# Infinite divisibility of probability distributions on the nonnegative reals

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# Definition

A random variable $X$ is said to be infinitely divisible if for every $n \in \mathbb{N}$ it can be written (in distribution) as

$$X \stackrel{d}{=} X_{n,1} + \cdots + X_{n,n},$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent and identically distributed random variables. Infinite divisibility is a property of the probability measure (distribution) induced by $X$. It follows that a probability measure $\mu$ on $\mathbb{R}$ is infinitely divisible if and only if, for every $n \in \mathbb{N}$, there is a probability measure $\mu_n$ such that $\mu$ is equal to the $n$-fold convolution of $\mu_n$.
Reference: F.W. Steutel and K. Van Harn, Infinite divisibility of probability distributions on the real line, Marcel-Dekker, New York, 2004.

# Example

In this talk we are interested in infinite divisibility of probability measures on $[0,\infty)$ which are given by a probability density function (pdf) $f(x)$ with $f(x)=0$ for $x<0$.
For $r>0$ and $\lambda>0$ consider the $\mathrm{gamma}(r,\lambda)$ distribution with density function

$$f(x;r,\lambda)=\frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x}\quad \text{for } x>0.$$

Then $f(x;r,\lambda)$ is the $n$-fold convolution of $f(x;\frac{r}{n},\lambda)$ so the $\mathrm{gamma}(r,\lambda)$ distribution is infinitely divisible.

# Student's t distribution

The student t distribution with $r > 0$ degrees of freedom is given by the pdf

$$g(x;r) = \frac{\Gamma(\frac{1}{2}(r+1))}{\sqrt{r\pi}\,\Gamma(\frac{r}{2})}\left(1+\frac{x^2}{r}\right)^{-\frac{1}{2}(r+1)}.$$

It is known that this distribution is infinitely divisible.
We are interested in the infinite divisibility of the half-student t distribution with density

$$f(x;r) = \begin{cases} 2g(x;r) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

# An integral equation

Suppose that $\mu$ is a probability measure on $[0, \infty)$ with density function $f(x)$. Suppose that $f$ is continuously differentiable on $[0, \infty)$ and $f(0) > 0$. Then the Volterra integral equation of the first kind

$$xf(x) = \int_0^x f(x-y)k(y)\,dy, \quad x \geq 0,$$

has a unique continuous solution $k : [0, \infty) \to \mathbb{R}$. It is known that $f(x)$ is infinitely divisible if and only if $k(x) \geq 0$ for all $x \geq 0$; see Steutel and van Harn.

# Equivalent integral equation

By differentiating both sides of the integral equation, we obtain the Volterra integral equation of the second kind

$$f(x) + xf'(x) = f(0)k(x) + \int_0^x f'(y-x)k(y)\,dy, \quad x \geq 0,$$

or, equivalently,

$$k(x) = g(x) + \int_0^x h(x-y)k(y)\,dy, \quad x \geq 0,$$

where $g(x) = f(0)^{-1}(f(x) + xf'(x))$, $h(x) = -f(0)^{-1}f'(x)$.

# Numerical approach

We solve the integral equation of the second kind numerically on $x \in [0, b]$ by using the trapezoidal rule. Let $N \in \mathbb{N}$ and set $\delta = \frac{b}{N}$. Let

$$x_i = i\frac{b}{N}, \quad i = 0, 1, \dots, N$$

and set

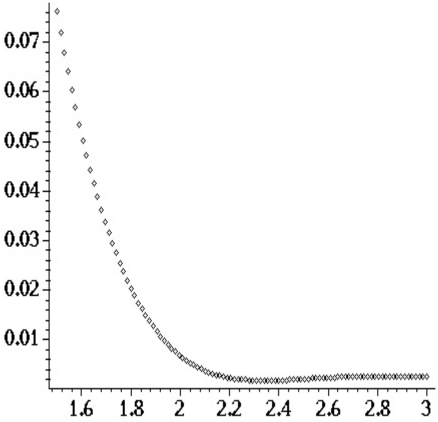
$$g_i = g(x_i),\ h_i = h(x_i), \quad i = 0, 1, \dots, N.$$

Then we define $k_0 = g_0$ and recursively compute $k_i$ from the equations

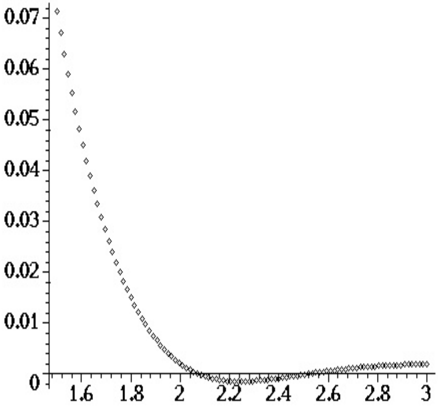
$$k_i = g_i + \tfrac{1}{2}\delta h_i k_0 + \delta \sum_{j=1}^{i-1} h_{i-j} k_j + \tfrac{1}{2}\delta h_0 k_i, \quad i = 1, 2, \dots, N.$$

We require $N$ to be so large that $\frac{1}{2}\delta h_0 < 1$. We then consider $k_i$ as an approximation of $k(x_i)$.

