Recent Asymptotic Results for Euler–Maclaurin Expansions, Gauss–Legendre Quadrature, and Legendre Polynomial Expansions

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Euler–Maclaurin expansions and generalizations

Classical Euler–Maclaurin (E–M) expansion

$$I[f] = \int_a^b f(x)dx \approx h \sum_{i=0}^{n}{}'' f(a+ih) = T_n[f]; \quad h = \frac{b-a}{n}$$

In case $f \in C^\infty[a,b]$, there holds

$$T_n[f] \sim I[f] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\Big[f^{(2k-1)}(b) - f^{(2k-1)}(a)\Big]h^{2k} \quad (h \to 0).$$

$$f'(a) \neq f'(b) \quad \Rightarrow \quad T_n[f] - I[f] = O(h^2).$$

$$f^{(2k-1)}(a) = f^{(2k-1)}(b),\ 1 \leq k \leq m-1 \ \Rightarrow T_n[f] - I[f] = O(h^{2m}).$$

If $f^{(2k-1)}(a) = f^{(2k-1)}(b)$, $k = 1, 2, \ldots,$ then

$$T_n[f] - I[f] = O(h^{2\mu}) \quad \forall \mu > 0.$$

This happens, e.g., when $f \in C^\infty(-\infty, \infty)$ and is $(b-a)$-periodic.

Navot's generalizations of classical E–M expansion

In case $f(x) = (x-a)^\alpha g_a(x) = (b-x)^\beta g_b(x)$,
$g_a \in C^\infty[a,b)$, $g_b \in C^\infty(a,b]$,
$\Re\alpha, \Re\beta > -1$ and not necessarily integers,
there holds [Navot (1961, 1962), Lyness & Ninham (1967)]

$$T_n[f] \sim I[f] + \sum_{k=0}^{\infty} \frac{\zeta(-\alpha-k)}{k!} g_a^{(k)}(a) h^{\alpha+k+1} + \sum_{k=0}^{\infty} (-1)^k \frac{\zeta(-\beta-k)}{k!} g_b^{(k)}(b) h^{\beta+k+1} \quad (h \to 0).$$

Here, $T_n[f] = h \sum_{i=0}^{n}{}'' f(a+ih)$ does not include undefined $f(a)$ and $f(b)$.
$\zeta(\omega)$: the Riemann Zeta function, continued to the complex plane.
With the help of

$$\zeta(0) = -\frac{1}{2}; \;\; \zeta(-2m) = 0, \;\; \zeta(1-2m) = -\frac{B_{2m}}{2m}, \;\; m = 1, 2, \ldots,$$

we recover the (classical) E–M expansion when $\alpha, \beta \to 0$.

Recent generalizations of E–M expansions [S. (2004)]

Let $f \in C^\infty(a,b)$ and

$$f(x) \sim \sum_{s=0}^{\infty} c_s (x-a)^{\gamma_s} \;\; (x \to a), \quad f(x) \sim \sum_{s=0}^{\infty} d_s (b-x)^{\delta_s} \;\; (x \to b)$$

$$\Re\gamma_0 \le \Re\gamma_1 \le \cdots ; \quad \Re\delta_0 \le \Re\delta_1 \le \cdots ;$$

$$\gamma_s, \delta_s \ne -1, \;\; \lim_{s\to\infty} \Re\gamma_s, \Re\delta_s = \infty.$$

Assume that these asymptotic expansions can be differentiated term by term infinitely many times. Define

$$\check{T}_n[f] = h \sum_{i=1}^{n-1} f(a+ih); \quad h = \frac{b-a}{n}.$$

Then

$$\check{T}_n[f] \sim I[f] + \sum_{\substack{s=0 \\ \gamma_s \notin \{2,4,\ldots\}}}^{\infty} c_s \zeta(-\gamma_s) h^{\gamma_s+1} + \sum_{\substack{s=0 \\ \delta_s \notin \{2,4,\ldots\}}}^{\infty} d_s \zeta(-\delta_s) h^{\delta_s+1}$$

$\zeta(\omega)$: the Riemann Zeta function, continued to the complex plane.

If, more generally,

$$f(x) \sim \sum_{s=0}^{\infty} P_s(\log(x-a))(x-a)^{\gamma_s} \quad (x \to a); \quad P_s(y) \text{ polynomial},$$

$$f(x) \sim \sum_{s=0}^{\infty} Q_s(\log(b-a))(b-x)^{\delta_s} \quad (x \to b); \quad Q_s(y) \text{ polynomial},$$

then, with $D_\omega = d/d\omega$,

$$\check{T}_n[f] \sim I[f] + \sum_{s=0}^{\infty} P_s(D_{\gamma_s})\Big[\zeta(-\gamma_s)h^{\gamma_s+1}\Big] + \sum_{s=0}^{\infty} Q_s(D_{\delta_s})\Big[\zeta(-\delta_s)h^{\delta_s+1}\Big]$$

<u>Note:</u> If $R \in \pi_m$, then

$$R(D_\omega)\Big[\zeta(-\omega)h^{\omega+1}\Big] = h^{\omega+1}W(\log h), \quad W \in \pi_m.$$

These generalizations hold also when $I[f]$ is not defined in the regular sense, but is defined in the sense of *Hadamard finite part*; i.e., for all γ_s, δ_s

$$\gamma_s, \delta_s \neq -1.$$

(In such a case, $\{\check{T}_n[f]\}_{n=1}^{\infty}$ is divergent.)

Asymptotic expansions for Gauss–Legendre quadrature

$$I[f] = \int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_{ni} f(x_{ni}) = G_n[f];\;\; G_n[f] = I[f],\;\; \forall f \in \pi_{2n-1}.$$

x_{ni} : abscissas, w_{ni} weights; $P_n(x_{ni}) = 0$, $i = 1, \ldots, n$.

If $f \in C^\infty[-1,1]$, then

$$G_n[f] - I[f] = O(n^{-\mu}) \;\; (n \to \infty), \;\; \forall \mu > 0.$$

If $f(z)$ analytic in an open domain of z-plane containing $[-1,1]$, then

$$G_n[f] - I[f] = O(e^{-\sigma n}) \;\; (n \to \infty), \;\; \text{for some } \sigma > 0.$$

Poor convergence in case of endpoint singularities.
If, for example, $f(x) = (1-x)^\alpha g(x)$, $\alpha > -1$, $g \in C^\infty[-1,1]$, then

$$G_n[f] - I[f] = O(n^{-2\alpha-2}) \;\; (n \to \infty).$$

[See Davis & Rabinowitz (1984).]

Asymptotic expansions of Verlinden (1997)

If $f(x) = (1-x)^{\alpha} g(x)$, $\Re\alpha > -1$ but $\alpha \neq 0, 1, \ldots$, and $g(z)$ analytic in an open set containing $[-1, 1]$, then,

$$G_n[f] - I[f] \sim \sum_{k=1}^{\infty} a_k(\alpha) h^{\alpha+k} \quad (n \to \infty); \;\; h = (n + 1/2)^{-2}.$$

Here, $a_k(\alpha)$ independent of n and analytic for $\Re\alpha > -1$.

If $f(x) = \log(1-x)\,(1-x)^{\alpha} g(x)$, then

$$G_n[f] - I[f] \sim \sum_{k=1}^{\infty} D_{\alpha}\Big[a_k(\alpha) h^{\alpha+k}\Big] \quad (n \to \infty).$$

If $f(x) = \log(1-x) g(x)$, then [because $a_k(0) = 0$, $k = 0, 1, \ldots,$]

$$G_n[f] - I[f] \sim \sum_{k=1}^{\infty} b_k(\alpha) h^{k} \;\; (n \to \infty).$$

Recent asymptotic expansions [S. (2009)]

Let $f \in C^{\infty}(-1,1)$ and

$$f(x) \sim \sum_{s=0}^{\infty} A_s(1-x)^{\alpha_s} \quad (x \to 1); \quad f(x) \sim \sum_{s=0}^{\infty} B_s(1+x)^{\beta_s} \quad (x \to -1),$$

$$-1 < \Re\alpha_0 \le \Re\alpha_1 \le \cdots, \quad -1 < \Re\beta_0 \le \Re\beta_1 \le \cdots; \quad \lim_{s\to\infty} \Re\alpha_s, \Re\beta_s = \infty.$$

Assume that these asymptotic expansions can be differentiated term by term infinitely many times.

Then, with $h = (n+1/2)^{-2}$, $\mathbb{Z}^+ = \{0,1,2,\ldots\}$,

$$G_n[f] \sim I[f] + \sum_{\substack{s=0 \\ \alpha_s \notin \mathbb{Z}^+}}^{\infty} A_s \sum_{k=1}^{\infty} c_k(\alpha_s) h^{\alpha_s+k} + \sum_{\substack{s=0 \\ \beta_s \notin \mathbb{Z}^+}}^{\infty} B_s \sum_{k=1}^{\infty} c_k(\beta_s) h^{\beta_s+k} \quad (n \to \infty).$$

$c_k(\omega)$ analytic functions of ω for $\Re\omega > -1$. Also $c_k(\omega) = 0$ if $\omega \in \mathbb{Z}^+$.

Note that the $c_k(\omega)$ are defined via $f_\omega^\pm(x) = (1 \pm x)^\omega$ and

$$G_n[f_\omega^\pm] \sim I[f_\omega^\pm] + \sum_{k=1}^{\infty} c_k(\omega) h^{\omega+k} \quad (n \to \infty). \qquad (*)$$

[This follows from Verlinden (1997).]

To prove our result, show that, with $E_n[g] = G_n[g] - I[g]$, there holds

$$E_n[f] \sim \sum_{s=0}^{\infty} A_s E_n[f_{\alpha_s}^-] + \sum_{s=0}^{\infty} B_s E_n[f_{\beta_s}^+] \quad (n \to \infty),$$

and then invoke $(*)$ and re-expand.

If $\alpha_0, \beta_0 \notin \mathbb{Z}^+$, then

$$E_n[f] = O(h^\omega), \quad \omega = \min\{\Re\alpha_0 + 1, \Re\beta_0 + 1\}$$

If $\alpha_0, \beta_0 \in \mathbb{Z}^+$, then

$$E_n[f] = O(h^\omega), \quad \omega = \min\{\Re\alpha_1 + 1, \Re\beta_1 + 1\}$$

If, more generally,

$$f(x) \sim \sum_{s=0}^{\infty} U_s(\log(1-x))(1-x)^{\alpha_s} \quad (x \to 1), \quad U_s(y) \text{ polynomial},$$

$$f(x) \sim \sum_{s=0}^{\infty} V_s(\log(1+x))(1+x)^{\beta_s} \quad (x \to -1), \quad V_s(y) \text{ polynomial},$$

then, with $D_\omega = d/d\omega$,

$$\begin{aligned} G_n[f] \sim I[f] &+ \sum_{s=0}^{\infty} U_s(D_{\alpha_s})\Big[\sum_{k=1}^{\infty} c_k(\alpha_s) h^{\alpha_s+k}\Big] \\ &+ \sum_{s=0}^{\infty} V_s(D_{\beta_s})\Big[\sum_{k=1}^{\infty} c_k(\beta_s) h^{\beta_s+k}\Big] \quad (n \to \infty). \end{aligned}$$

<u>Note:</u> If $R \in \pi_m$, then

$$R(D_\omega)\Big[c_k(\omega) h^{\omega+k}\Big] = h^{\omega+k} W(\log h), \quad W \in \pi_m.$$

Asymptotic expansions for Legendre series coefficients

$$f(x) = \sum_{n=0}^{\infty} e_n[f] P_n(x), \quad e_n[f] = (n + 1/2) \int_{-1}^{1} f(x) P_n(x)\, dx.$$

$$f \in L_2(-1,1) \quad \Rightarrow \quad e_n[f] = o(n^{1/2}) \quad (n \to \infty).$$

$$f \in C^{\infty}[-1,1] \quad \Rightarrow \quad e_n[f] = o(n^{-\mu}) \quad (n \to \infty).$$

f analytic on $D \supset [-1,1] \quad \Rightarrow \quad e_n[f] = O(e^{-\sigma n}) \quad (n \to \infty); \quad \sigma > 0$ const

$$f \in C^r[-1,1], \quad r \geq 0 \quad \Rightarrow \quad e_n[f] = O\Big(n^{-r+1/2} \omega_{f^{(r)}}(2/n)\Big) \quad (n \to \infty).$$

In case of endpoint singularities, last result not optimal. For example, if $f(x) = (1-x)^{r+\nu}$, $0 < \nu < 1$, then it gives $e_n[f] = O(n^{-(r+\nu)+1/2})$, whereas

$$e_n[f] \sim K n^{-2(r+\nu)-1} \quad \text{exactly.}$$

Asymptotics in presence of endpoint singularities [S. (2009)]

Let $f \in C^\infty(-1,1)$ and

$$f(x) \sim \sum_{s=0}^{\infty} A_s (1-x)^{\alpha_s} \quad (x \to 1); \quad f(x) \sim \sum_{s=0}^{\infty} B_s (1+x)^{\beta_s} \quad (x \to -1),$$

$$-1 < \Re\alpha_0 \le \Re\alpha_1 \le \cdots, \quad -1 < \Re\beta_0 \le \Re\beta_1 \le \cdots; \quad \lim_{s\to\infty} \Re\alpha_s, \Re\beta_s = \infty.$$

Assume that these asymptotic expansions can be differentiated term by term infinitely many times.

Then, with $\widehat{n} = n + 1/2$, and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$,

$$e_n[f] \sim \sum_{\substack{s=0 \\ \alpha_s \notin \mathbb{Z}^+}}^{\infty} A_s \sum_{k=1}^{\infty} c_k(\alpha_s)/\widehat{n}^{2(\alpha_s+k)}$$

$$+ (-1)^n \sum_{\substack{s=0 \\ \beta_s \notin \mathbb{Z}^+}}^{\infty} B_s \sum_{k=1}^{\infty} c_k(\beta_s)/\widehat{n}^{2(\beta_s+k)} \quad (n \to \infty).$$

$$c_k(\omega) = 2^{\omega+1}\frac{\Gamma(1+\omega)}{\Gamma(-\omega)}\frac{B_{2k}^{(\sigma)}(\sigma/2)}{(2k)!}\frac{\Gamma(2k+2\omega+2)}{\Gamma(2\omega+2)}, \quad \sigma = -2\omega - 1,$$

analytic functions of ω for $\Re\omega > -1$. Also $c_k(\omega) = 0$ if $\omega \in \mathbb{Z}^+$.

$B_k^{(\sigma)}(u)$: k^{th} generalized Bernoulli polynomial.

If, more generally,

$$f(x) \sim \sum_{s=0}^{\infty} U_s(\log(1-x))(1-x)^{\alpha_s} \quad (x \to 1), \quad U_s(y) \text{ polynomial},$$

$$f(x) \sim \sum_{s=0}^{\infty} V_s(\log(1+x))(1+x)^{\beta_s} \quad (x \to -1), \quad V_s(y) \text{ polynomial},$$

then, with $D_\omega = d/d\omega$,

$$e_n[f] \sim \sum_{s=0}^{\infty} U_s(D_{\alpha_s})\Big[\sum_{k=1}^{\infty} c_k(\alpha_s)/\widehat{n}^{2(\alpha_s+k)}\Big] + (-1)^n \sum_{s=0}^{\infty} V_s(D_{\beta_s})\Big[\sum_{k=1}^{\infty} c_k(\beta_s)/\widehat{n}^{2(\beta_s+k)}\Big] \quad (n \to \infty).$$

<u>Note:</u> If $R \in \pi_m$, then

$$R(D_\omega)\Big[c_k(\omega)/\widehat{n}^{2(\omega+k)}\Big] = W(\log \widehat{n})/\widehat{n}^{2(\omega+k)}, \quad W \in \pi_m.$$

Asymptotics in presence of interior and endpoint singularities [S. (2010)]

Letting $x = \cos\theta$, $\widehat{n} = n + 1/2$,

$$e_n[f] = \widehat{n}\int_0^\pi F(\theta)P_n(\cos\theta)\,d\theta; \quad F(\theta) = \sin\theta\, f(\cos\theta).$$

Asymptotics of $P_n(x)$, $-1 < x < 1$

$$P_n(\cos\theta) \sim \Re\left\{e^{\mathrm{i}\widehat{n}\theta}\sum_{k=0}^{\infty}\frac{\phi_k(e^{\mathrm{i}\theta})}{\widehat{n}^{k+1/2}}\right\} \quad \text{as } n\to\infty\text{, uniformly for } \epsilon \le \theta \le \pi-\epsilon,$$

$$\phi_k(e^{\mathrm{i}\theta}) = (-1)^k\frac{2}{\pi^{1/2}}\binom{-1/2}{k}\, e^{\mathrm{i}(\theta-\pi)/2}B_k^{(1/2)}((\mathrm{i}/2)D_\theta)\Big[(1-e^{\mathrm{i}2\theta})^{-1/2}\Big].$$

An intermediate result

Let $f \in C^\infty(c,d)$, $[c,d] \subset (-1,1)$, $\alpha = \cos^{-1} d > 0, \beta = \cos^{-1} c < \pi$,

$$F(\theta) \sim \sum_{s=0}^{\infty} U_s(\theta-\alpha)^{\rho_s} \quad \text{as } \theta \to \alpha+; \quad F(\theta) \sim \sum_{s=0}^{\infty} V_s(\beta-\theta)^{\sigma_s} \quad \text{as } \theta \to \beta-,$$

$$-1 < \Re\rho_0 \le \Re\rho_1 \le \cdots, \quad -1 < \Re\sigma_0 \le \Re\sigma_1 \le \cdots; \quad \lim_{s\to\infty} \Re\rho_s, \Re\sigma_s = \infty.$$

Assume that these asymptotic expansions can be differentiated term by term infinitely many times. Then

$$\int_c^d f(x) P_n(x)\, dx \sim$$

$$e^{\mathrm{i}\widehat{n}\alpha} \sum_{s=0}^{\infty} U_s \sum_{\mu=0}^{\infty} \frac{G_\mu^{(+)}(\alpha;\rho_s)}{\widehat{n}^{\rho_s+\mu+3/2}} + e^{-\mathrm{i}\widehat{n}\alpha} \sum_{s=0}^{\infty} U_s \sum_{\mu=0}^{\infty} \frac{\widehat{G}_\mu^{(+)}(\alpha;\rho_s)}{\widehat{n}^{\rho_s+\mu+3/2}}$$

$$+e^{\mathrm{i}\widehat{n}\beta} \sum_{s=0}^{\infty} V_s \sum_{\mu=0}^{\infty} \frac{G_\mu^{(-)}(\beta;\sigma_s)}{\widehat{n}^{\sigma_s+\mu+3/2}} + e^{-\mathrm{i}\widehat{n}\beta} \sum_{s=0}^{\infty} V_s \sum_{\mu=0}^{\infty} \frac{\widehat{G}_\mu^{(-)}(\beta;\sigma_s)}{\widehat{n}^{\sigma_s+\mu+3/2}}.$$

$$G_\mu^{(+)}(\theta;\omega) = \frac{1}{2} \sum_{\substack{j,k\geq 0\\ j+k=\mu}} \mathrm{i}^{\omega+j+1}\, \phi_{kj}(\theta)\, \Gamma(\omega+j+1),$$

$$\widehat{G}_\mu^{(+)}(\theta;\omega) = \frac{1}{2} \sum_{\substack{j,k\geq 0\\ j+k=\mu}} (-\mathrm{i})^{\omega+j+1}\, \overline{\phi_{kj}(\theta)}\, \Gamma(\omega+j+1),$$

$$G_\mu^{(-)}(\theta;\omega) = \frac{1}{2} \sum_{\substack{j,k\geq 0\\ j+k=\mu}} (-1)^j\, (-\mathrm{i})^{\omega+j+1}\, \phi_{kj}(\theta)\, \Gamma(\omega+j+1),$$

$$\widehat{G}_\mu^{(-)}(\theta;\omega) = \frac{1}{2} \sum_{\substack{j,k\geq 0\\ j+k=\mu}} (-1)^j\, \mathrm{i}^{\omega+j+1}\, \overline{\phi_{kj}(\theta)}\, \Gamma(\omega+j+1).$$

Here

$$\phi_{kj}(\theta) = \frac{1}{j!}\frac{d^j}{d\theta^j}\phi_k(e^{\mathrm{i}\theta}), \quad j,k = 0,1,\dots\ .$$

Full asymptotic expansion for $e_n[f]$

Let $f \in PC^\infty(-1,1)$ with possible endpoint singularities at $x = \pm 1$

$$f(x) \sim \sum_{s=0}^{\infty} A_s(1-x)^{\alpha_s} \quad (x \to 1); \quad f(x) \sim \sum_{s=0}^{\infty} B_s(1+x)^{\beta_s} \quad (x \to -1),$$

and with interior singularities at $x_1 > x_2 > \ldots > x_m$,

$$F(\theta) \sim \sum_{s=0}^{\infty} T_{rs}^{(\pm)} |\theta - \theta_r|^{\gamma_{rs}^{(\pm)}} \quad \text{as } \theta \to \theta_r\pm, \quad \text{r=1,...,m}; \quad F(\theta) = \sin\theta\, f(\cos\theta).$$

$$0 < \theta_1 < \theta_2 < \cdots < \theta_m < \pi, \quad \theta_r = \cos^{-1} x_r \quad \forall r.$$

Assume that these asymptotic expansions can be differentiated term by term infinitely many times.

Then

$$
\begin{aligned}
&e_n[f] \sim \\
&\sum_{r=1}^{m} \Bigg\{ e^{\mathrm{i}\widehat{n}\theta_r} \Bigg[\sum_{s=0}^{\infty} T_{rs}^{(+)} \sum_{\mu=0}^{\infty} \frac{G_{\mu}^{(+)}(\theta_r; \gamma_{rs}^{(+)})}{\widehat{n}^{\gamma_{rs}^{(+)}+\mu+1/2}} + \sum_{s=0}^{\infty} T_{rs}^{(-)} \sum_{\mu=0}^{\infty} \frac{G_{\mu}^{(-)}(\theta_r; \gamma_{rs}^{(-)})}{\widehat{n}^{\gamma_{rs}^{(-)}+\mu+1/2}} \Bigg] \\
&\quad + e^{-\mathrm{i}\widehat{n}\theta_r} \Bigg[\sum_{s=0}^{\infty} T_{rs}^{(+)} \sum_{\mu=0}^{\infty} \frac{\widehat{G}_{\mu}^{(+)}(\theta_r; \gamma_{rs}^{(+)})}{\widehat{n}^{\gamma_{rs}^{(+)}+\mu+1/2}} + \sum_{s=0}^{\infty} T_{rs}^{(-)} \sum_{\mu=0}^{\infty} \frac{\widehat{G}_{\mu}^{(-)}(\theta_r; \gamma_{rs}^{(-)})}{\widehat{n}^{\gamma_{rs}^{(-)}+\mu+1/2}} \Bigg] \Bigg\} \\
&\quad + \sum_{\substack{s=0 \\ \alpha_s \notin \mathbb{Z}^+}}^{\infty} A_s \sum_{k=0}^{\infty} \frac{c_k(\alpha_s)}{\widehat{n}^{2(\alpha_s+k+1/2)}} + (-1)^n \sum_{\substack{s=0 \\ \beta_s \notin \mathbb{Z}^+}}^{\infty} B_s \sum_{k=0}^{\infty} \frac{c_k(\beta_s)}{\widehat{n}^{2(\beta_s+k+1/2)}}.
\end{aligned}
$$

Variable transformations and Gauss–Legendre quadrature

Consider $I[f] = \int_0^1 f(x)dx \approx \sum_{i=1}^{n} w'_{ni} f(x'_{ni}) = G_n[f]$
We recall that if $f \in C^\infty(0,1)$ and

$$f(x) \sim \sum_{s=0}^{\infty} A_s x^{\alpha_s} \quad (x \to 0); \quad f(x) \sim \sum_{s=0}^{\infty} B_s (1-x)^{\beta_s} \quad (x \to 1),$$

then, with $h = (n + 1/2)^{-2}$, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$,

$$E_n[f] = G_n[f] - I[f] \sim \sum_{\substack{s=0 \\ \alpha_s \notin \mathbb{Z}^+}}^{\infty} A_s \sum_{k=1}^{\infty} c_k(\alpha_s) h^{\alpha_s + k} + \sum_{\substack{s=0 \\ \beta_s \notin \mathbb{Z}^+}}^{\infty} B_s \sum_{k=1}^{\infty} c_k(\beta_s) h^{\beta_s + k} \quad (n \to \infty).$$

Thus

$$\alpha_0, \beta_0 \notin \mathbb{Z}^+ \quad \Rightarrow \quad E_n[f] = O(h^\omega), \quad \omega = \min\{\Re\alpha_0 + 1, \Re\beta_0 + 1\}$$

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In case of endpoint singularities, to improve performance of G–L quadrature proceed as follows:

1. transform the variable x:

$$x = \psi(t), \quad \psi : [0, 1] \to [0, 1], \quad \psi \in \mathbb{C}^{\infty}(0, 1),$$

$$\psi \in C^1\,[0, 1]; \quad \psi(0) = 0, \quad \psi(1) = 1; \quad \psi'(t) > 0 \quad \text{for } t \in (0, 1).$$

$$I[f] = I[\widehat{f}] = \int_0^1 \widehat{f}(t)dt, \quad \widehat{f}(t) = f(\psi(t))\psi'(t).$$

Choose $\psi(t)$ such that $\psi^{(i)}(0) = \psi^{(i)}(1) = 0,\ i = 1, \ldots, p$, p large. This causes $f \in C^q[0, 1]$ for some large q depending on p.

2. apply G–L quadrature to $I[\widehat{f}]$:

$$I[f] \approx G_n[\widehat{f}] = \sum_{i=1}^{n} w'_{ni}\widehat{f}(x'_{ni}) = \widehat{G}_n[f].$$

Because $f \in C^q[0, 1]$ for some large q, $G_n[\widehat{f}]$ is very accurate.

Definition [S. (2009)]. $\psi \in \widetilde{\mathcal{S}}_{p,q}$, $p, q > 0$, not necessarily integers, if

$$\psi'(t) \sim \sum_{i=0}^{\infty} \epsilon_i^{(0)} t^{p+i} \quad \text{as } t \to 0+; \quad \epsilon_0^{(0)} > 0,$$

$$\psi'(t) \sim \sum_{i=0}^{\infty} \epsilon_i^{(1)} (1-t)^{q+i} \quad \text{as } t \to 1-; \quad \epsilon_0^{(1)} > 0.$$

Consequently,

$$\psi(t) \sim \sum_{i=0}^{\infty} \epsilon_i^{(0)} \frac{t^{p+i+1}}{p+i+1} \quad \text{as } t \to 0+,$$

$$\psi(t) \sim 1 - \sum_{i=0}^{\infty} \epsilon_i^{(1)} \frac{(1-t)^{q+i+1}}{q+i+1} \quad \text{as } t \to 1- .$$

Example. Extended Korobov ($K^{p,q}$) transformation

$$\psi(t) = \frac{\int_0^t u^p (1-u)^q \, du}{\int_0^1 u^p (1-u)^q \, du} = I_t(p+1, q+1).$$

[$I_t(a,b)$: regularized incomplete Beta function.]

$$\widehat{f}(t) \sim \sum_{s=0}^{\infty} A_s[\psi(t)]^{\alpha_s}\psi_s'(t) = \sum_{s=0}^{\infty}\sum_{i=0}^{\infty} U_{si}t^{\alpha_s(p+1)+p+i} \quad (t \to 0),$$

$$\widehat{f}(t) \sim \sum_{s=0}^{\infty} B_s[1-\psi(t)]^{\beta_s}\psi_s'(t) = \sum_{s=0}^{\infty}\sum_{i=0}^{\infty} V_{si}(1-t)^{\beta_s(q+1)+q+i} \quad (t \to 1).$$

Therefore, as $n \to \infty$,

$$E_n[\widehat{f}] \sim \sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \widehat{U}_{sik}h^{\alpha_s(p+1)+p+i+k} + \sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} \widehat{V}_{sik}h^{\beta_s(q+1)+q+i+k}$$

If $[\alpha_0(p+1)+p], [\beta_0(q+1)+q] \notin \mathbb{Z}^+$,

$$E_n[\widehat{f}] = O(h^{\omega}); \quad \omega = \min\{(\Re\alpha_0+1)(p+1), (\Re\beta_0+1)(q+1)\}.$$

Recall that integer powers of t and $(1-t)$ do not contribute to $E_n[\widehat{f}]$, and assume that α_0 and β_0 are real.
Choose p, q such that $[\alpha_0(p+1)+p]$ and $[\beta_0(q+1)+q]$ are in $\mathbb{Z}^+$.
Then so are all $[\alpha_0(p+1)+p+i]$ and $[\beta_0(q+1)+q+i]$. Therefore,

$$E_n[\widehat{f}] = O(h^{\omega}); \quad \omega = \min\{(\Re\alpha_1+1)(p+1), (\Re\beta_1+1)(q+1)\}.$$

Example. $\quad f(x) = \dfrac{x^{-3/4}\,(1-x)^{-1/4}}{1+x}, \quad \int_0^1 f(x) = \pi\, 2^{1/4}.$

$$\alpha_s = -3/4 + s, \quad \beta_s = -1/4 + s, \quad s = 0, 1, \ldots\ .$$

For *optimal* results choose p, q such that

$$\alpha_0(p+1) + p = k, \quad \beta(q+1) + q = l, \quad k, l \in \mathbb{Z}^+.$$

This gives $p = 4k + 3, \quad q = (4l + 1)/3, \quad k, l \in \mathbb{Z}^+$.

For *nonoptimal* results choose $p = 4k + 3 + 0.1, \quad q = (4l + 1)/3 + 0.1,$ $k, l \in \mathbb{Z}^+$.

Below we take $k = l = j$, $j = 0, 1, \ldots\ .$

Also $\widehat{E}_n[f] = O(n^{-\rho}) \quad (n \to \infty)$, where

$$\rho = \lim_{n \to \infty} \left\{ \frac{1}{\log 2} \cdot \log \left(\frac{|\widehat{E}_n[f]|}{|\widehat{E}_{2n}[f]|} \right) \right\}.$$

n	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
2	$2.87D-02$	$1.91D-01$	$6.13D-01$	$8.67D-01$	$9.63D-01$
4	$6.47D-03$	$6.37D-04$	$1.08D-02$	$1.40D-02$	$7.88D-02$
8	$2.10D-03$	$1.20D-04$	$6.14D-04$	$5.30D-04$	$2.81D-03$
16	$5.25D-04$	$8.58D-06$	$5.60D-07$	$9.35D-07$	$4.66D-06$
32	$1.26D-04$	$5.09D-07$	$7.28D-09$	$2.21D-10$	$1.52D-11$
64	$2.99D-05$	$2.98D-08$	$1.06D-10$	$7.91D-13$	$9.87D-15$

Errors $\widehat{E}_n[f]$ obtained with $n = 2^m$, $m = 1(1)6$, and with the $K^{p,q}$ transformation. In column j, we have chosen $p = 4j + 3 + 0.1$ and $q = (4j+1)/3 + 0.1$. (Nonoptimal p, q.)

m	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
1	2.149	8.225	5.826	5.957	3.611
2	1.623	2.409	4.136	4.719	4.809
3	2.000	3.806	10.099	9.146	9.236
4	2.057	4.075	6.265	12.048	18.229
5	2.079	4.094	6.105	8.125	10.586
∞	2.05	4.05	6.05	8.05	10.05

The numbers $\rho_{p,q,m} = \frac{1}{\log 2} \cdot \log\left(\frac{|\widehat{E}_{2m}[f]|}{|\widehat{E}_{2m+1}[f]|}\right)$, with $p, q,\ f(x)$, and $\widehat{E}_n[f]$ as in preceding table, for $m = 1(1)5$.

n	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
2	$5.00D-02$	$1.72D-01$	$5.81D-01$	$8.32D-01$	$9.25D-01$
4	$1.67D-03$	$3.61D-03$	$6.45D-03$	$1.77D-02$	$8.16D-02$
8	$5.81D-05$	$2.16D-05$	$4.71D-04$	$4.21D-04$	$3.04D-03$
16	$2.51D-06$	$1.42D-08$	$5.65D-08$	$8.53D-07$	$3.71D-06$
32	$1.04D-07$	$2.03D-11$	$4.22D-16$	$6.70D-14$	$4.42D-12$
64	$4.23D-09$	$3.20D-14$	$1.69D-30$	$1.40D-21$	$4.88D-25$

Errors $\widehat{E}_n[f]$ obtained with $n = 2^m$, $m = 1(1)6$, and with the $K^{p,q}$ transformation. In column j, we have chosen $p = 4j + 3$ and $q = (4j + 1)/3$. (Optimal p, q.)

Note that for $j = 2$, the asymptotic expansion of $\widehat{E}_n[f]$ is empty.

m	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
1	4.906	5.577	6.494	5.558	3.503
2	4.842	7.384	3.776	5.389	4.748
3	4.534	10.571	13.025	8.948	9.675
4	4.590	9.450	26.995	23.603	19.681
5	4.622	9.309	47.825	25.508	43.040
∞	$4\frac{2}{3}$	$9\frac{1}{3}$	∞	$18\frac{2}{3}$	$23\frac{1}{3}$

The numbers $\rho_{p,q,m} = \frac{1}{\log 2} \cdot \log\left(\frac{|\widehat{E}_{2m}[f]|}{|\widehat{E}_{2m+1}[f]|}\right)$, with $p, q, f(x)$, and $\widehat{E}_n[f]$ as in preceding table, for $m = 1(1)5$.