100000000+009	1.1500000000+010	7.410000000+009	-5.2900000000+007
73	1.000000000-002	-8.420000000+009	1.2000000000+010	7.740000000+009	-5.4200000000+007

Table 2 (Continued)

r	a ₀ (r)	b ₀ (r+1)	b ₁ (r+1)	b ₂ (r+1)	b ₃ (r+1)
74	-.66372769655-002	-.8688316899+009	1321188869+010	78194448899+009	-.56420000000+007
75	-.68424252996-002	-.9035600749+009	13921073669+011	8132273669+009	-.57924370000+007
76	-.69569169291-002	-.9326282198+009	1452845117+010	8458831678+009	-.59449400000+007
77	-.70769322005-002	-.9619936048+009	1513645411+011	8782729240+009	-.60998200000+007
78	-.720200032005-002	-.9918301310+010	1622067676+010	9120573497+009	-.62558300000+007
79	-.73323362665-002	-.1051890015+010	1704705224+010	9466971346+009	-.64142000000+007
80	-.74676999999-002	-.1190100000+010	1792945669+010	9822029716+009	-.65748300000+007
81	-.76082336666-002	-.1338131211+010	1887065494+010	1018555553+010	-.67369400000+007
82	-.77547275429-002	-.1496121111+010	198800424+010	1055855556+010	-.69012000000+007
83	-.79076266219-002	-.1664215511+010	2096272727+010	1094028274+010	-.70679600000+007
84	-.80669992609-002	-.1842818211+010	2212121212+010	1133310067+010	-.72385900000+007
85	-.82329992609-002	-.2032414444+010	2336736523+010	1172461111	-.74096100000+007
86	-.84056992609-002	-.2233616666+010	2469465708+010	12146111	-.75826300000+007
87	-.85843992609-002	-.2446916666+010	2611911638+010	12594611	-.77575200000+007
88	-.87692992609-002	-.2672416666+010	2763636595+010	1306769715	-.79346940000+007
89	-.89605992609-002	-.2910416666+010	2925636666+010	1356571052+010	-.81107100000+007
90	-.91585992609-002	-.3161416666+010	3098636666+010	1408822464+010	-.82872300000+007
91	-.93635992609-002	-.3425916666+010	3283836666+010	1463529846+010	-.84669400000+007
92	-.95759992609-002	-.3704416666+010	3481336666+010	1520727454+010	-.86483530000+007
93	-.97954992609-002	-.4006416666+010	3691336666+010	1580426999+010	-.88319700000+007
94	-.10022992609-002	-.4332416666+010	3914166666+010	1642727454+010	-.90170000000+007
95	-.10219992609-002	-.4682916666+010	4150166666+010	1707699999+010	-.92036940000+007
96	-.10439992609-002	-.5058416666+010	4400166666+010	1775366666+010	-.93920000000+007
97	-.10684992609-002	-.5459416666+010	4663666666+010	1845699999+010	-.95820000000+007
98	-.10949992609-002	-.5886416666+010	4940666666+010	1918833333+010	-.97736940000+007
99	-.11234992609-002	-.6340416666+010	5231666666+010	2000000000+010	-.99670000000+007
100	-.11539992609-002	-.6822916666+010	5538666666+010	2090000000+010	-.10127000000+007
101	-.11864992609-002	-.7335416666+010	5861666666+010	2189000000+010	-.10399000000+007
102	-.12209992609-002	-.7878416666+010	6201666666+010	2296000000+010	-.10686000000+007
103	-.12574992609-002	-.8452416666+010	6561666666+010	2411000000+010	-.10989000000+007
104	-.12959992609-002	-.9058416666+010	6941666666+010	2534000000+010	-.11309000000+007
105	-.13364992609-002	-.9696416666+010	7341666666+010	2665000000+010	-.11646000000+007
106	-.13789992609-002	-.1036416666+010	7761666666+010	2804000000+010	-.12000000000+007
107	-.14234992609-002	-.1124416666+010	8201666666+010	2951000000+010	-.12371000000+007
108	-.14699992609-002	-.1220916666+010	8661666666+010	3106000000+010	-.12759000000+007
109	-.15184992609-002	-.1326416666+010	9141666666+010	3269000000+010	-.13164000000+007
110	-.15689992609-002	-.1440916666+010	9641666666+010	3440000000+010	-.13586000000+007
111	-.16214992609-002	-.1564416666+010	1016166666+010	3619000000+010	-.14035000000+007
112	-.16759992609-002	-.1696916666+010	1069166666+010	3806000000+010	-.14500000000+007
113	-.17324992609-002	-.1838416666+010	1123166666+010	4000000000+010	-.14981000000+007
114	-.17909992609-002	-.1989916666+010	1178166666+010	4201000000+010	-.15479000000+007
115	-.18514992609-002	-.2151416666+010	1234166666+010	4409000000+010	-.15994000000+007
116	-.19139992609-002	-.2323916666+010	1291166666+010	4624000000+010	-.16526000000+007
117	-.19784992609-002	-.2507416666+010	1349166666+010	4846000000+010	-.17075000000+007
118	-.20449992609-002	-.2701916666+010	1408166666+010	5075000000+010	-.17641000000+007
119	-.21134992609-002	-.2907416666+010	1468166666+010	5311000000+010	-.18224000000+007
120	-.21839992609-002	-.3123916666+010	1529166666+010	5554000000+010	-.18824000000+007
121	-.22564992609-002	-.3351416666+010	1591166666+010	5804000000+010	-.19440000000+007
122	-.23309992609-002	-.3590916666+010	1654166666+010	6061000000+010	-.20071000000+007
123	-.24074992609-002	-.3842416666+010	1718166666+010	6325000000+010	-.20717000000+007
124	-.24859992609-002	-.4105916666+010	1783166666+010	6596000000+010	-.21378000000+007
125	-.25664992609-002	-.4381416666+010	1849166666+010	6874000000+010	-.22054000000+007
126	-.26489992609-002	-.4668916666+010	1916166666+010	7159000000+010	-.22745000000+007
127	-.27334992609-002	-.4968416666+010	1984166666+010	7451000000+010	-.23451000000+007
128	-.28199992609-002	-.5279916666+010	2053166666+010	7750000000+010	-.24172000000+007
129	-.29084992609-002	-.5603416666+010	2123166666+010	8056000000+010	-.24908000000+007
130	-.29989992609-002	-.5948916666+010	2194166666+010	8369000000+010	-.25659000000+007
131	-.30914992609-002	-.6316416666+010	2266166666+010	8689000000+010	-.26426000000+007
132	-.31859992609-002	-.6705916666+010	2339166666+010	9016000000+010	-.27209000000+007
133	-.32824992609-002	-.7117416666+010	2413166666+010	9350000000+010	-.28008000000+007
134	-.33809992609-002	-.7550916666+010	2488166666+010	9691000000+010	-.28823000000+007
135	-.34814992609-002	-.8006416666+010	2564166666+010	10039000000+010	-.29654000000+007
136	-.35839992609-002	-.8483916666+010	2641166666+010	10484000000+010	-.30501000000+007
137	-.36884992609-002	-.9084416666+010	2719166666+010	10946000000+010	-.31364000000+007
138	-.37949992609-002	-.9708916666+010	2798166666+010	11425000000+010	-.32244000000+007
139	-.39034992609-002	-.1036416666+010	2878166666+010	11921000000+010	-.33141000000+007
140	-.40139992609-002	-.1098916666+010	2959166666+010	12434000000+010	-.34054000000+007
141	-.41264992609-002	-.1166416666+010	3041166666+010	12964000000+010	-.34984000000+007
142	-.42409992609-002	-.1238916666+010	3124166666+010	13511000000+010	-.35930000000+007
143	-.43574992609-002	-.1316416666+010	3208166666+010	14074000000+010	-.36891000000+007
144	-.44759992609-002	-.1398916666+010	3293166666+010	14653000000+010	-.37867000000+007
145	-.45964992609-002	-.1486416666+010	3379166666+010	15248000000+010	-.38858000000+007
146	-.47189992609-002	-.1578916666+010	3466166666+010	15859000000+010	-.39864000000+007
147	-.48434992609-002	-.1676416666+010	3554166666+010	16486000000+010	-.40886000000+007
148	-.49699992609-002	-.1778916666+010	3643166666+010	17129000000+010	-.41924000000+007
149	-.50984992609-002	-.1886416666+010	3733166666+010	17788000000+010	-.42977000000+007
150	-.52289992609-002	-.1998916666+010	3824166666+010	18463000000+010	-.44046000000+007
151	-.53614992609-002	-.2116416666+010	3916166666+010	19154000000+010	-.45131000000+007
152	-.54959992609-002	-.2238916666+010	4009166666+010	19861000000+010	-.46232000000+007
153	-.56324992609-002	-.2366416666+010	4103166666+010	20584000000+010	-.47349000000+007
154	-.57709992609-002	-.2498916666+010	4198166666+010	21323000000+010	-.48482000000+007
155	-.59114992609-002	-.2636416666+010	4294166666+010	22078000000+010	-.49631000000+007
156	-.60539992609-002	-.2778916666+010	4391166666+010	22849000000+010	-.50796000000+007
157	-.62084992609-002	-.2926416666+010	4489166666+010	23636000000+010	-.51977000000+007
158	-.63649992609-002	-.3078916666+010	4588166666+010	24439000000+010	-.53174000000+007
159	-.65229992609-002	-.3236416666+010	4688166666+010	25258000000+010	-.54387000000+007
160	-.66834992609-002	-.3398916666+010	4789166666+010	26093000000+010	-.55616000000+007
161	-.68459992609-002	-.3566416666+010	4891166666+010	26944000000+010	-.56861000000+007
162	-.70104992609-002	-.3738916666+010	4994166666+010	27811000000+010	-.58122000000+007
163	-.71769992609-002	-.3916416666+010	5098166666+010	28694000000+010	-.59399000000+007
164	-.73454992609-002	-.4098916666+010	5203166666+010	29593000000+010	-.60692000000+007
165	-.75159992609-002	-.4286416666+010	5309166666+010	30508000000+010	-.61991000000+007
166	-.76884992609-002	-.4478916666+010	5416166666+010	31439000000+010	-.63296000000+007
167	-.78629992609-002	-.4676416666+010	5524166666+010	32386000000+010	-.64607000000+007
168	-.80394992609-002	-.4878916666+010	5633166666+010	33349000000+010	-.65924000000+007
169	-.82179992609-002	-.5086416666+010	5743166666+010	34328000000+010	-.67247000000+007
170	-.83984992609-002	-.5298916666+010	5854166666+010	35323000000+010	-.68576000000+007
171	-.85809992609-002	-.5516416666+010	5966166666+010	36334000000+010	-.69911000000+007
172	-.87654992609-002	-.5738916666+010	6079166666+010	37361000000+010	-.71252000000+007
173	-.89519992609-002	-.5966416666+010	6193166666+010	38404000000+010	-.72609000000+007
174	-.91399992609-002	-.6198916666+010	6308166666+010	39463000000+010	-.73982000000+007
175	-.93299992609-002	-.6436416666+010	6424166666+010	40538000000+010	-.75371000000+007
176	-.95219992609-002	-.6678916666+010	6541166666+010	41629000000+010	-.76776000000+007
177	-.97159992609-002	-.6926416666+010	6659166666+010	42736000000+010	-.78197000000+007
178	-.99119992609-002	-.7178916666+010	6778166666+010	43859000000+010	-.79634000000+007
179	-.10129992609-002	-.7436416666+010	6898166666+010	44998000000+010	-.81087000000+007
180	-.10359992609-002	-.7698916666+010	7019166666+010	46153000000+010	-.82556000000+007
181	-.10599992609-002	-.7966416666+010	7141166666+010	47324000000+010	-.84041000000+007

r	$a_0(r)$	$a_1(r)$	$b_0(r+2)$	$b_1(r+2)$	$b_2(r+2)$
0	.4940000000+004	-.8660000000+004	-.6800000000+003	.7092000000+004	-.9800000000+003
1	.1088000000+005	.5148243243+002	-.3459068105+006	.8777173372+005	-.4940000000+004
2	-.21171717+002	-.3171549407-001	.6698456698+005	.9931632546+005	-.1088000000+005
3	-.8297082852-001	-.3628350972+001	.5774950653+006	.1653500144+006	-.1088000000+005
4	.05011111070+000	-.0732854989+000	.6234442497+006	.3470012926+006	-.2870010000+005
5	.9426920379-001	-.172116481+001	.1306044193+007	.5524548570+006	-.4052000000+005
6	.127286041+000	-.8564866241+000	.2031289773+007	.0489786099+006	-.5444000000+005
7	.674743101-001	-.685181009+000	.3100925670+007	.1224344710+007	-.7023000000+005
8	.501106604-001	-.609161619	.4701912519+007	.1698029900+007	-.0510000000+005
9	.40787828292+004	-.5564866241+000	.6729702962+007	.2273009909+007	-.1079000000+006
10	.8263352931-001	-.5564866241+000	.9350461792+007	.2975922340+007	-.1296000000+006
11	.222724989-001	-.6121266545+000	.1267635347+008	.3202072886+007	-.1534400000+006
12	.222724989-001	-.478225161+000	.1682521303+008	.4767236353+007	-.1791000000+006
13	.18891119499991	-.3720364400+000	.2192522090+008	.5882101851+007	-.2069000000+006
14	.599551757-001	-.4071590138+000	.2311400004+008	.7157513000+007	-.2366000000+006
15	.1200754055-001	-.306084656032+000	.3553369039+008	.8608968929+007	-.2632000000+006
16	.1200754055-001	-.3576169924+000	.4485599583+008	.1023220654+008	-.3019400000+006
17	.9551610588-002	-.33699663175+000	.5473218974+008	.1205298585+008	-.3374300000+006
18	.8333272830-002	-.3184689239+000	.6634257072+008	.1407684113+008	-.3752000000+006
19	.155601156562	-.30103274312+000	.8037345680+008	.1631461184+008	-.4140300000+006
20	.7470831671-002	-.20988127124+000	.9701368018+008	.1877693753+008	-.4563300000+006
21	.619224974-002	-.27423367297+000	.11154828397+009	.2147450020+008	-.4999400000+006
22	.5563397166-002	-.260603030470+000	.1364797435+009	.2441801072+008	-.5454500000+006
23	.5084357140-002	-.24942386474+000	.1602327115+009	.2761814439+008	-.5936000000+006
24	.4664715419-002	-.2339703185+000	.1869745705+009	.3103559060+008	-.6425000000+006
25	.4296969696969696	-.2293370627+000	.2504038114+009	.3403104294+008	-.6939300000+006
26	.3965472663-002	-.2204336897+000	.2876048831+009	.3685519212+008	-.7474400000+006
27	.3672239999999999	-.2121788852+000	.3283202223+009	.3919372901+008	-.8023000000+006
28	.3412130685-002	-.1973614569+000	.3743317136+009	.4234264459+008	-.8603000000+006
29	.2964696710-002	-.1927614569+000	.4244230950+009	.5200672994+008	-.9197000000+006
30	.2723275160-002	-.1906839105+000	.4794105425+009	.5310257623+008	-.9810000000+006
31	.2599259362-002	-.1844429202+000	.4794105425+009	.6078141679+008	-.1044400000+007
32	.2442019699-002	-.1703080903+000	.5305861468+009	.7603579379+008	-.1109200000+007
33	.2292000409-002	-.1679065460+000	.6052730736+009	.1172100000+007	-.1246400000+007
34	.2292000409-002	-.1680242591+000	.7545067545+009	.8392791054+008	-.1317600000+007
35	.2166300443-002	-.1584187652+000	.8867361923+009	.9743704935+008	-.1390940000+007

Table 3. Numerical coefficients of the (0,2)-factorization of the JYIK operator with $x_1 = 1$ and $x_2 = 10$.

r	$a_0(r)$	$a_1(r)$	$b_0(r+2)$	$b_1(r+2)$	$b_2(r+2)$
37	.2045574795-002	-.15405831999+000	.9293463247+009	.1053524812+009	-.1466130000+007
38	.1934577302-002	-.1499230312+000	.102303054+010	.1136849059+009	-.1543400000+007
39	.1832324203-002	-.1460056740+000	.1134181733+010	.12244450148+009	-.1623000000+007
40	.1737925201-002	-.1422552717+000	.1243167437+010	.1316443499+009	-.1703730000+007
41	.1650595955-002	-.1387474906+000	.1370554416+010	.1412910523+009	-.1786940000+007
42	.1569661203-002	-.1353793425+000	.1501746643+010	.1513933652+009	-.1872303000+007
43	.1494504689-002	-.1321690319+000	.1642157546+010	.1619761290+009	-.1959200000+007
44	.1424594494-002	-.1291058229+000	.1792216040+010	.1730350359+009	-.2048300000+007
45	.1359455434-002	-.1261792338+000	.1952335533+010	.1845857777+009	-.2139300000+007
46	.1298664742-002	-.1233323363+000	.2132297908+010	.1966390462+009	-.2232440000+007
47	.1241345295-002	-.1207050172+000	.2304588996+010	.2092055333+009	-.2327430000+007
48	.1183309849-002	-.1181403007+000	.2497627253+010	.2222959303+009	-.2424500000+007
49	.1133306097-002	-.1156911330+000	.2702556432+010	.2359209306+009	-.2523850000+007
50	.1092012425-002	-.1133217499+000	.2919080132+010	.2500912246+009	-.2624435000+007
51	.104934261-002	-.1111111111+000	.3150064152+010	.2643175045+009	-.2727440000+007
52	.1006650913-002	-.1088777618+000	.3393615339+010	.2801104624+009	-.2832330000+007
53	.9679999999999999-003	-.1067332373+000	.3651042157+010	.2959807901+009	-.2939300000+007
54	.93008392339-003	-.1047665036+000	.3922862574+010	.3124391796+009	-.3044320000+007
55	.8994696022-003	-.1028244167+000	.4209604062+010	.3294963236+009	-.3159030000+007
56	.863334034-003	-.1009524233+000	.4511808592+010	.3471629112+009	-.3271940000+007
57	.8322073002-003	-.9914691230+000	.4833000766+010	.3654496373+009	-.3386700000+007
58	.8023718231-003	-.9776191019+000	.5164772233+010	.3843671927+009	-.3503600000+007
59	.7749042782-003	-.9672169989+000	.5516662896+010	.4039262995+009	-.3622400000+007
60	.7496605937-003	-.95969993882-003	.5836254369+010	.4241375595+009	-.3743100000+007
61	.7258366090-003	-.9540000000-003	.6274131420+010	.4450117547+009	-.3865940000+007
62	.7036999999999999-003	-.949332121-000	.6680337953+010	.4665595470+009	-.3990600000+007
63	.6767236966-003	-.9453148911-000	.7107127437+010	.4887916285+009	-.4117400000+007
64	.6559911591-003	-.9410663420-000	.7599595959	.5117186910+009	-.4246100000+007
65	.6345454425-003	-.9372566015-000	.8020517056+010	.5353514264+009	-.4376700000+007
66	.6140922600-003	-.9338072392-000	.8503921623+010	.5597003263+009	-.4509240000+007
67	.5960088325-003	-.9308934206-000	.9019310288+010	.5847766042+009	-.4644000000+007
68	.5799999999999999-003	-.9283311100-000	.9532357924+010	.6105905903+009	-.4780200000+007
69	.566024000159-003	-.9260704359-000	.1010870121+011	.6371529374+009	-.4919300000+007
70	.5539999999999999-003	-.9240209643-000	.1060901313+011	.6644744173+009	-.5059300000+007
71	.5429999999999999-003	-.9220764050-000	.1123998821+011	.6925657217+009	-.5202400000+007
72	.5319999999999999-003	-.9203473623-000	.1192430651+011	.7214375430+009	-.5346900000+007
73	.5209999999999999-003	-.9187333333466-000	.125800653355+011	.7511005760+009	-.5496300000+007

Table 3 (Continued)

r	$a_0(r)$	$a_1(r)$	$b_0(r+2)$	$b_1(r+2)$	$b_2(r+2)$
74	.4857710519-003	-.7599822409-001	.1336375539+011	.7815655037+009	-.5642000000+007
75	.4725461927-003	-.7496720711-001	.1397432359+011	.8128480270+009	-.5792340000+007
76	.4593531727-003	-.7396866641-001	.1471303520+011	.8449483850+009	-.5944940000+007
77	.4476633350-003	-.7298652500-001	.1543977677+011	.8778786195+009	-.6099380000+007
78	.4359320736-003	-.7203476491-001	.1627314537+011	.9116580726+009	-.6255300000+007
79	.4246381597-003	-.7110741142-001	.1710594353+011	.9462923363+009	-.6414300000+007
80	.41383641264-003	-.7020354096-001	.1796493945+011	.9817937524+009	-.6574530000+007
81	.403433979-003	-.6932227551-001	.1885590653+011	.1018171363+010	-.6736940000+007
82	.3934107331-003	-.6846278130-001	.1977962903+011	.1055406410+010	-.6901293000+007
83	.3837471233-003	-.6762426324-001	.2073690059+011	.1093399556+010	-.7067666000+007
84	.3743342722-003	-.6680959654-001	.2172352376+011	.1132671552+010	-.7235600000+007
85	.3654567392-003	-.6600716768-001	.2273531172+011	.1172660091+010	-.7335600000+007
86	.3567673936-003	-.6522713320-001	.2378130363+011	.1213504304+010	-.7406100000+007
87	.3484417220-003	-.6446535713-001	.2486127680+011	.1255447413+010	-.7484400000+007
88	.3403757916-003	-.6372106471-001	.2597011935+011	.1298261611+010	-.7522300000+007
89	.3325353407-003	-.6299870915-001	.2710069595+011	.1342033039+010	-.7522300000+007
90	.3250601330-003	-.6228272056-001	.2824450263+011	.1386787540+010	-.7522300000+007
91	.3177877477-003	-.6158554222-001	.2940119356+011	.1432522065+010	-.7522300000+007
92	.3107355553-003	-.6090766924-001	.3059760953+011	.1479248126+010	-.7522300000+007
93	.3038540637-003	-.6024262730-001	.3183361837+011	.1526990064+010	-.7522300000+007
94	.2972731309-003	-.5959189143-001	.3311759216+011	.1575723906+010	-.7522300000+007
95	.2910330093-003	-.5896550246-001	.3444209940+011	.1625503593+010	-.7522300000+007
96	.2849350790-003	-.5835015900-001	.3581275715+011	.1676315414+010	-.7522300000+007
97	.2790322177-003	-.5774233520-001	.3722366137+011	.1728175047+010	-.7522300000+007
98	.2622555313-003	-.5653727919-001	.4130029630+011	.1781093133+010	-.7522300000+007
100	.2567500999-003	-.5596408325-001	.4295352699+011	.1835080530+010	-.7522300000+007
101	.2516333952-003	-.5540179790-001	.4466621620+011	.1890147764+010	-.7522300000+007
102		-.5485807196-001	.4642443319+011	.1946305552+010	-.7522300000+007
				.2003356467+010	-.7522300000+007

Table 3 (Concluded).

r	$a_0(r)$	$a_1(r)$	$a_2(r)$	$b_0(r+3)$	$b_1(r+3)$
0	.4940600000+004	-.8665000000+004	-.8880000000+003	.7092000000+004	-.9800000000+002
1	.1000000000+005	-.4567200000+005	.2658200000+001	.3997004230+005	-.4940000000+004
2	.1890000000+005	-.1718556120+002	.1594028084+001	.1075944935+006	-.1000000000+005
3	.4046813311+001	-.6164007260+001	.1738469966+001	.2287543532+006	-.1880000000+005
4	.1015184176+001	-.4081667709+001	.1912950817+001	.3990883475+006	-.2870000000+005
5	.5059788255+000	-.3177155479+001	.2098037422+001	.6463656740+006	-.4080000000+005
6	.3140941670+000	-.2689668183+001	.2286196075+001	.9765018867+006	-.5444000000+005
7	.220509979+000	-.2385691458+001	.2476891031+001	.1401631945+007	-.7028000000+005
8	.166920697+000	-.2028655880+001	.2669360000+001	.1938782677+007	-.8810000000+005
9	.1328005695+000	-.1915033116+001	.2863158378+001	.2584634253+007	-.1079000000+006
10	.1094024355+000	-.1915033116+001	.3057964689+001	.3866266639+007	-.1296000000+006
11	.926577490+001	-.1826137504+001	.3253554679+001	.4290403971+007	-.1534000000+006
12	.8004046282+001	-.1754615377+001	.3449764714+001	.5868991898+007	-.1791800000+006
13	.7028558426+001	-.1695867554+001	.3646473889+001	.6613895102+007	-.2069000000+006
14	.62580016946+001	-.1646760796+001	.3843588540+001	.8036990831+007	-.2366000000+006
15	.5623774551+001	-.1605107302+001	.4041040949+001	.9650182145+007	-.2682000000+006
16	.5104102273+001	-.1569288572+001	.4238726602+001	.1146533591+008	-.3019400000+006
17	.4668409465+001	-.1538279208+001	.4436741776+001	.1349480981+008	-.3375800000+006
18	.4298364464+001	-.1511069853+001	.4634912674+001	.1574901382+008	-.3752000000+006
19	.3980233367+001	-.1487094275+001	.4833256726+001	.1824131792+008	-.4148000000+006
20	.3704420824+001	-.1465648677+001	.5031750502+001	.2098310211+008	-.4563000000+006
21	.3469531950+001	-.1446498870+001	.5230374609+001	.2398624686+008	-.4999400000+006
22	.3258861602+001	-.1429250793+001	.5429112357+001	.2726268068+008	-.5434800000+006
23	.3061987673+001	-.1413686181+001	.5627951628+001	.3082413505+008	-.5930000000+006
24	.2893241358+001	-.1399488885+001	.5826879384+001	.3468268947+008	-.6425000000+006
25	.2741623397+001	-.1386459488+001	.6025806288+001	.3885002394+008	-.6939800000+006
26	.2604712189+001	-.1374561648+001	.6224963390+001	.4338816844+008	-.7474400000+006
27	.2480491719+001	-.1363611588+001	.6424104914+001	.481895298+008	-.8028800000+006
28	.2367308970+001	-.1353500470+001	.6628880389+001	.5332425754+008	-.8608000000+006
29	.2268778178+001	-.1344135527+001	.6828552760+001	.5884596214+008	-.9197000000+006
30	.2165794857+001	-.1335437171+001	.7021049255+001	.6478594676+008	-.9810000000+006
31	.2081191659+001	-.1327336739+001	.7221188286+001	.7100609140+008	-.1044400000+007
32	.2000307170+001	-.1319774676+001	.7420566110+001	.7766827607+008	-.1109700000+007
33	.1928859962+001	-.1312669077+001	.7619979412+001	.8478488075+008	-.1177100000+007
34	.1855728113+001	-.1306066499+001	.7819425242+001	.9221620545+008	-.1246400000+007
35	.1790872761+001	-.1299838098+001	.8018990071+001	.1001258702+009	-.1317600000+007
36	.1730324795+001	-.1293960249+001	.8218404243+001	.1084750149+009	-.1390900000+007

Table 4. Numerical coefficients of the (0,3)-factorization of the JYK operator with $x_1 = 1$ and $x_2 = 10$.

r	a ₀ (r)	a ₁ (r)	a ₂ (r)	b ₀ (r+3)	b ₁ (r+3)
37	1.673674007-001	-.1260430044+001	.8417932944+001	.1172755996+009	-.14661200000+007
38	1.620560198-001	-.1263203568+001	.8617485170+001	.1263595044+009	-.15434000000+007
39	1.570665536-001	-.1270359019+001	.8817059205+001	.1362756092+009	-.16226600000+007
40	1.523709966-001	-.1273574197+001	.9016658494+001	.1465047940+009	-.17037000000+007
41	1.479463126-001	-.1269129171+001	.9216266625+001	.1572299388+009	-.17869400000+007
42	1.437643084-001	-.1264905999+001	.9415897317+001	.1684659236+009	-.18720000000+007
43	1.398111246-001	-.1260039435+001	.9615544401+001	.1802246284+009	-.19592000000+007
44	1.360669614-001	-.1257061969+001	.9815206899+001	.1926179332+009	-.20463000000+007
45	1.325158200-001	-.1253413155+001	.1001488856+002	.2053577189+009	-.21393000000+007
46	1.291482820-001	-.1249929948+001	.1021457377+002	.2163755823+009	-.22324400000+007
47	1.259868193-001	-.1246601330+001	.1041427661+002	.2272742477+009	-.23274000000+007
48	1.230331841-001	-.1243417237+001	.1061399132+002	.238242525+009	-.24245000000+007
49	1.199780131-001	-.1240368460+001	.1081371721+002	.24924192574+009	-.25235000000+007
50	1.17192108-001	-.1237440550+001	.1101345363+002	.26024169422+009	-.26244000000+007
51	1.145432413-001	-.1234643768+001	.1121319999+002	.27131696422+009	-.27274000000+007
52	1.095806133-001	-.1231952963+001	.1141295570+002	.28245377371+009	-.28323000000+007
53	1.072555081-001	-.1229367560+001	.1161272034+002	.2937440768+009	-.29393000000+007
54	1.050026207-001	-.1226831490+001	.1181249836+002	.30524675817+009	-.30482000000+007
55	1.008009527-001	-.1224409136+001	.1201227483+002	.3169425666+009	-.31590000000+007
56	1.000000000-001	-.1222183300+001	.1221206204+002	.328840607115+009	-.32719100000+007
57	1.000000000-001	-.1220077859-001	.1241185852+002	.3409384364+009	-.33857000000+007
58	1.000000000-001	-.12179665153+001	.1261166101+002	.35324000000+007	-.35036000000+007
59	1.000000000-001	-.121583362+001	.1281146996+002	.36574000000+007	-.36224000000+007
60	1.000000000-001	-.1213703665+001	.1301128503+002	.3784104911+009	-.37431000000+007
61	1.000000000-001	-.12115736523+001	.1321106006+002	.3912096360+009	-.38659400000+007
62	1.000000000-001	-.12094436577+001	.1341093268+002	.4040396360+009	-.39906000000+007
63	1.000000000-001	-.12073121212+001	.1361076453+002	.4169000000+007	-.41174000000+007
64	1.000000000-001	-.12051802282+001	.1381060152+002	.42979452307+009	-.42461000000+007
65	1.000000000-001	-.12030478385+001	.1401044333+002	.44271000000+007	-.43747000000+007
66	1.000000000-001	-.12009153025+001	.1421028983+002	.45564000000+007	-.45094000000+007
67	1.000000000-001	-.12000000000+001	.1441014033+002	.46857000000+007	-.46440000000+007
68	1.000000000-001	-.1199988545+001	.1460999608+002	.48150000000+007	-.47787000000+007
69	1.000000000-001	-.1199977032+001	.1480985531+002	.49443000000+007	-.49130000000+007
70	1.000000000-001	-.1199965416+001	.1500971850+002	.50736000000+007	-.50473000000+007
71	1.000000000-001	-.1199953309+001	.1520958334+002	.52029000000+007	-.51816000000+007
72	1.000000000-001	-.1199941202+001	.1540945596+002	.53322000000+007	-.53159000000+007
73	1.000000000-001	-.1199929095+001	.1560932994+002	.54615000000+007	-.54502000000+007

Table 4 (Continued)

r	a ₀ (r)	a ₁ (r)	a ₂ (r)	b ₀ (r+3)	b ₁ (r+3)
74	.75233429860-002	-.1191652008+001	.15809207224+002	.2659308800+009	-.5642000000+007
75	.7412522924-002	-.11909696942+001	.1600908771+002	.90858926649+009	-.5792400000+007
76	.73040262184-002	-.1189175734+001	.1620097126+002	.93926240093+009	-.5944900000+007
77	.72000950526-002	-.1187987173+001	.16408885775+002	.9758621943+009	-.6099400000+007
78	.709879243-002	-.1186829825+001	.1660874707+002	.1013400500+010	-.6255800000+007
79	.69999226521-002	-.11857202524+001	.1680863913+002	.1091340190+010	-.6414200000+007
80	.69005572202-002	-.11846460413+001	.1700853381+002	.11691340190+010	-.6574450000+007
81	.680222293-002	-.11835480686+001	.17208433104+002	.123176519+010	-.6733690000+007
82	.67189585833-002	-.11824480686+001	.1740838071+002	.13131765335+010	-.6901200000+007
83	.66300000217-002	-.11813174153+001	.17608323274+002	.1410101010	-.7067600000+007
84	.65459761063-002	-.11801478258+001	.1780818704+002	.1509004529+010	-.7235900000+007
85	.64697750356-002	-.11789480604+001	.1800804055+002	.16033443114+010	-.7406100000+007
86	.63977272525-002	-.11777826220+001	.1820795218+002	.17043919259+010	-.7578440000+007
87	.63297750356-002	-.1176630353+001	.1840786205+002	.1809543304+010	-.7752600000+007
88	.62621912727-002	-.11754973031+001	.1860777553+002	.191443015749+010	-.7928900000+007
89	.61943644212-002	-.11743773331+001	.18807699911+002	.20191661354+010	-.8107100000+007
90	.61269999233-002	-.11732633233+001	.19007606855+002	.21241339039+010	-.8287200000+007
91	.60600000000-002	-.1172150052+001	.1920752479+002	.222901034169+010	-.8469440000+007
92	.599260704935-002	-.11710326331+001	.1940744477+002	.2334134169+010	-.8653500000+007
93	.59250925001-002	-.11699141111	.19607336643+002	.2444134169+010	-.8839700000+007
94	.58569250001-002	-.11687961496+001	.1980728972+002	.2559179374+010	-.9027800000+007
95	.57893040667-002	-.116767026744+001	.2000721459+002	.2678665654+010	-.9217000000+007
96	.57233000000-002	-.1166541720+001	.2020714100+002	.280220109+010	-.9409900000+007
97	.56583000000-002	-.116541667877+001	.2040706089+002	.293055099+010	-.9603900000+007
98	.55949999999-002	-.1164294697+001	.2060699822+002	.30639542504+010	-.9800000000+007
99	.55339700000-002	-.11631744397+001	.2080692895+002	.320279445+010	-.9998000000+007
100	.54736000000-002	-.11620541720+001	.2100686104+002	.33468104314+010	-.1019790000+008
101	.5414527435-002	-.11609341627+001	.2120679445+002	.349627034+010	-.1039994000+008
102	.53566709681-002	-.11598111111	.2140672914+002	.3651626124+010	-.1060000000+008
103	.5300010187-002	-.11586995188+001	.2160666667+002	.381225104314+010	-.1080990000+008
104	.52440000000-002	-.11575941633+001	.21806600221+002	.3979735329+010	-.1101770000+008
104	.5190415506-002	-.11564973270+001	.2200654082+002	.415325104314+010	-.1123700000+008
105	.5137240910-002	-.11554075292+001	.2220647990+002	.4332444953+010	-.1145944000+008
105	.508515760100-002	-.11543240910-002	.2240642055+002	.45176660143+010	-.1168300000+008
106	.5034109359-002	-.1153250000+001	.22606366219+002	.4708400243+010	-.1190910000+008
107	.4984078233-002	-.1152176000+001			

Table 4 (Concluded).

r	z(r)	w ₁₀₀ (r)	y _r (100)	n _{r,1} (100)	(-) ^r K _r (10)
0	1.778006282-004)				- .3725270025-094
1	1.064877345-004)				.8786143716-004
2	2.215000911-004)				- .5754104998-004
3	2.808333506+000				.9540888710-004
4	1.727974285+001				- .1720257946-003
5	1.004+92354+001				.8862398998-003
6	2.45800004003+002				- .7100813388-003
7	7.835488045+002				.1614255800-002
8	2.457457646+003				
9	7.0120496883+003				
10	2.509471652+004				
20	2.668261020+010				
30	1.07827816+018				
40	7.271509245+026				
50	2.28191291				
60	5.28191291				
70	2.28191291				
80	5.28191291				
90	2.28191291				
100	5.28191291				
101	4.234861922+070				.1787442782+093
102	5.90+9222938809				.2080237818+010
103	2.28191291				.5938234631+016
104	5.28191291				.2061878773+020
105	2.28191291				.5241262205+028
106	5.28191291				.7087837258+049
107	2.28191291				.3943200218+061
108	5.28191291				
109	2.28191291				.7719051887+070
110	5.28191291				- .13932782042+078
111	2.28191291				.2544837707+076
112	5.28191291				- .4695587412+077
113	2.28191291				.6759148064+078
114	5.28191291				- .1651414683+080
115	2.28191291				.81464448827+081
116	5.28191291				- .6087691862+082
117	2.28191291				.1178840608+084
118	5.28191291				- .2315601068+085
119	2.28191291				.45966740824+085
120	5.28191291				
121	2.28191291				- .921650401+087
122	5.28191291				.1366888821+089
123	2.28191291				- .8616570724+090
124	5.28191291				.7838717277+091
125	2.28191291				- .1648000984+093
126	5.28191291				.8458237430+094
127	2.28191291				- .7847056982+095
128	5.28191291				.1577430016+097
129	2.28191291				- .8877006198+098
130	5.28191291				.819911236+099
131	2.28191291				- .9000000000
132	5.28191291				
133	2.28191291				
134	5.28191291				
135	2.28191291				
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252	5.28191291				
253	2.28191291				
254	5.28191291				
255	2.28191291				
256	5.28191291				
257	2.28191291				
258	5.28191291				
259	2.28191291				
260	5.28191291				
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262	5.28191291				
263	2.28191291				
264	5.28191291				
265	2.28191291				
266	5.28191291				
267	2.28191291				
268	5.28191291				
269	2.28191291				
270	5.28191291				
271	2.28191291				
272	5.28191291				
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332	5.28191291				
333	2.28191291				
334	5.28191291				
335	2.28191291				
336	5.28191291				
337	2.28191291				
338	5.28191291				
339	2.28191291				
340	5.28191291				
341	2.28191291				
342	5.28191291				
343	2.28191291				
344	5.28				

Example 4.1.2 In this example we compute the solutions $I_r(1)$, $J_r(10)$, $Y_r(10)$ and $(-)^r K_r(1)$ of the homogeneous equation defined in Figure 6, with $x_1 = 10$, $x_2 = 1$, for $r = 0, 1, 2, \dots, 100$. Both $I_r(1)$ and $K_r(1)$ are monotonic over this range of r . The solutions $J_r(10)$ and $Y_r(10)$ are monotonic only for $10 \leq r \leq 100$; for lower values they oscillate with approximately the same amplitude. However, since there is no "crossing over" of growth behavior in the four solutions in the nonasymptotic region $0 \leq r \leq 10$, we expect the algorithm to be stable with the following numbers of initial conditions: $j = 1$ for $I_r(1)$, $j = 2$ for $J_r(10)$, $j = 3$ for $Y_r(10)$ and $j = 4$ for $(-)^r K_r(1)$; compare (4.1.4). Tables 9 through 12 give the results of the computation of each of these solutions for selected values of r ; see §4.0 for a full description of these tables. Again, the stability of the algorithm for each solution is confirmed by comparison with Tables 9.4 and 9.11 of [16].

r	z(r)	w ₁₀₀ (r)	y _r (100)	n _{r,1} (100)	J _r (10)
0	(-)				
1	()				
2					
3					
4					
5					
6					
7					
8					
9					
10					
20					
30					
40					
50					
60					
70					
80					
90					
91					
92					
93					
94					
95					
96					
97					
98					
99					
100					
101					
102					
103					
104					
105					

Table 10. Computation of J_r(10) from the JYIK equation with x₁=10, x₂=1, i=0, m=100, j=2.

r	z(r)	w ₁₀₀ (r)	y _r (100)	n _{r,1} (100)	Y _r (10)
01	900+8809961698	6029011602	-	-	000+00000000
02	900+1828624822	601514180222	-	-	000+8859924226
03	900+1200120922	59886995901	-	-	000+52256602822
04	900+2982815698	601+58583180222	-	-	000+50599901498
05	900+60399988012	610-66492283	-	-	000+266882991
06	900+8226069612	601+6211228869	-	-	010+919122719522
07	900+8226069612	601+6211228869	-	-	000+04000000
08	900+8226069612	601+6211228869	-	-	000+8859924226
09	900+8226069612	601+6211228869	-	-	000+52256602822
10	900+8226069612	601+6211228869	-	-	000+50599901498
11	900+8226069612	601+6211228869	-	-	000+266882991
12	900+8226069612	601+6211228869	-	-	010+919122719522
13	900+8226069612	601+6211228869	-	-	000+04000000
14	900+8226069612	601+6211228869	-	-	000+8859924226
15	900+8226069612	601+6211228869	-	-	000+52256602822
16	900+8226069612	601+6211228869	-	-	000+50599901498
17	900+8226069612	601+6211228869	-	-	000+266882991
18	900+8226069612	601+6211228869	-	-	010+919122719522
19	900+8226069612	601+6211228869	-	-	000+04000000
20	900+8226069612	601+6211228869	-	-	000+8859924226
21	900+8226069612	601+6211228869	-	-	000+52256602822
22	900+8226069612	601+6211228869	-	-	000+50599901498
23	900+8226069612	601+6211228869	-	-	000+266882991
24	900+8226069612	601+6211228869	-	-	010+919122719522
25	900+8226069612	601+6211228869	-	-	000+04000000
26	900+8226069612	601+6211228869	-	-	000+8859924226
27	900+8226069612	601+6211228869	-	-	000+52256602822
28	900+8226069612	601+6211228869	-	-	000+50599901498
29	900+8226069612	601+6211228869	-	-	000+266882991
30	900+8226069612	601+6211228869	-	-	010+919122719522
31	900+8226069612	601+6211228869	-	-	000+04000000
32	900+8226069612	601+6211228869	-	-	000+8859924226
33	900+8226069612	601+6211228869	-	-	000+52256602822
34	900+8226069612	601+6211228869	-	-	000+50599901498
35	900+8226069612	601+6211228869	-	-	000+266882991
36	900+8226069612	601+6211228869	-	-	010+919122719522
37	900+8226069612	601+6211228869	-	-	000+04000000
38	900+8226069612	601+6211228869	-	-	000+8859924226
39	900+8226069612	601+6211228869	-	-	000+52256602822
40	900+8226069612	601+6211228869	-	-	000+50599901498
41	900+8226069612	601+6211228869	-	-	000+266882991
42	900+8226069612	601+6211228869	-	-	010+919122719522
43	900+8226069612	601+6211228869	-	-	000+04000000
44	900+8226069612	601+6211228869	-	-	000+8859924226
45	900+8226069612	601+6211228869	-	-	000+52256602822
46	900+8226069612	601+6211228869	-	-	000+50599901498
47	900+8226069612	601+6211228869	-	-	000+266882991
48	900+8226069612	601+6211228869	-	-	010+919122719522
49	900+8226069612	601+6211228869	-	-	000+04000000
50	900+8226069612	601+6211228869	-	-	000+8859924226
51	900+8226069612	601+6211228869	-	-	000+52256602822
52	900+8226069612	601+6211228869	-	-	000+50599901498
53	900+8226069612	601+6211228869	-	-	000+266882991
54	900+8226069612	601+6211228869	-	-	010+919122719522
55	900+8226069612	601+6211228869	-	-	000+04000000
56	900+8226069612	601+6211228869	-	-	000+8859924226
57	900+8226069612	601+6211228869	-	-	000+52256602822
58	900+8226069612	601+6211228869	-	-	000+50599901498
59	900+8226069612	601+6211228869	-	-	000+266882991
60	900+8226069612	601+6211228869	-	-	010+919122719522
61	900+8226069612	601+6211228869	-	-	000+04000000
62	900+8226069612	601+6211228869	-	-	000+8859924226
63	900+8226069612	601+6211228869	-	-	000+52256602822
64	900+8226069612	601+6211228869	-	-	000+50599901498
65	900+8226069612	601+6211228869	-	-	000+266882991
66	900+8226069612	601+6211228869	-	-	010+919122719522
67	900+8226069612	601+6211228869	-	-	000+04000000
68	900+8226069612	601+6211228869	-	-	000+8859924226
69	900+8226069612	601+6211228869	-	-	000+52256602822
70	900+8226069612	601+6211228869	-	-	000+50599901498
71	900+8226069612	601+6211228869	-	-	000+266882991
72	900+8226069612	601+6211228869	-	-	010+919122719522
73	900+8226069612	601+6211228869	-	-	000+04000000
74	900+8226069612	601+6211228869	-	-	000+8859924226
75	900+8226069612	601+6211228869	-	-	000+52256602822
76	900+8226069612	601+6211228869	-	-	000+50599901498
77	900+8226069612	601+6211228869	-	-	000+266882991
78	900+8226069612	601+6211228869	-	-	010+919122719522
79	900+8226069612	601+6211228869	-	-	000+04000000
80	900+8226069612	601+6211228869	-	-	000+8859924226
81	900+8226069612	601+6211228869	-	-	000+52256602822
82	900+8226069612	601+6211228869	-	-	000+50599901498
83	900+8226069612	601+6211228869	-	-	000+266882991
84	900+8226069612	601+6211228869	-	-	010+919122719522
85	900+8226069612	601+6211228869	-	-	000+04000000
86	900+8226069612	601+6211228869	-	-	000+8859924226
87	900+8226069612	601+6211228869	-	-	000+52256602822
88	900+8226069612	601+6211228869	-	-	000+50599901498
89	900+8226069612	601+6211228869	-	-	000+266882991
90	900+8226069612	601+6211228869	-	-	010+919122719522
91	900+8226069612	601+6211228869	-	-	000+04000000
92	900+8226069612	601+6211228869	-	-	000+8859924226
93	900+8226069612	601+6211228869	-	-	000+52256602822
94	900+8226069612	601+6211228869	-	-	000+50599901498
95	900+8226069612	601+6211228869	-	-	000+266882991
96	900+8226069612	601+6211228869	-	-	010+919122719522
97	900+8226069612	601+6211228869	-	-	000+04000000
98	900+8226069612	601+6211228869	-	-	000+8859924226
99	900+8226069612	601+6211228869	-	-	000+52256602822
100	900+8226069612	601+6211228869	-	-	000+50599901498

Table 11. Computation of Y_r(10) from the JYIK equation with x₁=10, x₂=1, i=0, m=100, j=3.

r
(-) $K_r(1)$

r	$z(r)$
0	(.4210244002+000)
1	(-.6019072802+000)
2	(.1624688999+001)
3	(-.7101262825+001)
4	(.1318125992+006
5	-.2505066492+007
6	.4705148746+008
7	-.9195060280+009
8	.1911225918+011
9	-.4260146458+012
10	.1019945808+014
20	.1913110624+029
30	.8526972704+045
40	.1552042514+065
50	.7612506607+084
60	.2606296949+105
70	.4540658381+126
80	.8205727667+148
90	.7733913482+170
91	-.1422205621+173
92	.2647501185+175
93	-.4904096321+177
94	.9470808627+179
95	-.1819590993+182
96	.8582668868+184
97	-.6920752517+186
98	.1922811189+189
99	-.2747423684+191
100	.5508858191+193

.44382-1600+002
-.8609605996+003
.8651028212+004
-.4420702093+005
.622521230+006
-.1008804999+008
.1807182899+009
.6294889861+023
.4706148522+040
.1142206514+059
.8406966954+078
.7966785327+098
.1006478960+120
.8390549908+141
.1018936801+164
-.1834125434+166
.8828210277+168
-.6142290018+170
.1142536831+173
-.2148800190+175
.4081871827+177
-.7886446327+179
.1520811289+182
-.2979889471+184
.5900888180+186

Table 12. Computation of $(-)^r K_r(1)$ from the JYIK equation with $x_1=10$, $x_2=1$, $i=0$, $m=100$, $j=4$.

Example 4.1.3 In this example we compute the solutions[†] $A_r(1)$, $J_r(1)$, $I_r(1)$, $Y_r(1)$ and $(-)^r K_r(1)$ of the homogeneous equation defined in Figure 6, with $x_1 = 1$, $x_2 = 1$, for $r = 0, 1, 2, \dots, 100$. Since each solution is monotonic (in magnitude) except possibly at the very beginning of the range of r , the algorithm is expected to be stable provided that the proper number of initial values is taken in each case. This number is $j = 1$ for $A_r(1)$, $j = 2$ for $J_r(1)$ and $I_r(1)$, and $j = 4$ for $Y_r(1)$ and $(-)^r K_r(1)$. However, in the case of $A_r(1)$, for reasons given earlier in this section, we take $j = 2$ instead of $j = 1$, thereby exchanging mild instability for a much improved rate of convergence.

Because of this mild instability, each individual rounding error, introduced at say $r = s$, is amplified in direct proportion to the number of steps taken beyond the point s , since in (4.1.7) $A_r/I_r = O(1/r)$. Thus, after 100 steps, approximately two decimal places of precision will be lost. On the other hand, if we were to take the number of initial conditions theoretically required (in this case it is $j=1$, since A_r is dominated by all of the other homogeneous solutions), then the convergence would be extremely slow: A terminal point of the order of 10^8 would be required to obtain 8 significant figures of accuracy in $A_{100}(1)$. Tables 13 through 17 give the results of these computations for selected values of r ; see §4.0 for a full description of these tables. Again, these results agree with the entries in Table 9.4 and 9.11 of [16]. In the case of $A_r(1)$, the method we have used is equivalent to computing $J_r(1)$ and $I_r(1)$ as type 2 solutions and then forming their difference.

[†]The function $A_r(1)$ is defined by $A_r(1) = J_r(1) - I_r(1)$; compare (4.1.6).

r	z(r)	w ₁₀₀ (r)	y _r (100)	η _{r,1} (100)	A _r (1)
0	(-.5000681614+0000)				-.20044-0486-001
1	(-.1251085103+0000)				-.25000707090909
2	.4999999999666666				-.26641251134497
3	.1248914565126202				-.2117425240212
4	1.00-46458126202				-.92452450212
5	.260200787				-.5552323475
6	.00-2190128002				-.96601426600
7	.2170031001				-.588279810
8	.1550099166				-.9001090226266
9	.068820201				-.209183000
10	.5882444634				-.122333168
20	.1567290958-022				-.010-0245134412
30	.2106906666				-.1022326266
40	.0912121				-.381001408
50	.220-6090666666				-.931001408
60	.2560857402				-.2009-080086
70	.989972666666				-.381001408
80	.1619999999				-.123
90	.9786730566-163				-.2505-4723810-167
91	.5637072397-164				-.1623582278-169
92	.2987403342-167				-.823000000-172
93	.1628588190-168				-.463000000-174
94	.8723968800-172				-.244000000-176
95	.4648000270-174				-.12237132231
96	.2448720000-176				-.9160132231
97	.12272770885-178				-.25474609959
98	.6566000000-180				-.82662327208
99	.8347238776-183				-.160036607
100	.1699000000-185				-.3484725418
101	.6452780050-188				-.4184521849-190
102	.4184521665-190				-.5128603289-199
103	.2051236033-192				-.2464717068-199
104					-.5696408736-208

Table 13. Computation of A_r(1) = J_r(1)-I_r(1) from the JYIK equation with x₁=x₂=1, i=0, m=100, j=2.

r	z(r)	w ₁₀₀ (r)	y _r (100)	η _{r,1} (100)	J _r (1)
0	.7651976866+000)				.11490030049+000
1	.4400505887+000)				.1956888698-001
2	.2758206999+001				.2476638964-002
3	-.9375529130+000				.2497487302-003
4	-.1977173463+000				.2693838800-004
5	-.2994885860-001				.1802828817-005
6	-.3516459756-002				.9422844171-007
7	-.3964624510-003				.5269210179-008
8	-.2713885526-004				.2680615123-009
9	-.1689614090-005				
10	-.1157424989-006				
20	-.6507469604-021				.8878508008-024
30	-.1295626969-037				.9482839798-041
40	-.7267926914-056				.1107918081-059
50	-.2964124868-075				.2906094943-079
60	-.1519800281-095				.1030114976-099
70	-.1400820206-116				.7046879418-121
80	-.2906882030-138				.1152211443-142
90	-.1778296075-160				
91	-.9977052458-163				.5422149230-165
92	-.5541644717-165				.2979291888-167
93	-.3044285047-167				.1619237661-169
94	-.1654143867-169				.8705774029-172
95	-.8891405888-172				.4680060514-174
96	-.4728608868-174				.2487861823-176
97	-.2488172427-176				.1269498866-178
98	-.1295788947-178				.6545962340-181
99	-.2229922299-181				.8888887231-183
100	-.8406458817-183				.1686829015-185
101	-.1720124610-185		.8481828772-188	.1043672861-196	.8481828772-183
102	-.8599208974-188	.2379592624-004	.8481828788-188	.5065008585-197	.4174275965-190
103	-.4256828888-190	.1121932040-010	.8481828788-188	.1182007093-205	.2046261419-192
104			.8481828788-188		.6000000000
					.6000000000

Table 14. Computation of J_r(1) from the JYIK equation with x₁=x₂=1, i=0, m=100, j=2.

r	$z(r)$	$Y_r(1)$
0	0025696420-001	- .0027042300+002
1	000+0100101001	- .2602000000+000
2	000+0000000001	- .2570700244+004
3	000+0000000001	- .3000000000+000
4	000+0000000002	- .42507246185+006
5	000+0000000000	- .6700000000+000
6	200+0000000001	- .1216100400+000
7	000+0000000001	- .0000000000+000
8	000+0000000001	- .4110000000+000
9	000+0000000001	- .0040100000+000
10	000+0000000001	- .7000000000+000
11	000+0000000001	- .2100000000+000
12	000+0000000001	- .2000000000+000
13	000+0000000001	- .5500000000+000
14	000+0000000001	- .6500000000+000
15	000+0000000001	- .6500000000+000
16	000+0000000001	- .1720000000+000
17	000+0000000001	- .2000000000+000
18	000+0000000001	- .3000000000+000
19	000+0000000001	- .7000000000+000
20	000+0000000001	- .6500000000+000
21	000+0000000001	- .6500000000+000
22	000+0000000001	- .6500000000+000
23	000+0000000001	- .6500000000+000
24	000+0000000001	- .6500000000+000
25	000+0000000001	- .6500000000+000
26	000+0000000001	- .6500000000+000
27	000+0000000001	- .6500000000+000
28	000+0000000001	- .6500000000+000
29	000+0000000001	- .6500000000+000
30	000+0000000001	- .6500000000+000
31	000+0000000001	- .6500000000+000
32	000+0000000001	- .6500000000+000
33	000+0000000001	- .6500000000+000
34	000+0000000001	- .6500000000+000
35	000+0000000001	- .6500000000+000
36	000+0000000001	- .6500000000+000
37	000+0000000001	- .6500000000+000
38	000+0000000001	- .6500000000+000
39	000+0000000001	- .6500000000+000
40	000+0000000001	- .6500000000+000
41	000+0000000001	- .6500000000+000
42	000+0000000001	- .6500000000+000
43	000+0000000001	- .6500000000+000
44	000+0000000001	- .6500000000+000
45	000+0000000001	- .6500000000+000
46	000+0000000001	- .6500000000+000
47	000+0000000001	- .6500000000+000
48	000+0000000001	- .6500000000+000
49	000+0000000001	- .6500000000+000
50	000+0000000001	- .6500000000+000
51	000+0000000001	- .6500000000+000
52	000+0000000001	- .6500000000+000
53	000+0000000001	- .6500000000+000
54	000+0000000001	- .6500000000+000
55	000+0000000001	- .6500000000+000
56	000+0000000001	- .6500000000+000
57	000+0000000001	- .6500000000+000
58	000+0000000001	- .6500000000+000
59	000+0000000001	- .6500000000+000
60	000+0000000001	- .6500000000+000
61	000+0000000001	- .6500000000+000
62	000+0000000001	- .6500000000+000
63	000+0000000001	- .6500000000+000
64	000+0000000001	- .6500000000+000
65	000+0000000001	- .6500000000+000
66	000+0000000001	- .6500000000+000
67	000+0000000001	- .6500000000+000
68	000+0000000001	- .6500000000+000
69	000+0000000001	- .6500000000+000
70	000+0000000001	- .6500000000+000
71	000+0000000001	- .6500000000+000
72	000+0000000001	- .6500000000+000
73	000+0000000001	- .6500000000+000
74	000+0000000001	- .6500000000+000
75	000+0000000001	- .6500000000+000
76	000+0000000001	- .6500000000+000
77	000+0000000001	- .6500000000+000
78	000+0000000001	- .6500000000+000
79	000+0000000001	- .6500000000+000
80	000+0000000001	- .6500000000+000
81	000+0000000001	- .6500000000+000
82	000+0000000001	- .6500000000+000
83	000+0000000001	- .6500000000+000
84	000+0000000001	- .6500000000+000
85	000+0000000001	- .6500000000+000
86	000+0000000001	- .6500000000+000
87	000+0000000001	- .6500000000+000
88	000+0000000001	- .6500000000+000
89	000+0000000001	- .6500000000+000
90	000+0000000001	- .6500000000+000
91	000+0000000001	- .6500000000+000
92	000+0000000001	- .6500000000+000
93	000+0000000001	- .6500000000+000
94	000+0000000001	- .6500000000+000
95	000+0000000001	- .6500000000+000
96	000+0000000001	- .6500000000+000
97	000+0000000001	- .6500000000+000
98	000+0000000001	- .6500000000+000
99	000+0000000001	- .6500000000+000
100	000+0000000001	- .6500000000+000

Table 16. Computation of $Y_r(1)$ from the JYIK equation with $x_1=x_2=1$, $i=0$, $m=100$, $j=4$.

r	z(r)	$(-)^r K_r(1)$
0	(.4210244222+000)	.4423241536+002
1	(-.6019072302+000)	-.360960896+003
2	(.162483899+001)	.265333312+004
3	(-.7101262825+001)	-.4420702033+005
4	.4423241536+002	.6225521231+006
5	-.360960896+003	-.1600504099+003
6	.365333312+004	.1807132899+009
7	-.4420702033+005	.6294869362+023
8	.6225521231+006	.4706145523+040
9	-.1600504099+003	.114220651+039
10	.1807132899+009	.3406396335+073
20	.6294869362+023	.7960735223+095
30	.4706145523+040	.1006475500+120
40	.114220651+039	.5390549903+141
50	.3406396335+073	
60	.7960735223+095	
70	.1006475500+120	
80	.5390549903+141	
90	.1018926301+164	
91	-.1834125434+166	
92	.3338210273+168	
93	-.6142490315+170	
94	.1142536531+173	
95	-.2143080196+175	
96	.4031371627+177	
97	-.7336448327+179	
98	.1520311739+182	
99	-.297939471+184	
100	.5900333105+186	

Table 17. Computation of $(-)^r K_r(1)$ from the JYIK equation with $x_1=x_2=1$, $i=0$, $m=100$, $j=4$.

4.2 Examples Involving the Inhomogeneous JYIK Equation

For fixed x and large r the Anger-Weber function has the asymptotic forms

$$(4.2.1) \quad E_{2r}(x) \sim \frac{2x}{(4r^2-1)\pi} \quad , \quad E_{2r+1}(x) \sim \frac{2}{(2r+1)\pi} \quad ;$$

compare [21]. Comparison of (4.2.1) with (4.1.1) and (4.1.2) shows that

$$(4.2.2) \quad \left\{ \begin{array}{l} I_r(x_2) \\ J_r(x_1) \end{array} \right\} < E_r(x) < \left\{ \begin{array}{l} (-)^r K_r(x_2) \\ Y_r(x_1) \end{array} \right\} \quad , \quad r \rightarrow \infty \quad ,$$

for any given set of values of x_1 , x_2 , x . Thus $E_r(x)$ is a solution of type 2 of the first inhomogeneous equation defined in Figure 6 and the correct number of initial values of $E_r(x)$ to be used in every case is $j = 2$.

The asymptotic form of the Struve function, also taken from [21], is

$$(4.2.3) \quad H_r(x) \sim \frac{x}{\sqrt{2}\pi r} \left(\frac{ex}{2r} \right)^r$$

for fixed x and large r . Comparison of (4.2.3) with (4.1.1) and (4.1.2) shows that

$$(4.2.4) \quad H_r(x) \sim \frac{x}{\sqrt{\pi r}} \left\{ \begin{array}{l} I_r(x) \\ J_r(x) \end{array} \right\} \quad , \quad r \rightarrow \infty \quad .$$

Thus $H_r(x)$ is a solution of type 1 of the second inhomogeneous equation defined in Figure 6 when $x_1 = x_2$; compare (4.1.8). However, since the separation of $H_r(x)$ from solutions of type 2 is even weaker than

that of the solution $A_r(x)$ defined by (4.1.6) it is again advantageous in most applications to specify the number of initial values appropriate to solutions of type 2, i.e., $j = 2$ instead of $j = 1$; compare Example 4.1.3.

Example 4.2.1 For the inhomogeneous fourth-order difference equations defined in Figure 6, we compute $E_r(1)$ with $x_1 = x_2 = 1$ for $r = 0, 1, 2, \dots, 100$; and $H_r(0.1)$ with $x_1 = x_2 = 0.1$, for $r = 0, 1, 2, \dots, 50$.[†] The appropriate number of initial values in these cases is $j = 2$ for $E_r(1)$ and $j = 1$ for $H_r(0.1)$. However, for reasons given above we actually use $j = 2$ for $H_r(0.1)$. Initial values for $E_r(1)$ and $H_r(0.1)$ were computed from series expansions given in [21] and checked against values found in [21] and [23]. Tables 18 and 19 give the results of these computations for selected values of r ; see §4.0 for a full description of these tables. They agree with the 10-figure values of $E_r(1)$ and $H_r(0.1)$ $r = 2, 3, 4, 5$, given in [23]; the 7 through 9-figure values of $E_r(1)$, $r = 2, 3, \dots, 10$, given in [21]; and the 9-figure values of $H_r(0.1)$, $r = 1, 2, \dots, 13$, given in [21]. In addition, they agree with series calculation of the functions to 10 significant figures in the case of $E_r(1)$, $r = 2, 3, \dots, 100$, and to 9 significant figures in the case of $H_r(0.1)$, $r = 2, 3, \dots, 50$. The loss of precision in the Struve function computation is due to mild instability which is acceptable in lieu of impossibly slow convergence.

[†]We restrict the sequence of $H_r(0.1)$ to $r \leq 50$ in order to avoid under-flow in the computer. As suggested at the end of §4.0, this difficulty could be overcome by using the software described in [11].

r	z(r)	w ₁₀₀ (r)	y _r (100)	n _{r,1} (100)	E _r (1)
0	(-.5686566270+000)				
1	(.4301624062+000)				.172419547+000
2	-.4073956117+001				.2400050824+000
3	-.1170473198+002				.4705079509+001
4	-.0758495822+001				.1840009783+000
5	-.1597396855+002				.1091940340+001
6	-.0140049310+001				.9000234082+001
7	-.2076353882+002				.1029301131+001
8	-.3947692529+001				.7166030007+001
9	-.2574056800+002				.6502199216+002
10	-.2040097021+001				
20	-.2604454606+001				.1599629050+002
30	-.2605916191+001				.7009870029+002
40	-.2612656179+001				.8900305946+003
50	-.2599001067+001				.2540531103+003
60	-.2590010425+001				.1769322837+003
70	-.2580660036+001				.1296704972+003
80	-.2570922335+001				.9050295357+004
90	-.2575255665+001				
91	-.25426973368+003				.70614435420+004
92	-.2574619590+001				.6996609089+002
93	-.2390900239+003				.7523270700+004
94	-.2574011044+001				.6046165019+002
95	-.2444025297+003				.72000947046+004
96	-.2573422356+001				.6702003695+002
97	-.2573750565+003				.6900266606+004
98	-.2572269631+001				.6563703941+002
99	-.2574075991+003				.6600074372+004
100	-.2572333697+001				.6431159193+002
101	-.2597601595+003		.6367162628+004	.1576066173+003	.6309704260+003
102	-.2571019090+001	.2879592624+004	.6307020285+004	.1514022220+003	.6120167453+004
103	-.2540527050+003	.1121932640+010	.6307471717+004	.7855123329+013	.6101215807+003
104	-.2571025406+001	.5447703348+009	.6367471725+004	.346733279+013	.5336783121+004
105	-.2699450265+003	.3700095333+015	.6367471725+004	.2477680711+017	.6000000000
106			.6367471726+004		.6000000000

Table 18. Computation of E_r(1) from the JYIK equation with x₁=x₂=1, i=0, m=100, j=2.

r	z(r)	w ₅₀ (r)	y _r (50)	η _{r,1} (50)	H _r (0.1)
0	(.6359126999-001)				
1	(.2120651601-002)				.4242111240-004
2	-.1018106632-006				.6060000285-008
3	-.2909183739-003				.6734676050-003
4	-.5887739747-010				.6122718211-010
5	-.7347261821-012				.4709942660-012
6	-.7912706100-014				.3140044932-014
7	-.7033709520-016				.1047123376-016
8	-.5319715276-013				.9721864421-019
9	-.3499871171-020				.4629523147-021
10	-.2036990177-022				.2012859400-023
11	-.1062789807-024				.3051517471-026
12	-.5024146893-027				.2932068393-023
13	-.2170946155-029				.1023307245-030
14	-.5637780352-032				.3317141763-033
15	-.3134456090-034				.1003200275-035
16	-.1093657099-036				.2872018372-033
17	-.3515347053-039				.7762240109-041
18	-.1061874447-041				.1990356275-043
19	-.3025295937-044				.4854472685-046
20	-.3153514110-047				
25	-.5553626624-050				.2136010240-059
30	-.1323786920-073				.3572007049-073
35	-.1339398907-087				.2657337515-087
40	-.6462471983-102				.9851329249-102
41	-.3175424095-105				.1186903260-104
42	-.1008783294-107				.1396363913-107
43	-.1214677153-110				.1605017379-110
44	-.142336304-113				.1803391790-113
45	-.1640839654-116				.1931750790-116
46	-.1842316533-119				.2130916436-119
47	-.2024147771-122				.2243071533-122
48	-.2175549674-125				.2312446507-125
49	-.229089923-123				.2365806110-123
50	-.2358934415-131				.2312660799-131
51	-.2391833159-134			.4492442073-143	.2245322540-134
52	-.2357876234-137			.2096474211-143	.0000000000
53				.2312680799-131	.0000000000

Table 19. Computation of H_r(0.1) from the JYIK equation with x₁=x₂=0.1, i=0, m=50, j=2.

4.3 Examples Involving the Homogeneous JYPQ Equation

For large values of r and fixed values of μ and x , the associated Legendre functions have the asymptotic forms

$$(4.3.1) \quad P_r^\mu(x) = \frac{\Gamma(r+\mu+1)}{\Gamma(r+3/2)} \left(\frac{1}{2} \pi \sin \theta\right)^{-1/2} \cos\left(r\theta + \frac{1}{2}\theta - \frac{1}{4}\pi + \frac{1}{2}\mu\pi\right) + O\left(\frac{1}{r}\right)$$

and

$$(4.3.2) \quad Q_r^\mu(x) = \frac{\Gamma(r+\mu+1)}{\Gamma(r+3/2)} \left(\frac{\pi}{2 \sin \theta}\right)^{1/2} \cos\left(r\theta + \frac{1}{2}\theta + \frac{1}{4}\pi + \frac{1}{2}\mu\pi\right) + O\left(\frac{1}{r}\right)$$

where $x = \cos \theta$ with $0 < \theta < \pi$; see [16, Chapter 8]. Consequently, for values of x less than 1 in magnitude, neither associated Legendre function dominates the other. Comparing (4.3.1) and (4.3.2) with (4.1.1) we obtain

$$(4.3.3) \quad J_r(x_1) < \left\{ \begin{array}{l} P_r^\mu(x_2) \\ Q_r^\mu(x_2) \end{array} \right\} < Y_r(x_1) \quad , \quad r \rightarrow \infty$$

for all admissible values of μ , x_1 , x_2 . Thus the JYPQ equation has three distinct types of solution, and the dimensions of the subdominant subspaces of types 1, 2 and 3 are 1, 3 and 4 respectively; this contrasts with (4.1.3), (4.1.4), and (4.1.8).

Example 4.3.1 In this example we compute the solutions $J_r(1)$, $P_r(0.5)$, $Q_r(0.5)$ and $Y_r(1)$ of the homogeneous equation[†] defined in Figure 7, with $\mu = 0$, $x_1 = 1$, $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 100$. Since these solutions exhibit their characteristic asymptotic separation, described by (4.3.3), from the very beginning of this range of r , the algorithm

[†]In accordance with custom, when the order of the associated Legendre functions is zero we drop the superscript in the notation.

will be stable provided the proper number of initial values is used in each case. This number is $j = 1$ for $J_r(1)$, $j = 3$ for $P_r(0.5)$ and $Q_r(0.5)$, and $j = 4$ for $Y_r(1)$.

Table 20 gives the numerical coefficients of the JYPQ operator with $\mu = 0$, $x_1 = 1$ and $x_2 = 0.5$ for $r = 0, 1, 2, \dots, 104$. Tables 21 and 22 give the numerical coefficients of the (0,1)-factorization and the (0,3)-factorization. Since the (i,j)-factorization $D_j^i = A_j^i B^{i+j}$ produced by the test program has the principal diagonal of A_j^i equal to (1,1,1,...), we do not include values of $a_j(r)$.

Tables 23 through 26 give the results of the computation of the four desired solutions for selected values of r ; see §4.0 for a full description of these tables. Initial values were obtained from Table 9.4 of [16] in the case of the Bessel functions and by hand calculation of the explicit elementary forms given in [16], Section 8.4 in the case of the Legendre functions. The computed Bessel function values agree satisfactorily with the values given in [16]. The Legendre function values were compared with data available for $r \leq 10$ in [16] and (in the case of the $P_r(0.5)$) [17]. Agreement is satisfactory to the precision available in these references; this ranges from 6 to 8 significant figures. In addition, all of the computed Legendre function values agree satisfactorily with double-precision values computed by an unpublished program of Dr. J. M. Smith of the National Bureau of Standards. This is sufficient to demonstrate stability.

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
0	-.1137500000+003	.2926250000+003	-.3972500000+003	.2031250000+003	-.2300000000+002
1	-.1001500000+004	.4898975000+004	-.6207500000+004	-.5817875000+004	-.5687500000+003
2	-.4459250000+004	.3069712500+005	-.3720675000+005	.3464662500+005	-.3604500000+004
3	-.1049030000+005	.1223863750+006	-.1437410000+006	.1379233750+006	-.1049030000+005
4	-.3422875000+005	.3728996250+006	-.4238322500+006	.4169141250+006	-.2759380000+005
5	-.2272350000+005	.6469788750+006	-.6738385000+007	.6527373750+007	-.6161175000+005
6	-.4380305000+006	.2128580125+007	-.2817373750+007	.2339181625+007	-.1226875000+006
7	-.2559500000+006	.2288093375+007	-.4773180000+007	.4721190375+007	-.2241303000+006
8	-.4380667250+006	.9158322625+007	-.8917307250+007	.889997125+007	-.3838970500+006
9	-.0000900000+006	.6660608558+008	-.1502393330+008	.1583832787+008	-.6222043750+006
10	.7000125000+007	.2487961512+008	-.2629480475+008	.2614010062+008	-.1443040250+006
11	.1079902000+007	.3969626657+008	-.4227631500+008	.4209663453+008	-.9665035000+006
12	-.1100000000+007	.61494911	-.6562719425+008	.6534680412+008	-.2105370000+007
13	-.3200796500+007	.9293470887+008	-.9662519435+008	.9647867007+008	-.2980948750+007
14	.2000000000+007	.1388283321+009	-.4800000597+009	.4453844916+009	-.4125598500+007
15	.2800222000+007	.9675651512+009	-.2078509460+009	.2072629244+009	-.5594939250+007
16	-.2279946750+007	.6000441500+009	-.2919046681+009	.2917115911+009	-.7460370000+007
17	.0086259550+008	.6000696900+009	-.4025284755+009	.4016249779+009	-.9789000750+007
18	.0000000000+008	.3831596859+009	-.7296040391+009	.5449469946+009	-.1266539450+008
19	-.5999999999+008	.6779595227+009	-.7298599610+009	.7285286554+009	-.1618949428+008
20	.0000000000+008	.6000921312+009	-.9662536712+009	.9660941998+009	-.2060517800+008
21	-.2678222000+008	.1203626500+009	-.1253954922+010	.1252059573+010	-.2554034375+008
22	.3380666666+008	.0100606032+010	-.1615416282+010	.1613179722+010	-.3161744950+008
23	-.3380666666+008	.2597000000+010	-.2059779456+010	.2057157610+010	-.8879899825+008
24	-.49192964375+008	.0100000000+010	-.2601596537+010	.2598542069+010	-.4722090000+008
25	-.5999999999+008	.0100000000+010	-.3287257353+010	.3253713072+010	-.5706936675+008
26	.71000012	.0100000000+010	-.4045182289+010	.4041055137+010	-.6847091250+008
27	-.71000012	.0100000000+010	-.6100000000+010	.6098166186+010	-.8169337855+008
28	.1600001601	.0100000000+010	-.6100000000+010	.6096546329+010	-.9674392000+008
29	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008
30	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008
31	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008
32	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008
33	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008
34	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008
35	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008
36	.6000000000+010	.0100000000+010	-.8964520120+010	.8962651896+010	-.1400154174+008

Table 20. Numerical coefficients of the JYPQ operator with $\mu=0$, $x_1=1$, and $x_2=0.5$.

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
22	- .3699107905+009	.2845565517+011	-.2917104185+011	.2915610035+011	-.35969933157+009
23	- .4192525702+009	.5311506839+011	-.3392867503+011	.3391217079+011	-.4085171095+009
24	- .4740103000+009	.3039612621+010	-.3981595232+011	.5929776581+011	-.4622565892+009
25	- .5345889632+009	.5756796953+011	-.4539788103+011	.4537783579+011	-.5215111330+009
26	- .60008662215+009	.5107522780+011	-.5224416964+011	.5223223463+011	-.5866651187+009
27	- .67828896623+009	.5296940343+011	-.5992999677+011	.5990595520+011	-.6582330045+009
28	- .7586989690+009	.6709694034+011	-.8355566123+011	.8350937311+011	-.7366569882+009
29	- .84000557825+009	.8733319159+011	-.7814712984+011	.7811852322+011	-.8232039850+009
30	- .9100999229+010	.9573602922+011	-.1007616880+012	.1007278079+012	-.9156231217+009
31	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
32	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
33	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
34	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
35	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
36	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
37	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
38	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
39	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
40	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
41	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
42	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
43	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
44	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
45	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
46	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
47	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
48	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
49	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
50	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
51	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
52	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
53	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
54	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
55	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
56	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
57	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
58	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
59	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
60	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
61	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
62	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
63	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
64	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
65	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
66	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
67	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
68	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
69	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
70	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
71	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
72	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010
73	- .01048891511	.1111111111	-.1007616880+012	.1007278079+012	-.1017313182+010

Table 20. (Continued)

r	$d_0(r)$	$d_1(r)$	$d_2(r)$	$d_3(r)$	$d_4(r)$
74	-.1027157665+011	.1550938747+013	-.1571147466+013	.1570939420+013	-.1013465693+011
75	-.109689616664+011	.16773643269+013	-.1693925187+013	.1693716079+013	-.1031959410+011
76	-.1169246276+011	.1812254379+013	-.1835268775+013	-.1835033171+013	-.1154004910+011
77	-.1245956092+011	.195670715+013	-.1980599423+013	-.1930356332+013	-.1229906343+011
78	-.1326640673+011	.2109273715+013	-.2135595979+013	.2135141033+013	-.1309851593+011
79	-.1411451566+011	.227347769+013	-.2300914534+013	.2299077531+013	-.1393812336+011
80	-.1500545861+011	.2445795237+013	-.2475852998+013	.2475071031+013	-.1482024144+011
81	-.1594082251+011	.2630137415+013	-.26661543415+013	.2661244403+013	-.1574646067+011
82	-.1692226400+011	.2825915093+013	-.2869260556+013	.285951197+013	-.16713742371+011
83	-.1795145939+011	.3033638332+013	-.3069066350+013	.3063744136+013	-.1773779409+011
84	-.1903013023+011	.3254039095+013	-.3291551161+013	.3291211342+013	-.1889679073+011
85	-.2016003943+011	.3487568335+013	-.3527914266+013	.3526957730+013	-.1992566531+011
86	-.2134566910+011	.3734399434+013	-.3776984447+013	.3776612500+013	-.2109771574+011
87	-.2258803973+011	.3996677370+013	-.4041210290+013	.4040821184+013	-.2232427795+011
88	-.2388754703+011	.4270566461+013	-.4320662891+013	.4320235614+013	-.2360733210+011
89	-.2523877032+011	.4566265334+013	-.4616036265+013	.4615611239+013	-.2494610033+011
90	-.2664277650+011	.4875400139+013	-.4923694783+013	.4923704075+013	-.2635004952+011
91	-.2811948221+011	.5201949812+013	-.5237439215+013	.5256973946+013	-.2781363756+011
92	-.2966695039+011	.5546436613+013	-.5604976211+013	.5604492352+013	-.2934336359+011
93	-.3128923079+011	.5909727423+013	-.5971449382+013	.5970948733+013	-.3093669013+011
94	-.3298692079+011	.6292634536+013	-.6292634536+013	.6292634536+013	-.3268990950+011
95	-.3466632692+011	.6695963904+013	-.6695963904+013	.6695963904+013	-.3453337476+011
96	-.3651727046+011	.7120677662+013	-.7120677662+013	.7120677662+013	-.3644603107+011
97	-.3833351543+011	.7572999922+013	-.7572999922+013	.7572999922+013	-.384023137233+011
98	-.4039669604+011	.8037595100+013	-.8037595100+013	.8037595100+013	-.3999380397+011
99	-.4244733057+011	.8531699190+013	-.8531699190+013	.8531699190+013	-.4202391077+011
100	-.4458561299+011	.9050359631+013	-.9050359631+013	.9050359631+013	-.4414827339+011
101	-.4681129601+011	.9596032342+013	-.9596032342+013	.9596032342+013	-.4638591944+011
102	-.4912539297+011	.1016840517+014	-.1026548090+014	.1026478036+014	-.4864702637+011
103	-.5152700272+011	.1076399353+014	-.1087072939+014	.1086997939+014	-.5103161697+011
104	-.5402325503+011	.1139364632+014	-.1150542300+014	.1150464912+014	-.5350031057+011

Table 20. (Concluded).

r	a ₀ (r)	b ₀ (r+1)	b ₁ (r+1)	b ₂ (r+1)	b ₃ (r+1)
0	1.137500000+003	.2926250000+003	-.3972500000+003	.2031250000+003	-.2300000000+002
1	-.3524969321+001	.3493572992+004	-.5491406534+004	.5236500246+004	-.5687500000+003
2	-.1237364051+001	.2362433223+003	-.3046245052+003	.3391415105+003	-.3094500000+003
3	-.504000000+000	.1045435939+006	-.1239321237+006	.1361159122+006	-.1040900000+005
4	-.2214653676+000	.3323233520+006	-.3337369999+006	.4134765797+006	-.2759800000+005
5	-.0633333333+000	.8649079154+006	-.6046266966+006	.1046666637+007	-.6161175000+005
6	-.1266666666+000	.4042175529+007	-.4470500504+007	.2629022393+007	-.12326375000+006
7	-.0633333333+000	.7600054176+007	-.4901601629+007	.4705251997+007	-.12341382500+006
8	-.0633333333+000	.1370439101+008	-.1490151403+008	.1554333666+008	-.5833970000+006
9	-.0633333333+000	.2322245270+008	-.2303959596+008	.2609133697+008	-.6220467500+006
10	-.0633333333+000	.3770004266+008	-.3700004266+008	.4200003333+008	-.9665005000+006
11	-.0633333333+000	.5909090909+009	-.5909090909+009	.6526603933+008	-.1443456250+007
12	-.0633333333+000	.8222222222+009	-.8222222222+009	.9000000000+009	-.3105872000+007
13	-.0633333333+000	.1190909091+010	-.1190909091+010	.1443373232+009	-.2900945750+007
14	-.0633333333+000	.1822222222+010	-.1822222222+010	.2070765504+009	-.4125595500+007
15	-.0633333333+000	.2666666667+010	-.2666666667+010	.2909336447+009	-.5596939250+007
16	-.0633333333+000	.3703703704+010	-.3703703704+010	.4013373050+009	-.7466275000+007
17	-.0633333333+000	.5000000000+010	-.5000000000+010	.5445954413+009	-.12666539450+008
18	-.0633333333+000	.6666666667+010	-.6666666667+010	.7201031395+009	-.1610049425+008
19	-.0633333333+000	.8888888889+010	-.8888888889+010	.9664314333+009	-.3033517000+008
20	-.0633333333+000	.1190909091+010	-.1190909091+010	.1251451006+010	-.3554004375+008
21	-.0633333333+000	.1610049425+010	-.1610049425+010	.1612461506+010	-.3161744930+008
22	-.0633333333+000	.2142857143+010	-.2142857143+010	.2053145153+010	-.307999925+008
23	-.0633333333+000	.2857142857+010	-.2857142857+010	.2597253355+010	-.4722379900+008
24	-.0633333333+000	.3809523810+010	-.3809523810+010	.3252577235+010	-.5706046675+008
25	-.0633333333+000	.5000000000+010	-.5000000000+010	.4079973333+010	-.8247091250+008
26	-.0633333333+000	.6666666667+010	-.6666666667+010	.4979955015+010	-.8168274325+008
27	-.0633333333+000	.8888888889+010	-.8888888889+010	.6099966666+010	-.9674399900+008
28	-.0633333333+000	.1190909091+010	-.1190909091+010	.7410300312+010	-.140133447+009
29	-.0633333333+000	.1610049425+010	-.1610049425+010	.8955532472+010	-.1333697035+009
30	-.0633333333+000	.2142857143+010	-.2142857143+010	.1075933333+009	-.1593333312+009
31	-.0633333333+000	.2857142857+010	-.2857142857+010	.1205501799+009	-.1310773270+009
32	-.0633333333+000	.3809523810+010	-.3809523810+010	.1523333333+009	-.2008800040+009
33	-.0633333333+000	.5000000000+010	-.5000000000+010	.1900000000+009	-.2410707000+009
34	-.0633333333+000	.6666666667+010	-.6666666667+010	.2129137474+009	-.2765191945+009
35	-.0633333333+000	.8888888889+010	-.8888888889+010	.2499646790+009	-.3160135500+009

Table 21. Numerical coefficients of the (0,1)-factorization of the JYPQ operator with $\mu=0$, $x_1=1$ and $x_2=0.5$.

r	a ₀ (r)	b ₀ (r+1)	b ₁ (r+1)	b ₂ (r+1)	b ₃ (r+1)
74	-.2219198799-002	.1540531100+013	-.15609670907+013	.1570870946+013	-.1018465692+011
75	-.7017212725-002	.1690404183+013	-.16887275564+013	.1693643953+013	-.1081939410+011
76	-.7017212725-002	.1690404183+013	-.16887275564+013	.1693643953+013	-.1154064910+011
77	-.6920029669-002	.1995840414+013	-.19929700769+013	.1993277016+013	-.1229986943+011
78	-.6920029669-002	.2095840414+013	-.20929700769+013	.2093277016+013	-.1309361593+011
79	-.6644696669-002	.2250660565+013	-.22480070227+013	.22474979200+013	-.1393301235+011
80	-.6644696669-002	.2250660565+013	-.22480070227+013	.22474979200+013	-.1462033144+011
81	-.6644696669-002	.2250660565+013	-.22480070227+013	.22474979200+013	-.1574646367+011
82	-.6644696669-002	.2250660565+013	-.22480070227+013	.22474979200+013	-.1671842327+011
83	-.6644696669-002	.2250660565+013	-.22480070227+013	.22474979200+013	-.1773779403+011
84	-.6810705512+013	.3233762827+013	-.32272185052+013	.32291099603+013	-.1890639073+011
85	-.6810705512+013	.3233762827+013	-.32272185052+013	.32291099603+013	-.1992566653+011
86	-.6810705512+013	.3233762827+013	-.32272185052+013	.32291099603+013	-.1992566653+011
87	-.6810705512+013	.3233762827+013	-.32272185052+013	.32291099603+013	-.2109771574+011
88	-.6600401019+013	.4244027229+013	-.42396969999+013	.4240122019+013	-.2282427795+011
89	-.6600401019+013	.4244027229+013	-.42396969999+013	.4240122019+013	-.2360225313+011
90	-.5938693499-002	.4540759322+013	-.45300077899+013	.45461547113+013	-.2494850033+011
91	-.5938693499-002	.4540759322+013	-.45300077899+013	.45461547113+013	-.2635004952+011
92	-.5768217259-002	.51733526313+013	-.52233062039+013	.52356323127+013	-.2731383786+011
93	-.5768217259-002	.51733526313+013	-.52233062039+013	.52356323127+013	-.2934206639+011
94	-.5668366205-002	.5373127220+013	-.5374833702+013	.5370779466+013	-.30936649013+011
95	-.5668366205-002	.6259934237+013	-.62524210092+013	.6256977932+013	-.3259959593+011
96	-.5543950379-002	.66609441517+013	-.67262699335+013	.67658771433+013	-.3480037476+011
97	-.5422957426-002	.7053735631+013	-.71551270203+013	.7192029517+013	-.3614006137+011
98	-.5364334465-002	.7423575597+013	-.7604436886+013	.7642643639+013	-.3802318723+011
99	-.5309940696-002	.7996792355+013	-.8076373337+013	.81161655323+013	-.3996303097+011
100	-.5252366461-002	.843339471+013	-.8572466609+013	.8614696212+013	-.4202291347+011
101	-.5197335226-002	.9005538714+013	-.9093700703+013	.91333059133+013	-.4414523309+011
102	-.5144301599-002	.9546314829+013	-.9644079267+013	.9687683433+013	-.4635261944+011
103	-.5092301549-002	.1071687347+014	-.1031564233+014	.102645240+014	-.4864702667+011
104	-.5040952510-002	.1104411033+014	-.1031846039+014	.1036973203+014	-.5103161697+011
			-.1145063419+014	.1150409137+014	-.5350331057+011

Table 21. (Concluded).

r	a ₀ (r)	a ₁ (r)	a ₂ (r)	b ₀ (r+3)	b ₁ (r+3)
0	-.1197500000+003	.2926250000+003	-.3972500000+003	.2631250000+003	-.2800000000+003
1	-.1031500000+004	.4093375000+004	-.3056000000+002	.4614995000+004	-.5682500000+003
2	-.4499350000+004	.1511243077+003	-.7305976699+001	.3043964450+005	-.3094500000+004
3	-.6793350000+002	.2616908829+002	-.4226250523+001	.1248452428+006	-.1049635000+005
4	-.7416855273+001	.1206972078+002	.3131241531+001	.3340415686+006	-.2759000000+005
5	-.2419360379+001	.7549482485+001	-.2688769583+001	.9313616523+006	-.6161175000+005
6	-.1142946627+001	.5511216766+001	-.2262647193+001	.2199778972+007	-.1226318000+006
7	-.6655577451+000	.4391577334+001	-.2026851722+001	.4470169597+007	-.2241333500+006
8	-.4383536956+000	.3695733495+001	-.13993456512+001	.8415510563+007	-.3083976000+006
9	-.8180003213+000	.3226850225+001	-.1748933183+001	.1490078256+003	-.6220467500+006
10	-.4383536956+000	.2366645365+001	-.1690283036+001	.2808866556+003	-.9565045660+006
11	-.1872641790+000	.2639774969+001	-.1619626301+001	.4050097555+003	-.1443306250+007
12	-.7416855273+000	.2449272709+001	-.1562021843+001	.4008586455+003	-.2105322000+007
13	-.1278983833+000	.2229157267+001	-.1514169624+001	.9529978055+003	-.2938094750+007
14	-.1099090517+000	.2166330010+001	-.1473795333+001	.1401411875+009	-.5128893800+007
15	-.0466607126+000	.1975618011+001	-.1409433259+001	.2013250455+009	-.5596992250+007
16	-.3339101205+000	.1000110011+001	-.1383378044+001	.2332838050+009	-.7460276000+007
17	-.7890896880+000	.1000110011+001	-.1366423582+001	.5913046345+009	-.9739000750+007
18	-.6633226180+000	.1000110011+001	-.1320059693+001	.5316297255+009	-.1612004643+008
19	-.6633226180+000	.1000110011+001	-.1320059693+001	.7115111511	-.3043517800+008
20	-.5022505000+000	.1000110011+001	-.1321806037+001	.9395535005+009	-.3554803437+008
21	-.5022505000+000	.1000110011+001	-.1290751593+001	.2255331001+010	-.3161744950+008
22	-.4647838356+000	.1000110011+001	-.1277341262+001	.5802104802+010	-.3397399915+008
23	-.4016377434+000	.1000110011+001	-.1266113474+001	.2016771346+010	-.4723899000+008
24	-.3225523100+000	.1000110011+001	-.1243912053+001	.2549457033+010	-.5706316675+008
25	-.3225523100+000	.1000110011+001	-.1243912053+001	.3194449703+010	-.6847091250+008
26	-.3525541330+000	.1000110011+001	-.1236262040+001	.3970003936+010	-.8163346335+008
27	-.3825523100+000	.1000110011+001	-.1234129695+001	.4096559400+010	-.9674339200+008
28	-.3166404031+000	.1000110011+001	-.1217202515+001	.5996514641+010	-.1142946627+009
29	-.2970000000+000	.1000110011+001	-.1217202515+001	.7295914453+010	-.1336640595+009
30	-.2559616284+000	.1000110011+001	-.1209233374+001	.8319738636+010	-.1559345312+009
31	-.2559616284+000	.1000110011+001	-.1202555723+001	.1066016453+011	-.1810773270+009
32	-.2445440000+000	.1000110011+001	-.1183757762+001	.3267274939+011	-.2093636407+009
33	-.2445440000+000	.1000110011+001	-.1183757762+001	.1907111866+011	-.2410787905+009
34	-.2345038000+000	.1000110011+001	-.1183757762+001	.1703625956+011	-.2765191943+009
35	-.2244274594+000	.1000110011+001	-.1173213132+011	.2101172132+011	-.3160135500+009
36	-.2154566666+000	.1000110011+001	-.1173213132+011	.2464440301+011	

Table 22. Numerical coefficients of the (0,3)-factorization of the JYPQ operator with $\mu=0$, $x_1=1$ and $x_2=0.5$.

r	a ₀ (r)	a ₁ (r)	a ₂ (r)	b ₀ (r+3)	b ₁ (r+3)
37	-.20722248848-001	.1354008987+001	-.1168465899+001	.2872635046+011	-.8598988157+009
38	-.1995851946-001	.1343501443+001	-.1168868613+001	.3349330226+011	-.4085171095+009
39	-.1928785000-001	.1338596637+001	-.1159515401+001	.3832408834+011	-.4622585892+009
40	-.1866882761-001	.1332343365+001	-.1155338734+001	.4434374007+011	-.5215111830+009
41	-.1794847470-001	.1325992329+001	-.1151467417+001	.5162173155+011	-.5866881187+009
42	-.1735784707-001	.1320229293+001	-.1147740282+001	.5923258961+011	-.6582260045+009
43	-.1680495797-001	.1299077664+001	-.1144192064+001	.6725624458+011	-.7365090852+009
44	-.1628214511-001	.1291531629+001	-.1140810187+001	.727823679+011	-.8222008830+009
45	-.1588186440-001	.1284885572+001	-.1137583228+001	.8108998190+011	-.9156821217+009
46	-.1540095029-001	.1277583302+001	-.1134500785+001	.9063908888+011	-.1017940187+010
47	-.1494947147-001	.1271009946+001	-.1131588869+001	.1127793863+012	-.1127960643+010
48	-.1449985165-001	.1264794575+001	-.1128782800+001	.1222718846+012	-.1267008937+010
49	-.1408840470-001	.1258866950+001	-.1126029628+001	.1432833809+012	-.1377874287+010
50	-.137259901-001	.1253178952+001	-.1123454458+001	.1609394458+012	-.1518484410+010
51	-.1339899557-001	.1247743894+001	-.1120958847+001	.180372812+012	-.1670260006+010
52	-.1309258645-001	.1242588666+001	-.1118561902+001	.2017221901+012	-.1838992482+010
53	-.128127001-001	.1237583327+001	-.1116266583+001	.2251374312+012	-.2010818736+010
54	-.1254248090-001	.1232758332+001	-.1114055621+001	.2507743367+012	-.2140360024+010
55	-.1229133509-001	.1228147161+001	-.1111939847+001	.2788000228+012	-.2240360024+010
56	-.1205449501-001	.1223721762+001	-.1109880824+001	.3093876812+012	-.2282158825+010
57	-.1183296606-001	.1219191454+001	-.1107990947+001	.3423218885+012	-.2385621677+010
58	-.1162359831-001	.1214965578+001	-.1106199999+001	.3789968136+012	-.2402733333+010
59	-.1142449702-001	.1211146726+001	-.1104459807+001	.4184150832+012	-.2437450844+010
60	-.1123538212-001	.1207600976+001	-.1102888883+001	.4611923458+012	-.2486991880+010
61	-.1105665809-001	.1204387174+001	-.1101466584+001	.5075588886+012	-.2540580789+010
62	-.108882370-001	.1196967706+001	-.1100066580+001	.5573048669+012	-.2600580789+010
63	-.107308270-001	.1190696770+001	-.1097899808+001	.6119832806+012	-.2666650789+010
64	-.1058355802-001	.1184888888+001	-.1095858882+001	.6705523874+012	-.2738044058+010
65	-.1044665802-001	.1179588888+001	-.1093867888+001	.7337333333+012	-.2814044058+010
66	-.103199015-002	.1174888888+001	-.1091946462+001	.8017929519+012	-.2894044058+010
67	-.102022227-002	.1170647188+001	-.1090088854+001	.8750825015+012	-.2978044058+010
68	-.1009355558+002	.1166821022+001	-.1088378881+001	.9537333313+012	-.3066044058+010
69	-.1000000000+002	.1163270521+002	-.1086788881+001	.1038811434+013	-.3158044058+010
70	-.9922222222+002	.1159919499+001	-.1085262789+001	.1129014162+013	-.3254044058+010
71	-.9855555555+002	.1156666666+001	-.1083666666+001	.1236223333+013	-.3354044058+010
72	-.9799999999+002	.1153535353+001	-.1082066666+001	.1350812296+013	-.3458044058+010
73	-.9755555555+002	.1150555555+001	-.1080466666+001	.1474166181+013	-.3566044058+010

Table 22. (Continued)

r	$a_0(r)$	$a_1(r)$	$a_2(r)$	$b_0(r+3)$	$b_1(r+3)$
74	-.8276591600-002	1.1657925559+001	-.1082644943+001	.1560670497+013	-.10184665698+011
75	-.8241946409-002	1.1684401443+001	-.1031522283+001	.1687755221+013	-.1081959410+011
76	-.8111584663-002	1.1611584663+001	-.1080426769+001	.1623343576+013	-.1154064910+011
77	-.7955929794-002	1.1509297940+001	-.1079366620+001	.1967990298+013	-.1229906243+011
78	-.7856985565-002	1.1562767666+001	-.1078338356+001	.2121877220+013	-.1309851990+011
79	-.7725108666-002	1.1546641360+001	-.1077320379+001	.2285766226+013	-.1393819350+011
80	-.7572191611-002	1.1526158211+001	-.1076378785+001	.2466069709+013	-.1482023414+00+011
81	-.7403423088-002	1.1506459474+001	-.1075378320+001	.26645311035+013	-.1574646007+011
82	-.7223434321-002	1.1487764665+001	-.1074378271+001	.2882002513+013	-.1671834887+011
83	-.7033233088-002	1.1469976677+001	-.1073383286+001	.3119202818+013	-.1772770403+011
84	-.6832431262-002	1.1453276677+001	-.1072383271+001	.33723188253+013	-.1878706306+011
85	-.6621191509-002	1.1437666677+001	-.1071383271+001	.3645996999+013	-.1989696969+011
86	-.6399201509-002	1.1423193281+001	-.1070383271+001	.3935273524+013	-.2106227346+011
87	-.6166924309-002	1.1409830281+001	-.1069383271+001	.4239688510+013	-.2228236363+011
88	-.5925245609-002	1.1397555683+001	-.1068383271+001	.4559006600+013	-.2355699000+011
89	-.5675245609-002	1.1386379380+001	-.1067383271+001	.4893333613+013	-.2488333613+011
90	-.5418191609-002	1.1376221755+001	-.1066383271+001	.5242666613+013	-.2625999999+011
91	-.5155191609-002	1.1367169100+001	-.1065383271+001	.5607000000+013	-.2768333756+011
92	-.4887229526-002	1.1359144622+001	-.1064383271+001	.5986333613+013	-.2915666666+011
93	-.4614229526-002	1.1352229526+001	-.1063383271+001	.6379666613+013	-.3067999999+011
94	-.4337229526-002	1.1346379380+001	-.1062383271+001	.6787000000+013	-.3235333613+011
95	-.4057229526-002	1.1341630281+001	-.1061383271+001	.7208333613+013	-.3407666666+011
96	-.3774229526-002	1.1337929526+001	-.1060383271+001	.7643666613+013	-.3584999999+011
97	-.3489229526-002	1.1335229526+001	-.1059383271+001	.8093000000+013	-.3767333613+011
98	-.3203229526-002	1.1333555683+001	-.1058383271+001	.8556333613+013	-.3954666666+011
99	-.2917229526-002	1.1332830281+001	-.1057383271+001	.9033666613+013	-.4146999999+011
100	-.2632229526-002	1.13330281+001	-.1056383271+001	.9525000000+013	-.4344333613+011
101	-.2348229526-002	1.1334193281+001	-.1055383271+001	.1008333613+013	-.4546666666+011
102	-.2065229526-002	1.1335403281+001	-.1054383271+001	.1092666613+013	-.4753999999+011

Table 22. (Concluded).

r	z (r)	w ₁₀₀ (r)	y _r (100)	η _{r,1} (100)	J _r (1)
01	000-66483830455				26300-603
02	100-20000000000				200-000
03	000+18309099999				44223-173
04	100+5902199927				15023-380
05	200+40222828291				20930-000
06	300+3095999992				24276-630
07	000+4100032000				100-000
08	000+3526191968				11490-349
09	000+4034610000				19593-558
10	000+4093514000				100-000
11	000+0999261992				44000-550
12	081-6691093992				151-000
13	601-9989544912				121-000
14	000-0609960982				690-222
15	090-0225904040				29000-000
16	100-19919914				11079-100
17	100-6996000000				140-000
18	510-999454040				30730-000
19	081-6691093992				151-000
20	601-9989544912				121-000
21	000-0609960982				690-222
22	090-0225904040				29000-000
23	100-19919914				11079-100
24	100-6996000000				140-000
25	510-999454040				30730-000
26	081-6691093992				151-000
27	601-9989544912				121-000
28	000-0609960982				690-222
29	090-0225904040				29000-000
30	100-19919914				11079-100
31	100-6996000000				140-000
32	510-999454040				30730-000
33	081-6691093992				151-000
34	601-9989544912				121-000
35	000-0609960982				690-222
36	090-0225904040				29000-000
37	100-19919914				11079-100
38	100-6996000000				140-000
39	510-999454040				30730-000
40	081-6691093992				151-000
41	601-9989544912				121-000
42	000-0609960982				690-222
43	090-0225904040				29000-000
44	100-19919914				11079-100
45	100-6996000000				140-000
46	510-999454040				30730-000
47	081-6691093992				151-000
48	601-9989544912				121-000
49	000-0609960982				690-222
50	090-0225904040				29000-000
51	100-19919914				11079-100
52	100-6996000000				140-000
53	510-999454040				30730-000
54	081-6691093992				151-000
55	601-9989544912				121-000
56	000-0609960982				690-222
57	090-0225904040				29000-000
58	100-19919914				11079-100
59	100-6996000000				140-000
60	510-999454040				30730-000
61	081-6691093992				151-000
62	601-9989544912				121-000
63	000-0609960982				690-222
64	090-0225904040				29000-000
65	100-19919914				11079-100
66	100-6996000000				140-000
67	510-999454040				30730-000
68	081-6691093992				151-000
69	601-9989544912				121-000
70	000-0609960982				690-222
71	090-0225904040				29000-000
72	100-19919914				11079-100
73	100-6996000000				140-000
74	510-999454040				30730-000
75	081-6691093992				151-000
76	601-9989544912				121-000
77	000-0609960982				690-222
78	090-0225904040				29000-000
79	100-19919914				11079-100
80	100-6996000000				140-000
81	510-999454040				30730-000
82	081-6691093992				151-000
83	601-9989544912				121-000
84	000-0609960982				690-222
85	090-0225904040				29000-000
86	100-19919914				11079-100
87	100-6996000000				140-000
88	510-999454040				30730-000
89	081-6691093992				151-000
90	601-9989544912				121-000
91	000-0609960982				690-222
92	090-0225904040				29000-000
93	100-19919914				11079-100
94	100-6996000000				140-000
95	510-999454040				30730-000
96	081-6691093992				151-000
97	601-9989544912				121-000
98	000-0609960982				690-222
99	090-0225904040				29000-000
100	100-19919914				11079-100
101	100-6996000000				140-000
102	510-999454040				30730-000
103	081-6691093992				151-000
104	601-9989544912				121-000
105	000-0609960982				690-222
106	090-0225904040				29000-000
107	100-19919914				11079-100
108	100-6996000000				140-000
109	510-999454040				30730-000
110	081-6691093992				151-000
111	601-9989544912				121-000
112	000-0609960982				690-222
113	090-0225904040				29000-000
114	100-19919914				11079-100
115	100-6996000000				140-000
116	510-999454040				30730-000
117	081-6691093992				151-000
118	601-9989544912				121-000
119	000-0609960982				690-222
120	090-0225904040				29000-000
121	100-19919914				11079-100
122	100-6996000000				140-000
123	510-999454040				30730-000
124	081-6691093992				151-000
125	601-9989544912				121-000
126	000-0609960982				690-222
127	090-0225904040				29000-000
128	100-19919914				11079-100
129	100-6996000000				140-000
130	510-999454040				30730-000
131	081-6691093992				151-000
132	601-9989544912				121-000
133	000-0609960982				690-222
134	090-0225904040				29000-000
135	100-19919914				11079-100
136	100-6996000000				140-000
137	510-999454040				30730-000
138	081-6691093992				151-000
139	601-9989544912				121-000
140	000-0609960982				690-222
141	090-0225904040				29000-000
142	100-19919914				11079-100
143	100-6996000000				140-000
144	510-999454040				30730-000
145	081-6691093992				151-000
146	601-9989544912				121-000
147	000-0609960982				690-222
148	090-0225904040				29000-000
149	100-19919914				11079-100
150	100-6996000000				140-000
151	510-999454040				30730-000
152	081-6691093992				151-000
153	601-9989544912				121-000
154	000-0609960982				690-222
155	090-0225904040				29000-000
156	100-19919914				11079-100
157	100-6996000000				140-000
158	510-999454040				30730-000
159	081-6691093992				151-000
160	601-9989544912				121-000

Table 23. Computation of J_r(1) from the JYPQ equation with μ=0, x₁=1, x₂=0.5, i=0, m=100, j=1.

r	z(r)	w ₁₀₀ (r)	y _r (100)	η _{r,1} (100)	Q _r (0.5)
0	(.54988061448+000)				
1	(-.2258469278+000)				
2	(-.8186682680+000)				
3	-.5047576455+002				
4	.1718708983+004				
5	.1640476225+005				
6	.3184088692+005				
7	-.1197502686+006				
8	-.4811982211+006				
9	-.2918088574+006				
10	.1217698851+007				
20	-.1117664768+009				
30	.3256688786+009				
40	.4227104546+010				
50	-.2059749110+011				
60	.1577881422+011				
70	.9017998828+011				
80	-.2046915722+012				
90	.1498895899+012				
100	-.4241970700+012				
110	-.6201147816+012				
120	-.1777688888+012				
130	.5081575540+012				
140	.7121211447				
150	.2148287447+012				
160	-.6052042840+012				
170	-.8810012469+012				
180	-.2541581695+012				
190	.7169441082+012				
101	.1041659452+013	.4707947541-002	.9427982240-001	.6448970616-003	.1291470462+000
102	.3990484912+013	.2195844033-004	.9492421946-001	.8658374800-006	.3461941243-001
103	-.8450170483+013	.1014722037-006	.9492502530-001	-.1127575337-007	-.9854043140-001
104	-.1225580995+013	.4646278180-009	.94925807402-001	-.7487943837-010	-.1271185535+000
105	-.3520495553+012	.2103200739-011	.94925807395-001	-.9759976886-013	.6000000000
106			.94925807395-001		

Table 25. Computation of Q_r(0.5) from the JYPQ equation with μ=0, x₁=1, x₂=0.5, i=0, m=100, j=3.

r	z(r)	$Y_r(1)$
0	(.8925696420-001)	-.8327313303+062
1	(-.7812123213+000)	-.2604013366+093
2	(.1659032907+001)	-.2570730973+093
3	(-.582157606+001)	-.8056329705+095
4	.2665403272+003	-.4256246134+096
5	.1431033266+006	-.6730204933+097
6	.8211305133+009	-.1216110143+099
7	.11747276313+011	
8	.4177402913+012	
9	.1491449323+014	
01	.9699132490+000	
02	.2037031522+003	-.4113970314+093
03	.2227051907+002	-.3043133743+093
04	.2227051907+002	-.0990430799+093
05	.2094812321+001	-.7134376299+093
06	.1524010291+004	-.2191123311+093
07	.8914020230+004	-.5111093773+093
08	.6514132949+003	-.4430519343+093
09	.1076253391+074	-.6523234115+093
10	.2621195933+073	-.1174727631+093
11	.5044463193+073	-.2136319363+093
12	.935910143+073	-.3931240003+070
13	.1926913335+073	-.7312333339+072
14	.3332222245+073	-.3332222245+073
15	.7667409922	-.2611933001+077
16	.1551200252+073	-.5014003243+073
17	.3121521294+072	-.8723333307+073
18	.6546606412+074	-.1900060149+074
19	.1307404991	-.3727333310+073
001		

Table 26. Computation of $Y_r(1)$ from the JYPQ equation with $\mu=0$, $x_1=1$, $x_2=0.5$, $i=0$, $m=100$, $j=4$.

Example 4.3.2 In this example we compute the solutions $P_r^1(0.5)$ and $Q_r^1(0.5)$ of the homogeneous equation defined in Figure 7, with $\mu = 1$, $x_1 = 1$, $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 100$. The algorithm will be stable provided that we take three initial values for each of these functions; compare (4.3.3). Tables 27 and 28 give the results of these computations for selected values of r ; see §4.0 for a full description of these tables. Initial values were obtained by hand calculation of the explicit elementary forms given in Tables 8.2 and 8.4 of [16]. The computed function values agreed satisfactorily for $r \leq 10$ with the limited data available in [16] and [17]. In addition, satisfactory agreement was obtained with double-precision values produced by an unpublished program of Dr. J. M. Smith; compare Example 4.3.1. Consequently, the stability of the method is confirmed for this example.

r	z(r)	w ₁₀₀ (r)	y _r (100)	η _{r,1} (100)	P _r ¹ (0.5)
0	(.000000000000)				
1	(-.86602540380+0000)				
2	(-.12999998106+0001)				
3	(-.16602922222+0003)				
4	(.46603799999+0004)				
5	(.7231831845+0005)				
6	(.95510336943+0006)				
7	(-.21111111111+0007)				
8	(.74719385445+0007)				
9	(.20045060818+0007)				
10	(.20045060818+0007)				
20	(-.1800619476+010)				
30	(.5932095701+010)				
40	(.1071445304+012)				
50	(-.6435491124+012)				
60	(.587673111115+012)				
70	(.4562923092+013)				
80	(-.140099999992+014)				
90	(.8444063043+013)				
91	(-.2452069769+014)				
92	(-.3609924379+014)				
93	(-.10469727531+014)				
94	(.30349322740+014)				
95	(.4456303087+014)				
96	(.1299225912+014)				
97	(-.8729525971+014)				
98	(-.5245694307+014)				
99	(-.1577323567+014)				
100	(.4555000150+014)				
101	(.666000000000+014)	.470295729-002	.6049207730+001	.4164026073-001	.000000000000
102	(.1918119230+014)	.2195426657-004	.6090347990+001	.5592476447-004	.000000000000
103	(.6500992030+014)	.1014485504-006	.6090903915+001	-.7450174283-006	.000000000000
104	(-.8870343643+014)	.4644545465-009	.6090903170+001	-.4977635242-008	.000000000000
105	(-.2319243770+014)	.2107237256-011	.6090903165+001	-.6490344953-011	.000000000000
106			.6090903165+001		

Table 27. Computation of P_r¹(0.5) from the JYPQ equation with μ=1, x₁=1, x₂=0.5, i=0, m=100, j=3.

r	z(r)	w ₁₀₀ (r)	y _r (100)	η _{r,1} (100)	Q _r ¹ (0.5)
0	(-.1154700500+001)				.2691878995+001
1	(-.1058068845+001)				.1984006612+001
2	(.7298860590+000)				-.9839668000+000
3	.76885918472+003				-.8888709067+001
4	.6756871488+004				-.2587206800+001
5	-.1640816000+005				.1118811645+001
6	-.8541824004+006				.4008405651+001
7	-.7689728225+006				.8028234180+001
8	.7891965473+007				.1640757029+001
9	.1272983300+008				-.7198807871+001
10	.5566925550+009				.6028114936+001
20	-.8878449965+011				.8516720988+001
30	.1703365594+012				-.1012070116+002
40	.26800543247+012				.7971604891+001
50	-.8099429000+013				.8150077898+001
60	.6897088827+013				-.1388386420+002
70	.599981816685+013				-.9687961614+001
80	-.4201806107+014				.8681708092+001
90	-.8878479041+014				.1258669816+002
91	.1504789899+014				.9236404948+001
92	.6079816164+014				-.6485238883+001
93	.4788779601+014				-.1278897418+002
94	-.1888221546+014				-.9682868894+001
95	-.7490857483+014				.8487079026+001
96	-.5885020988+014				.1393897609+002
97	.8379498105+014				.9528668669+001
98	.9166417449+014				
99	.7106587002+014				
100					
101	-.8770207800+014	.4707495729-002	.9544185226+001	-.1737295218-001	-.8589749051+001
102	-.1114000099+015	.2195426657-004	.9526762274+001	-.8250818121-003	-.1817821487+002
103	-.82837566745+014	.1014435504-006	.9526437242+001	-.1175296869-005	-.9668437522+001
104	.8887942831+014	.4644545486-009	.9526436667+001	.2077387541-003	.8826579851+001
105	.1246270742+015	.2107227236-011	.9526436069+001	.3773098262-010	.3000070060
106			.9526436069+001		

Table 28. Computation of Q_r¹(0.5) from the JYPQ equation with μ=1, x₁=1, x₂=0.5, i=0, m=100, j=3.

4.4 Examples Involving the Inhomogeneous JYPQ Equation

Using Stirling's formula, we find

$$(4.4.1) \quad \frac{\Gamma(r+\mu+1)}{\Gamma(r+3/2)} \sim r^{\mu-1/2}, \quad r \rightarrow \infty$$

for fixed values of $\mu = 0, 1, 2, \dots$. Comparison of (4.2.1) with (4.3.1), (4.3.2) and (4.4.1) shows that

$$(4.4.2) \quad E_r(x_1) < \left\{ \begin{array}{l} P_r^\mu(x_2) \\ Q_r^\mu(x_2) \end{array} \right\}, \quad r \rightarrow \infty$$

for fixed μ , x_1 and x_2 such that $x_1 > 0$ and $x_2 < 1$. Accordingly, the Anger-Weber function is a solution of type 1 and asymptotically the correct number of initial values to be used in every case is $j = 1$; compared (4.3.3). However, the separation of $E_r(x_1)$ from the solutions of type 2 is weak when $\mu = 0$ (being of relative order $r^{-1/2}$ at the odd terms of the sequence and $r^{-3/2}$ at the even terms of the sequence). The separation is also weak when $\mu = 1$ (being of relative order $r^{-3/2}$ at the odd terms of the sequence and $r^{-5/2}$ at the even terms of the sequence). Accordingly, we treat $E_r(1)$ as a solution of type 2 in both cases, even though when $\mu = 1$ greater numerical instability is incurred than in any of the other examples. This is especially true at the even terms of the sequence, which are separated from the Legendre functions by the relative order of $r^{-5/2}$.

From (4.2.4) and (4.3.3) it is seen that the type of the Struve function $H_r(x_1)$ with respect to the JYPQ operator is 0 for all admissible values of x_1 and x_2 . But again because of the weakness of the separation, we regard $H_r(x_1)$ as belonging to type 1.

Example 4.4.1 For the inhomogeneous fourth-order difference equations defined in Figure 7, we compute $E_r(1)$ with $\mu = 1$, $x_1 = 1$ and $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 100$; and $H_r(0.1)$ with $\mu = 1$, $x_1 = 0.1$ and $x_2 = 0.5$, for $r = 0, 1, 2, \dots, 50$. (We restrict the sequence of $H_r(0.1)$ to $r \leq 50$ for the same reasons that were described in Example 4.2.1.) For reasons given above, we use 3 initial values (rather than 1) for $E_r(1)$ and 1 initial value (rather than 0) for $H_r(0.1)$. Initial values for $E_r(1)$ and $H_r(0.1)$ were computed from series expansions given in [21] and checked against values found in [21] and [23].

Tables 29 and 30 give extracts from the results of these computations for selected values of r ; see §4.0 for a full description of these tables. Comparison of the function values produced by the algorithm with values given in [21] and [23] show good agreement over the available ranges, which, however, do not extend beyond $r = 13$. Comparison of the values of $E_r(1)$ in Table 2.9 with values computed to 10 significant figures from the series expansion shows a steady deterioration of precision at the even values of r , culminating in a relative error of $0.5(10)^{-5}$ at $r = 100$. At the odd values of r the precision also deteriorates steadily, but at a slower rate, culminating in a relative error of $0.8(10)^{-7}$ at $r = 99$. These rates of loss of precision are consistent with the separation ratios of $r^{-5/2}$ and $r^{-3/2}$ which govern the growth of the error in this example.

That is, an individual relative rounding error of, say, 10^{-10} in one of the stored initial values of $E_r(1)$ will be amplified by a factor of approximately $100^{5/2} = 10^5$ at $E_{100}(1)$ and by $100^{3/2} = 10^3$ at $E_{99}(1)$. (As noted in Eq. (4.2.1), the function is in behavior different at even and odd values of r .) These amplifications after 100 steps are greater than the factor of 100 that was predicted and observed in the computation of

$A_r(1)$ in Example 4.1.3 and Table 13, where $A_r(1)$ was computed as if it were a function of the next higher type than it actually is.

Finally, comparison of the values of $H_r(0.1)$ in Table 30 with values computed to 10 significant figures from the series expansion shows agreement to 9 or more significant figures at all values of r . The loss of precision in the Struve function computation is slight, due to the separation ratio of only $r^{-1/2}$, but it offsets impossibly slow convergence which would be evidenced if the theoretically correct number of initial values were to be taken.

r	z(r)	w ₁₀₀ (r)	y _r (100)	η _{r,1} (100)	E _r (1)
0	(-.5666566370+000)				.2400000000+000
1	(.4381624362+000)				.4785019803+001
2	(.1717419846+000)				.1340009783+000
3	.7690941103+002				.1891944657+001
4	.1296888822+003				.9808234291+001
5	.3580689967+004				.1029381123+001
6	.1251440409+004				.7166003790+001
7	.3210589930+005				.6502159113+002
8	.5129388429+004				
9	.1487618467+006				
10	.1473523876+005				
20	.8187300182+006				.1599629741+002
30	.1737479379+007				.7009381200+003
40	.5737318992+007				.390303076+003
50	.1437594106+008				.2548819496+003
60	.3032734086+008				.1769316716+003
70	.5637574469+008				.1299702382+003
80	.97913808226+008				.9980274129+004
90	.1579689528+009				.7061498977+004
91	.297858373+011				.6996607801+002
92	.00+0424999				.7828248961+004
93	.7226899999972+011				.6346195274+002
94	.8319796520+009				.7206441899+004
95	.000073332929+009				.6702000926+002
96	.8607356090+011				.6909811959+004
97	.408956504+009				.6563235664+002
98	.3230942203+009				.6630041815+004
99	.46556180156+011				.643116331+002
100	.2421221073+009				.6067430993+004
101	.5041524599+011				
102	.2620992263+009	.4707495729+002	.3215467135+004	.3151820350+004	.63037045701+002
103	.55673749687+006	.2195426657+004	.6367287435+004	.7649079584+009	.5128234485+004
104	.2388087131+009	.1014485594+006	.6367383975+004	.7500401654+009	.6131307682+002
105	.513821399696+011	.4644545486+009	.6367488979+004	.1750585505+013	.5936824737+004
106	.3065559790+009	.2102227256+011	.6567438931+004	.1717202266+013	.6068345301+002
107		.9474675870+014	.6367438933+004	.3857278079+013	.6060000000
			.6367438933+004		

Table 29. Computation of E_r(1) from the JYPQ equation with μ=1, x₁=1, x₂=0.5, i=0, m=100, j=3.

r	z(r)	w ₅₀ (r)	y _r (50)	n _{r,1} (50)	H _r (0.1)
0	.6359126999-001)				.2120651601-002
1	.1764475402+002				.4242111249-004
2	.3725796347+001				.6060800286-006
3	.2006152537+000				.6734676051-008
4	.1130603957-001				.6122718212-010
5	.2951956860-003				.4709944262-012
6	.5555247554-005				.3140044933-014
7	.8055968245-007				.1847123376-016
8	.9421280136-009				.9721864425-019
9	.9181233205-011				.4629523149-021
10	.7641248782-013				.2012859489-023
11	.5535693734-015				.8051517475-026
12	.3544096079-017				.2982066699-028
13	.2030075239-019				.1623307246-030
14	.1051029043-021				.3317141764-033
15	.4960510130-024				.1005200276-035
16	.2149332779-026				.2872015573-038
17	.8609246729-029				.7762240113-041
18	.3203064964-031				.1990326276-043
19	.1112427579-033				.4854472688-046
20	.3621645659-036				
25	.5651366921-049				.2136010241-059
30	.268669010-062				.3572007851-073
35	.4864654216-076				.2657537517-087
40	.3911351153-090				.9851329257-102
41	.5440692753-093				.1186900261-104
42	.7364236054-096				.1396363919-107
43	.9707556739-099				.1605017381-110
44	.1247075179-101				.1803391800-113
45	.1563259694-104				.1981756791-116
46	.1909659297-107				.2130916437-119
47	.2279042662-110				.22438071557-122
48	.2656949149-113				.2312446509-125
49	.3027455931-116				.2335806112-128
50	.3373306313-119				.2312660001-131
51	.3677234303-122	.8829267290+000	.2310457009-131	.2223791084-134	.2245322552-134
52	.3923673771-125	.9595439215-003	.2312680800-131	.2573697593-140	.2136403697-137
53	.4099657133-128	-.6907674373+000	.2312680803-131	-.1939641672-140	.1996692400-140
54	.4196379447-131	-.6142402122+000	.2312680801-131	-.1765443724-143	.000000000000
55			.2312680801-131		.000000000000
56					.000000000000

Table 30. Computation of H_r(0.1) from the JYPQ equation with μ=1, x₁=0.1, x₂=0.5, i=0, m=50, j=1.

CHAPTER 5. SUMMARY AND CONCLUSIONS

This thesis is concerned with the numerical solution of a broad class of linear difference equations of arbitrary order, and of either homogeneous or inhomogeneous form. An algorithm is presented for the stable computation of any solution of such an equation. The results apply directly to scalar linear difference equations but could be adapted to similar problems posed in vector form.

Historically, the algorithm is an outgrowth of algorithms of Miller and Olver. The underlying idea of Miller's algorithm is the following: If a solution $y(r)$ has growth behavior, as $r \rightarrow \infty$, that is less than or equal to the growth behavior of all independent solutions, then in the reverse direction its growth behavior is greater than or equal to the growth behavior of all independent solutions. Consequently, because of the superposition principle, errors introduced in computing $y(r)$ by backward recurrence cannot grow faster than $y(r)$ itself, and in this sense the backward recurrence process is stable. However, this is true only in a region in which the solutions actually maintain the growth behavior indicated by their asymptotic forms as $r \rightarrow \infty$. For this reason we call this the asymptotic region; evidently it depends on the asymptotic structure of the given difference equation for large r .

Miller's algorithm starts the backward recurrence process at a value of r that is larger than any r for which a valid approximation of $y(r)$ is wanted. Arbitrary starting values are assumed. The computed

solution is essentially a scalar multiple of $y(r)$ for early values of r when (i) the difference equation is homogeneous; (ii) $y(r)$ is subdominant compared to every independent solution of the equation. Under these conditions the algorithm is stable and $y(r)$ is obtained from the computed solution for early values of r by a normalization procedure, such as matching the value of the computed solution at $r = 0$ with $y(0)$ in order to determine the correct scale factor. Modified normalization procedures are required when the asymptotic behaviors of more than one fundamental solution are similar.

Miller's algorithm involves no forward recurrence, and is stable only for solutions whose growth behavior is less than or equal to the growth behavior of all independent solutions. Olver's algorithm was applied originally to second-order inhomogeneous difference equations, in cases where a particular solution $y(r)$ has growth behavior as $r \rightarrow \infty$ that lies between that of a pair of linearly independent solutions of the homogeneous equation. To construct a stable algorithm, Olver posed a boundary value problem having one condition at each end. The initial condition is the value of $y(r)$ at a suitable point $r = i$. The terminal condition is zero, it being anticipated that isolated values of $y(r)$ for large r will not be easily obtained. The resulting boundary value problem is a tri-diagonal linear system of algebraic equations. Olver specified a method for solving this linear system that involves a forward recurrence stage and a backward recurrence stage. The method is equivalent to Gaussian elimination without pivoting, scaled in a

particular way. Analysis of the method shows that (i) the forward and backward recurrence stages are stable, at least in ranges beginning at a sufficiently large value of r ; (ii) the second-order linear difference operator that corresponds to the difference equation is, in effect, factored into the product of two first-order operators. Furthermore Olver derived an infinite series expansion of the truncation error, valid at any point r , that is incurred by choosing a particular terminal point for the boundary value problem. This enabled the optimal terminal point to be calculated automatically, in contrast to Miller's algorithm in which the terminal point is guessed and subsequently tested.

Extensions of Olver's algorithm to difference equations of higher order were proposed by Zahar, Oliver and Cash. Criteria for selecting the correct number of initial conditions (and therefore the correct number of terminal conditions, since the total number of conditions must equal the order of the difference equation) were given independently by Zahar and Oliver. Zahar proved convergence of the algorithm under assumed conditions on the solutions of the adjoint equation. Oliver investigated stability by analyzing the factorization of the finite boundary value problem. Cash provided an extension of the series expansion of the truncation error. The work of Zahar, Oliver and Cash, taken together, contains the essential elements of a practicable generalization of Olver's algorithm.

In this thesis Olver's algorithm has been extended and analyzed afresh. The view was taken that a linear difference operator is an

infinite upper-triangular band matrix. For a specified initial point i and number of initial conditions j , an (i,j) -factorization of a linear difference operator D is said to exist if Gaussian elimination without pivoting about the j -th super-diagonal of D , starting at the point i on this diagonal, can be carried out indefinitely. This process produces two lower-order difference operators (infinite band matrices). Their product is, of course, the original operator, and the sum of their orders is the order of the original operator. Furthermore, the solution of any finite boundary-value problem having initial point i , number of initial points j and terminal point n may be found by forward recursion with one of these lower-order operators followed backward recursion with the other (Theorem 1.2.2). Procedurally, a finite boundary value problem is solved by the algebraic method of forward elimination without pivoting followed by back substitution; this is accompanied by an analysis of the stability of this process in terms of linear difference operators.

Determination of the correct number of initial conditions for stability requires a knowledge of the relative rates of asymptotic growth of solutions of the homogeneous equation as $r \rightarrow \infty$. Since the number of linearly independent homogeneous solutions is equal to the order ℓ of the linear difference operator, there cannot be more than ℓ distinct growth rates as $r \rightarrow \infty$. Cases in which there are exactly ℓ distinct growth rates are the easiest to treat. In these cases a basis exists that is linearly ordered by growth rate, and we call the operator totally separable. However, there exist operators of practical

interest, including constant-coefficient operators for which one or more of the distinct characteristic roots are equal in magnitude, that are not totally separable. Accordingly, a more general class of operators, called separable operators, is introduced (Definition 2.2.1). The solutions of a homogeneous equation that has a separable, but not totally separable, operator exhibit less than ℓ distinct rates of growth as $r \rightarrow \infty$. Separable operators are characterized by the existence of certain bases, called optimally ranked. An optimally ranked basis includes $\sigma \leq \ell$ disjoint subsets such that (i) neither of two distinct solutions in the same subset dominates the other as $r \rightarrow \infty$; (ii) any two solutions taken from distinct subsets are such that one dominates the other; (iii) no nonzero linear combination of solutions from one subset is dominated by any solution in the subset. These conditions ensure that separable operators are unambiguously defined in the sense that any two distinct optimally ranked bases have similar structure (Lemma 2.2.1).

The disjoint subsets of an optimally ranked basis are arranged in linear order by increasing rate of growth. All possible solutions of an inhomogeneous difference equation with arbitrary right side are classifiable by comparison with the solutions in an optimally ranked basis of the corresponding homogeneous form. Thus a particular solution $y(r)$ is said to be of type s if (i) $y(r)$ is not dominated as $r \rightarrow \infty$ by any solution in the first s subsets of an optimally ranked basis; (ii) $y(r)$ is dominated by every solution in the remaining $\sigma - s$ subsets. The extreme cases $s = \sigma$ and $s = 0$ correspond to particular

solutions for which forward recurrence and backward recurrence of the given difference equation are stable, respectively. For the intermediate cases $0 < s < \sigma$, let j be the number of solutions in the first s subsets. If in addition to being separable the difference operator is (i,j) -factorizable for some initial point i , analysis of the boundary value problem shows that the equivalent pair of forward and backward recurrence problems (Theorem 1.2.2) are stable in the sense that rounding errors are not propagated more rapidly than the solutions, at least in the asymptotic region belonging to the given problem. Accordingly, the solution $y(r)$ of the boundary value problem is obtained in a stable manner by forward elimination without pivoting followed by back substitution (§3.3 and Theorem 3.3.1). Furthermore, under appropriate conditions it follows that (i) $y(r)$ is uniquely determined by j initial values (Theorem 3.1.1); (ii) the algorithm converges pointwise as the terminal point proceeds to infinity (Theorem 3.2.1).

The numerical examples included in this thesis were chosen so that known asymptotic results could be used in conjunction with the new classification theory for separable operators. This enabled the correct number of initial conditions for any chosen solution to be ascertained immediately. The examples also show that it is sometimes desirable to associate a solution with solutions of the next higher theoretical type. This artificial raising of the type is manifested in the algorithm by the specification of the number of initial conditions appropriate for the higher type. This artifice preserves unique determination

of the solution and also convergence of the algorithm, but introduces instability. It is employed only when the asymptotic separation between two types is weak, consequently the loss of precision resulting from the instability is mild. The compensating advantage is that the rate of convergence is increased very substantially, enabling a much smaller terminal point to be used.

The trade-off between numerical instability and slow convergence can be regarded heuristically in the following way. First, suppose $u(r) \sim e^{-r}$ and $v(r) \sim e^r$ are two fundamental solutions of a second-order equation. Then one initial value of $u(r)$ determines the function uniquely and the algorithm is both convergent and stable. Furthermore, the rate of convergence is very rapid. This can be seen from the equation

$$u_n(r) = u(r) - \frac{e^{-n}}{e^n} \cdot v(r)\{1+o(1)\}, \quad n \rightarrow \infty,$$

where $u_n(r)$ is the solution of the approximating boundary value problem having n as its terminal point (compare Eq. 3.2.6). The rapid convergence of $u_n(r)$ to $u(r)$ as $n \rightarrow \infty$ for fixed r is readily apparent. Thus the algorithm performs very well with regard to both numerical stability and rapidity of convergence when the separation of solutions is strong as in this example of exponential separation.

Next, suppose that $u(r) \sim r^{-1}$ and $v(r) \sim r^{-1/2}$. Here again, one initial value of $u(r)$ determines the function uniquely and the algorithm is both convergent and stable. But in this case

$$u_n(r) = u(r) - \frac{n^{-1}}{n^{-1/2}} \cdot v(r)\{1+o(1)\}, \quad n \rightarrow \infty$$

and it is apparent that the rate of convergence will be governed by the very slowly decreasing function $n^{-1/2}$. Suppose, however, that we give an additional initial value of $u(r)$. With two initial values our algorithm is equivalent to forward recurrence of the given difference equation and all questions of convergence disappear. This forward recurrence is, of course, unstable for the desired solution $u(r)$ but the instability is very weak; any small component of the more dominant solution $v(r)$ can grow no faster than $r^{1/2}$ relative to $u(r)$. Although this rate of growth (or any other positive rate of growth) is ultimately sufficient to obliterate the desired solution when computed in finite precision arithmetic, subsequences of the solution of considerable length can be computed before this happens. Thus in many practical situations where the separation of solutions is weak, a minor modification[†] of the algorithm performs acceptably well with regard to both stability and convergence.

Finally, suppose that the relative separation of $u(r)$ and $v(r)$ is like $r^{-3/2}$ or $r^{-5/2}$ (compare Example 4.4.1 which presents the computation of a function exhibiting both forms of behavior, $r^{-3/2}$ on the subsequence of odd terms and $r^{-5/2}$ on the subsequence of even terms). These separation ratios are too weak to allow rapid convergence of the unmodified algorithm, and at the same time too strong for the numerical instability in the modified algorithm to be truly insignificant. Computation of cases such as this are the most difficult for the algorithm to perform well, and performance can be improved only by recourse to extended-precision computation.

[†]In general the modified algorithm will require a back substitution as well as a forward elimination stage. Nevertheless, similar considerations govern the trade-off between numerical instability and slow convergence.

Perhaps it should be stressed that a complete knowledge of the asymptotic behavior of all of the solutions of the homogeneous equation is essential to determine the correct number of initial conditions. This information may be developed for a broad class of linear difference equations using known analytical results (§2.2). An alternative would be to attempt to construct the essential classification parameters - the number of distinct types of solution, and the number of linearly independent solutions of each type - by numerical experimentation. Under this approach, the algorithm presented in this thesis would be applied with chosen initial conditions appropriate to different possible realizations of the classification theory, that is, different possible forms of the optimally ranked basis. Presumably, incorrect choices would be revealed by discernible numerical instability. The effectiveness of such an approach has not been assessed, however.

In conclusion, the results presented in this thesis are distinguished from the work of earlier authors in the following ways. First, the interplay between linear difference operators and infinite band matrices achieves a fuller blending of analytic and algebraic ideas. This leads to a clearer understanding of stability, including the tradeoff choices that exist between slow convergence and mild instability when the asymptotic separation of adjacent solutions is weak. Secondly, the classification theory for separable operators is believed to be new. It enables the proper number of initial conditions that are prescribed by the convergence theorem to be determined directly from the asymptotic behavior of fundamental solutions of the given difference equation. In contrast, Zahar employs fundamental solutions of the adjoint equation and Oliver applies a

point-by-point growth criterion between pairs of solutions. Finally, the numerical examples presented here have been chosen in such a way that the solutions exhibit a much richer variety of growth rates than in any previous investigation.

APPENDIX

Proof of Lemma 2.1.1

The proof consists of (i) identifying the sequence defined by equation (2.1.4) as an almost periodic sequence, and (ii) proving that every nonzero almost periodic sequence has an infinite subsequence which is bounded away from zero. We begin by presenting the essential elements of the theory of almost periodic functions.

First we give the characterization of almost periodic functions that is taken as the definition by Corduneanu [7]. A function of the form

$$T(x) = \sum_{k=1}^n c_k e^{i\lambda_k x}, \quad -\infty < x < \infty,$$

where the c_k are complex numbers, the λ_k are real numbers and $i = \sqrt{-1}$, is called a complex trigonometric polynomial. A function $f(x)$ taking complex values and having all real numbers as domain is an almost periodic function provided that to each $\varepsilon > 0$ there corresponds some trigonometric polynomial $T_\varepsilon(x)$ such that

$$|f(x) - T_\varepsilon(x)| < \varepsilon$$

uniformly on $-\infty < x < \infty$. Clearly, every complex trigonometric polynomial is itself an almost periodic function.

An alternative characterization is the definition of H. Bohr [4]: A continuous complex valued function $f(x)$ defined on the real line is almost periodic if to each $\varepsilon > 0$ there corresponds a positive real number $\ell = \ell(\varepsilon)$ such that every open interval of the real line of length ℓ contains at least one point ξ for which

$$|f(x+\xi) - f(x)| < \varepsilon$$

uniformly for every x . Any number ξ which satisfies this condition is called a translation number of f corresponding to ε , or more briefly, an ε -translation number of f .

An ε -translation number of f is "almost" a period of f . Every periodic function is almost periodic as well. If p is the fundamental period of a periodic function f , then rp for $r = 0, \pm 1, \pm 2, \dots$ are translation numbers of f for each ε . Bohr's definition generalizes the fact that the periods of a periodic function form an arithmetic progression.

Bohr's definition and Corduneanu's definition define exactly the same class of functions. This is the "fundamental theorem" of the theory of almost periodic functions. See either [4] or [7] for a full account.

Turning now to sequences, we have the following definition presented in [7, §1.6]: A function $f(r)$, taking complex values and having the set of integers as domain, is an almost period sequence provided that to each $\varepsilon > 0$ there corresponds some positive integer $N = N(\varepsilon)$ such that, within any set of N consecutive integers, there exists an integer p for which

$$|f(r+p) - f(r)| < \varepsilon .$$

uniformly over all integers r . Corduneanu proves "A necessary and sufficient condition for a sequence $\{a_r\}$ to be almost periodic is the existence of an almost periodic function $f(x)$, $-\infty < x < \infty$, such that $a_r = f(r)$, $r = 0, \pm 1, \pm 2, \dots$ "; see [7, Theorem 1.27].

We are now ready to prove Lemma 2.1.1. If r were allowed to assume all real values, instead of being restricted to $r = 0, 1, 2, \dots$, then equation (2.1.4) defines a trigonometric polynomial. By Corduneanu's definition, every trigonometric polynomial is an almost periodic function.

By the theorem quoted above, the sequence $x(r)$, $r = 0, \underline{+1}, \underline{+2}, \dots$ defined by equation (2.1.4) is an almost periodic sequence. The only thing remaining is to prove the following assertion:

If an almost periodic sequence $\{f(r)\}$ is not identically zero, then there exists an infinite subsequence of $\{f(r)\}$ which is bounded away from zero.

Proof Assume $\{f(r)\}$ is not identically zero. Let r_0 be such that $f(r_0) \neq 0$ and put $A = |f(r_0)|$. Put $\varepsilon = A/(2+A)$. Then $\varepsilon \in (0,1)$, and by the definition of an almost periodic sequence there exists a positive integer N_1 , say, such that for some integer $p_1 \in \{1,2,\dots,N_1\}$ the inequality

$$|f(r_0+p_1) - f(r_0)| < \varepsilon$$

is satisfied. Put $r_1 = r_0 + p_1$. Next, there exists a positive integer N_2 such that for some integer $p_2 \in \{1,2,\dots,N_2\}$ the inequality

$$|f(r_1+p_2) - f(r_1)| < \varepsilon^2$$

is satisfied. Put $r_2 = r_1 + p_2$. Continuing in this manner, we see that for each $k = 3,4,5,\dots$ we can find an integer $r_k > r_{k-1}$ such that

$$|f(r_k) - f(r_{k-1})| < \varepsilon^k.$$

For each $k = 1,2,3,\dots$ we have

$$\begin{aligned} |f(r_k) - f(r_0)| &\leq \sum_{j=1}^k |f(r_j) - f(r_{j-1})| \\ &< \sum_{j=1}^k \varepsilon^j < \frac{\varepsilon}{1-\varepsilon} < \frac{1}{2} A. \end{aligned}$$

Since $A = |f(r_0)|$, we have

$$\begin{aligned} |f(r_k)| &= |f(r_0) + f(r_k) - f(r_0)| \\ &\geq A - |f(r_k) - f(r_0)| > \frac{1}{2} A , \end{aligned}$$

which finishes the proof of the assertion. \square

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<p>Consider a given linear difference equation $\sum_{s=0}^{\ell} d_s(r)y(r+s) = g(r)$ where $\ell \geq 2$ and $r = 0, 1, 2, \dots$. Suppose y is a solution of this equation and u, v are solutions of the homogeneous form of this equation such that $u(r)/v(r) \rightarrow 0$, $y(r)/v(r) \rightarrow 0$, $u(r)/y(r) \rightarrow 0$. Under these circumstances algorithms for the computation of y based on forward or backward recurrence, such as the Miller algorithm, are numerically unstable.</p> <p>Stable algorithms, such as the method of Olver, have been based on approximating $y(r)$ by the solutions of a certain sequence of boundary value problems. Specifically, $y_n(r)$ is a solution that coincides with $y(r)$ over some initial range of r, say $r=i, i+1, \dots, i+j-1$, and satisfies $y_n(r) = 0$ for $r=n, n+1, \dots, n+\ell-j-1$. Here j is an integer whose value depends on the asymptotic behavior of the chosen solution $y(r)$ and n is an arbitrary large integer. Boundary value problems of this type are shown to be equivalent to two initial value problems of order j and $\ell-j$ by factorization of the difference operator. The solution of the problem of order j is obtained by forward recurrence; the solution of the other problem is obtained by backward recurrence.</p> <p>The algorithm is specified completely for a broad class of operators including, for example, every constant-coefficient operator. Convergence of $y_n(r)$ to $y(r)$ as $n \rightarrow \infty$ for fixed r is proved and an expansion of the truncation error is derived. Numerical stability is demonstrated. The method is tested by numerical examples involving fourth-order equations with variable coefficients.</p>		14. Sponsoring Agency Code	
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