# $k(x)$ for $r = 14$



# $k(x)$ for $r = 16$



# Conjecture

Let $f(x;r)$ be the pdf of the half student t distribution with $r$ degrees of freedom. As $r \to \infty$, $f(x;r)$ converges to the half-normal pdf. It follows that there is $r_0 > 0$ such that $k(x;r)$ becomes negative for some $x > 0$. Very likely, there is $r_0 > 0$ such that $f(x;r)$ is infinitely divisible for $0 < r \leq r_0$ but not for $r > r_0$. Numerical calculations suggest that $14 < r_0 < 16$.

# Partial proof of conjecture

**Theorem**

*If $r \geq 16$ then the half-student t distribution with r degrees of freedom is not infinitely divisible.*

We will show that the solution $k(x)$ of the integral equation

$$xf(x) = \int_0^x f(x-y)k(y)\,dy$$

attains negative values when

$$f(x) = \left(1 + \frac{x^2}{q}\right)^{-q}, \quad q \geq \frac{17}{2}.$$

We start with the Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} (-1)^n f_n x^{2n}, \quad f_n = \frac{1}{n!}\frac{(q)_n}{q^n}$$

which converges for $|x| < \sqrt{q}$.

The solution $k(x)$ is also analytic in $|x| < \sqrt{q}$ with Taylor expansion

$$k(x) = \sum_{n=0}^{\infty} (-1)^n k_n x^{2n}.$$

Substituting the Taylor expansions in the integral equation and using the convolution

$$\frac{x^m}{m!} * \frac{x^n}{n!} = \frac{x^{m+n+1}}{(n+m+1)!},$$

we obtain, for every $n = 0, 1, \dots,$

$$(2n+1) f_n = \sum_{m=0}^{n} f_{n-m} k_m \frac{(2n-2m)!(2m)!}{(2n)!}.$$

This formula allows us to compute $k_n$ recursively.

The first coefficients are given by

$$\begin{aligned}
k_0 &= 1 \\
k_1 &= 2 \\
k_2 &= \frac{5}{3} + 2q^{-1} \\
k_3 &= \frac{37}{45} + \frac{14}{5}q^{-1} + 2q^{-2} \\
k_4 &= \frac{353}{1260} + \frac{193}{105}q^{-1} + \frac{249}{70}q^{-2} + 2q^{-3} \\
k_5 &= \frac{583}{8100} + \frac{145}{189}q^{-1} + \frac{1763}{630}q^{-2} + \frac{517}{126}q^{-3} + 2q^{-4}
\end{aligned}$$

In general, $k_n$ is a polynomial in $q^{-1}$ with rational coefficients.

By induction on $n$, we are able to show that

$$\frac{n+1}{n!}\frac{(q)_n}{q^n} \leq k_n \leq \frac{2n+1}{n!}\frac{(q)_n}{q^n}.$$

Based on this inequality we show the following. Let $x > 0$ and $0 < 2x^2 < q$. If we choose an even integer $N \geq 2$ so large that

$$(N+1)^2 q \geq (2N+3)(q+N)x^2.$$

then

$$\sum_{n=0}^{N-1}(-1)^n k_n x^{2n} \leq k(x) \leq \sum_{n=0}^{N}(-1)^n k_n x^{2n}.$$

Using a computer we calculate the partial sum

$$h(x) = \sum_{n=0}^{20} (-1)^n k_n x^{2n}$$

of the Taylor series for $k(x)$, and then $h(x_0)$, where $x_0^2 = \frac{5}{2}$. Now $q^{19} h(x_0)$ is a polynomial in $q$ of degree 19 with rational coefficients. If we substitute $q = \frac{17}{2} + p$ then all coefficients of $q^{19} h(x_0)$ when written in powers of $p$ are negative. Therefore, $h(x_0) < 0$ for $q \geq \frac{17}{2}$ and so

$$k(x_0) \leq h(x_0) < 0 \quad \text{for } q \geq \frac{17}{2}.$$

This completes the proof of the theorem.

# Solution by Laplace transform

Consider $f(x) = (1 + x^2)^{-q}$. We want to solve the integral equation

$$xf(x) = \int_0^x f(x-y)k(y)\,dy$$

by Laplace transform. Let $F(s), K(s)$ denote the Laplace transforms of $f, k$, respectively. Then

$$-F'(s) = F(s)K(s).$$

Let $\nu = \frac{1}{2} - q$. If $q \notin \mathbb{N}$ then

$$F(s) = \frac{\sqrt{\pi}}{2}\Gamma(1-q)\left(\frac{s}{2}\right)^{-\nu}\mathbf{K}_\nu(s).$$

The function $\mathbf{K}_\nu$ is the solution of the inhomogeneous form of Bessel's equation

$$w''(s) + \frac{1}{s}w'(s) + \left(1 - \frac{\nu^2}{s^2}\right) w(s) = \frac{\left(\frac{s}{2}\right)^{\nu-1}}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}$$

which vanishes at $s = +\infty$. We have

$$\mathbf{K}_\nu(s) = \mathbf{H}_\nu(s) - Y_\nu(s),$$

where $\mathbf{H}_\nu(s)$ is Struve's function; see F. Olver, Asymptotics and Special Functions.

# The case $q = 1$

In 1987 L. Bondesson showed that the half-student t distribution is infinitely divisible when $r = q = 1$. We will give another proof. The Laplace transform of $f(x) = \frac{1}{1+x^2}$ is

$$F(s) = \mathrm{Ci}(s) \sin s - \left(\mathrm{Si}(s) - \frac{\pi}{2}\right) \cos s,$$

where

$$\mathrm{Si}(s) = \int_0^s \frac{\sin x}{x}\, dx, \quad \mathrm{Ci}(s) = \gamma + \ln s + \int_0^s \frac{\cos x - 1}{x}\, dx$$

and $\gamma = 0.5772\ldots$ is Euler's constant.

# Zeros of $F$

**Theorem**

*The zeros of $F(s)$ in the upper half-plane are simple and they form a sequence $\{c_n\}_{n=1}^{\infty}$ such that*

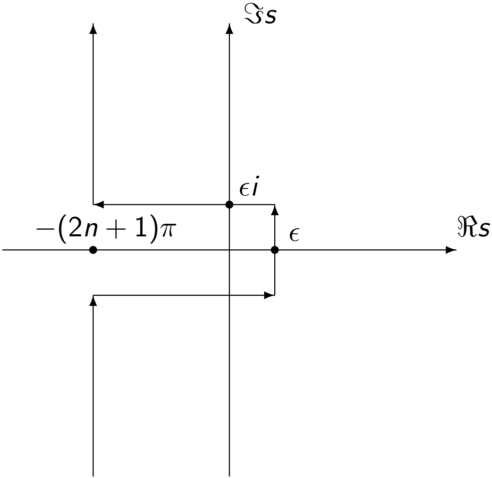
$$-2n\pi < \Re c_n < -(2n - \tfrac{1}{2})\pi \quad \textit{for } n \in \mathbb{N}.$$

The first three zeros of $F$ are (rounded to ten digits)

$$\begin{aligned} c_1 &= -5.830190833 + 3.056022944i, \\ c_2 &= -12.27934811 + 3.706041345i, \\ c_3 &= -18.63527112 + 4.098284913i. \end{aligned}$$

The equation $-F'(s) = F(s)K(s)$ shows that $K(s)$ is analytic in $-\pi \le \arg s \le \pi$ except for simple poles at the points $c_n$ and $\bar{c}_n$.

We compute the inverse Laplace transform $k(x)$ of $K(s)$ by applying the residue theorem to the contour




Then we let $n \to \infty$.

## Theorem

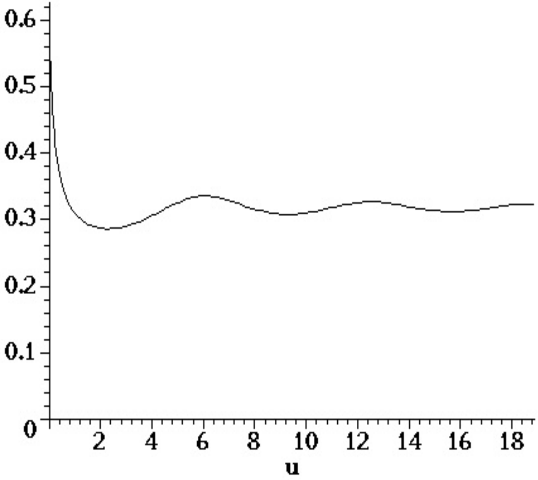
*For $x > 0$ we have*

$$k(x) = \int_0^\infty g(u)e^{-ux}\,du - 2\sum_{m=1}^\infty e^{-a_m x}\cos(b_m x),$$

*where $c_m = -a_m + ib_m$, $(2m - \frac{1}{2})\pi < a_m < 2m\pi$, $b_m > 0$, denote the zeros of $F(s)$, and*

$$\begin{aligned} g(u) &= \frac{1}{\pi}\Im K(-u - i0) \\ &= \frac{\mathrm{Si}(u) + \frac{\pi}{2}}{(F(u) - \pi\cos u)^2 + \pi^2\sin^2 u}. \end{aligned}$$

The graph of $g(u)$ looks like this



Using that $g(u) > \frac{1}{5}$ for all $u \geq 0$, we obtain

**Theorem**

*For $x > 0$ there holds the inequality*

$$k(x) > \frac{1}{5x} - \frac{2e^{-\frac{3}{2}\pi x}}{1 - e^{-2\pi x}}.$$

**Theorem**

*We have $k(x) > 0$ for all $x \geq 0$.*

The asymptotics of $k(x)$ as $x \to \infty$ is given by

$$k(x) = \frac{2}{\pi}\frac{1}{x} + \frac{4}{\pi^2}\frac{1 - 2\ln x}{x^2} + O(x^{-3}\ln^2 x).$$

We may use our formula to compute $k(x)$ numerically: