ANALYSIS AND APPLICATIONS OF SOME MODIFIED BESSEL FUNCTIONS

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Abstract. Some new properties of kernels of modified Kontorovitch–Lebedev integral transforms — modified Bessel functions of the second kind with complex order $K_{\frac{1}{2}+i\beta}(x)$ are presented. Inequalities giving estimations for these functions with argument x and parameter β are obtained. The polynomial approximations of these functions as a solutions of linear differential equations with polynomial coefficients and their systems are proposed.

Key words. Chebyshev polynomials, modified Bessel functions, Lanczos Tau method, Kontorovich-Lebedev integral transforms

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1. Some properties of the functions $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$. In this section new properties of the kernels of modified Kontorovitch–Lebedev integral transforms are deduced, and some of their known properties are collected, which are necessary later on.

It is possible to write the kernels of these transforms in the form

$$ReK_{\frac{1}{2}+i\beta}(x) = \frac{K_{\frac{1}{2}+i\beta}(x) + K_{\frac{1}{2}-i\beta}(x)}{2} \quad \text{and} \quad ImK_{\frac{1}{2}+i\beta}(x) = \frac{K_{\frac{1}{2}+i\beta}(x) - K_{\frac{1}{2}-i\beta}(x)}{2i},$$

where $K_\nu(x)$ is the modified Bessel function of the second kind (also called MacDonald function).

The functions $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$ have integral representations [1]

$$ReK_{\frac{1}{2}+i\beta}(x) = \int_0^\infty e^{-x\cosh t} \cosh\frac{t}{2} \cos(\beta t) dt, \tag{1.1}$$

$$ImK_{\frac{1}{2}+i\beta}(x) = \int_0^\infty e^{-x\cosh t} \sinh\frac{t}{2} \sin(\beta t) dt. \tag{1.2}$$

The vector-function $(y_1(x), y_2(x))$ with the components $y_1(x) = ReK_{\frac{1}{2}+i\beta}(x)$, $y_2(x) = ImK_{\frac{1}{2}+i\beta}(x)$ is the solution of the system of differential equations

$$\frac{d^2y_1}{dx^2} + \frac{1}{x}\frac{dy_1}{dx} - \left(1 + \frac{\frac{1}{4}-\beta^2}{x^2}\right) y_1 + \frac{\beta}{x^2} y_2 = 0,$$
$$\frac{d^2y_2}{dx^2} + \frac{1}{x}\frac{dy_2}{dx} - \frac{\beta}{x^2} y_1 - \left(1 + \frac{\frac{1}{4}-\beta^2}{x^2}\right) y_2 = 0.$$

The functions $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$ are even and odd functions, respectively of the variable β,

$$ReK_{\frac{1}{2}+i\beta}(x) = ReK_{\frac{1}{2}-i\beta}(x),$$

$$ImK_{\frac{1}{2}+i\beta}(x) = -ImK_{\frac{1}{2}-i\beta}(x).$$

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The functions $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$ are related to the modified Bessel functions of the first kind $I_\nu(x)$ as follows,

$$ReK_{\frac{1}{2}+i\beta}(x) = \frac{\pi}{\cosh(\pi\beta)} \frac{ReI_{-\frac{1}{2}-i\beta}(x) - ReI_{\frac{1}{2}+i\beta}(x)}{2}, \tag{1.3}$$

$$ImK_{\frac{1}{2}+i\beta}(x) = \frac{\pi}{\cosh(\pi\beta)} \frac{ImI_{-\frac{1}{2}-i\beta}(x) - ImI_{\frac{1}{2}+i\beta}(x)}{2}. \tag{1.4}$$

The expansion of $I_{\frac{1}{2}+i\beta}(x)$ in ascending powers of x has the form

$$\begin{aligned} I_{\frac{1}{2}+i\beta}(x) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \left(\cos\left(\beta \ln \frac{x}{2}\right) + i \sin\left(\beta \ln \frac{x}{2}\right)\right) \\ \times \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^k}{k!\Gamma\left(k+\frac{3}{2}+i\beta\right)} = \sum_{k=0}^{\infty} (a_k + ib_k), \end{aligned} \tag{1.5}$$

where a_k and b_k satisfy the following recurrence relations:

$$\begin{aligned} a_0 + ib_0 &= \left(\frac{x}{2}\right)^{\frac{1}{2}} \frac{\cos\left(\beta \ln \frac{x}{2}\right) + i \sin\left(\beta \ln \frac{x}{2}\right)}{\Gamma\left(\frac{3}{2}+i\beta\right)}, \\ m_k &= x^2 \frac{k+\frac{1}{2}}{4k\left(\left(k+\frac{1}{2}\right)^2 + \beta^2\right)}, \qquad n_k = x^2 \frac{\beta}{4k\left(\left(k+\frac{1}{2}\right)^2 + \beta^2\right)}, \\ a_k &= a_{k-1}m_k + b_{k-1}n_k, \qquad b_k = b_{k-1}m_k - a_{k-1}n_k. \end{aligned}$$

The expansion of $I_{-\frac{1}{2}-i\beta}(x)$ in ascending powers of x has the form

$$\begin{aligned} I_{-\frac{1}{2}-i\beta}(x) = \left(\frac{x}{2}\right)^{-\frac{1}{2}} \left(\cos\left(\beta \ln \frac{x}{2}\right) - i \sin\left(\beta \ln \frac{x}{2}\right)\right) \\ \times \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^k}{k!\Gamma\left(k+\frac{1}{2}-i\beta\right)} = \sum_{k=0}^{\infty} (c_k + id_k), \end{aligned} \tag{1.6}$$

where c_k and d_k satisfy the following recurrence relations:

$$\begin{aligned} c_0 + id_0 &= \left(\frac{x}{2}\right)^{-1/2} \frac{\cos\left(\beta \ln \frac{x}{2}\right) - i \sin\left(\beta \ln \frac{x}{2}\right)}{\Gamma\left(\frac{1}{2}-i\beta\right)}, \\ p_k &= x^2 \frac{k-\frac{1}{2}}{4k\left(\left(k-\frac{1}{2}\right)^2 + \beta^2\right)}, \qquad q_k = x^2 \frac{\beta}{4k\left(\left(k-\frac{1}{2}\right)^2 + \beta^2\right)}, \\ c_k &= c_{k-1}p_k - d_{k-1}q_k, \qquad d_k = d_{k-1}p_k + c_{k-1}q_k. \end{aligned}$$

The expansions (1.5) and (1.6) converge for all $0 < x < \infty$ and $0 \le \beta < \infty$.

It follows from (1.1)–(1.2) that it is possible to write $ReK_{\frac{1}{2}+i\beta}(x)$ in the form of the Fourier cosinus-transform

$$ReK_{\frac{1}{2}+i\beta}(x) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} F_C\left[e^{-x\cosh t} \cosh \frac{t}{2}; t \to \beta\right], \tag{1.7}$$

and $ImK_{\frac{1}{2}+i\beta}(x)$ in the form of the Fourier sinus-transform

$$ImK_{\frac{1}{2}+i\beta}(x) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} F_S\left[e^{-x\cosh t}\sinh\frac{t}{2}; t \to \beta\right]. \tag{1.8}$$

The inversion formulas have the respective forms

$$F_C\left[ReK_{\frac{1}{2}+i\beta}(x); \beta \to t\right] = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-x\cosh t}\cosh\frac{t}{2},$$
$$F_S\left[ImK_{\frac{1}{2}+i\beta}(x); \beta \to t\right] = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-x\cosh t}\sinh\frac{t}{2}$$

or, in integral form,

$$\int_0^\infty ReK_{\frac{1}{2}+i\beta}(x)\cos(t\beta)d\beta = \frac{\pi}{2}e^{-x\cosh t}\cosh\frac{t}{2}, \tag{1.9}$$
$$\int_0^\infty ImK_{\frac{1}{2}+i\beta}(x)\sin(t\beta)d\beta = \frac{\pi}{2}e^{-x\cosh t}\sinh\frac{t}{2}. \tag{1.10}$$

Differentiating equations (1.9) and (1.10) with respect to t, we obtain

$$\int_0^\infty \beta ReK_{\frac{1}{2}+i\beta}(x)\sin(t\beta)d\beta = \frac{\pi}{2}\left(x\sinh t\cosh\frac{t}{2} - \sinh\frac{t}{2}\right)e^{-x\cosh t},$$
$$\int_0^\infty \beta ImK_{\frac{1}{2}+i\beta}(x)\cos(t\beta)d\beta = \frac{\pi}{2}\left(\cosh\frac{t}{2} - x\sinh t\sinh\frac{t}{2}\right)e^{-x\cosh t}. \tag{1.11}$$

It follows from (1.9) that

$$\int_0^\infty ReK_{\frac{1}{2}+i\beta}(x)d\beta = \frac{\pi}{2}e^{-x},$$

and from (1.11) that

$$\int_0^\infty \beta ImK_{\frac{1}{2}+i\beta}(x)d\beta = \frac{\pi}{2}e^{-x}.$$

Differentiating (1.9) and (1.10) $2n$ times with respect to t, we obtain

$$\int_0^\infty \beta^{2n} ReK_{\frac{1}{2}+i\beta}(x)\cos(t\beta)d\beta = \frac{\pi}{2}(-1)^n D_t^{2n}\left(e^{-x\cosh t}\cosh\frac{t}{2}\right),$$
$$\int_0^\infty \beta^{2n} ImK_{\frac{1}{2}+i\beta}(x)\sin(t\beta)d\beta = \frac{\pi}{2}(-1)^n D_t^{2n}\left(e^{-x\cosh t}\sinh\frac{t}{2}\right),$$

from which there follows, for $t = 0$,

$$\int_0^\infty \beta^{2n} ReK_{\frac{1}{2}+i\beta}(x)d\beta = \frac{\pi}{2}(-1)^n D_t^{2n}\left(e^{-x\cosh t}\cosh\frac{t}{2}\right)_{t=0}.$$

Differentiating (1.9) and (1.10) $2n+1$ times with respect to t, we obtain

$$\int_0^\infty \beta^{2n+1} ReK_{1/2+i\beta}(x)\sin(t\beta)d\beta = \frac{\pi}{2}(-1)^{n+1} D_t^{2n+1}\left(e^{-x\cosh t}\cosh\frac{t}{2}\right),$$
$$\int_0^\infty \beta^{2n+1} ImK_{1/2+i\beta}(x)\cos(t\beta)d\beta = \frac{\pi}{2}(-1)^{n} D_t^{2n+1}\left(e^{-x\cosh t}\sinh\frac{t}{2}\right).$$

whence, for $t=0$,

$$\int_0^\infty \beta^{2n+1} ImK_{1/2+i\beta}(x)d\beta = \frac{\pi}{2}(-1)^n D_t^{2n+1}\left(e^{-x\cosh t}\sinh\frac{t}{2}\right)_{t=0}.$$

For the computation of certain integrals of the functions $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$, integral identities are useful. They reduce this problem to the computation of some other integrals of elementary functions.

PROPOSITION 1.1. *If f is absolutely integrable on $[0,\infty)$, then the following identities hold,*

$$\int_0^\infty ReK_{\frac{1}{2}+i\beta}(x)f(\beta)d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\int_0^\infty e^{-x\cosh t}\cosh\frac{t}{2}F_C(t)dt, \tag{1.12}$$

$$\int_0^\infty ImK_{\frac{1}{2}+i\beta}(x)f(\beta)d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\int_0^\infty e^{-x\cosh t}\sinh\frac{t}{2}F_S(t)dt, \tag{1.13}$$

where $F_C(t)$ is the Fourier cosinus-transform of $f(\beta)$, and $F_S(t)$ the Fourier sinus-transform of $f(\beta)$.

Proof. Multiplying both sides of the equalities (1.7) and (1.8) by $f(\beta)$, integrating with respect to β from 0 to ∞, and applying Fubini's theorem for singular integrals with parameter [2], we obtain (1.12) and (1.13). ∎

PROPOSITION 1.2. *If f is absolutely integrable on $[0,\infty)$, then the following identities hold*

$$\int_0^\infty ReK_{\frac{1}{2}+i\beta}(x)F_C(\beta)d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\int_0^\infty e^{-x\cosh t}\cosh\frac{t}{2}f(t)dt, \tag{1.14}$$

$$\int_0^\infty ImK_{\frac{1}{2}+i\beta}(x)F_S(\beta)d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\int_0^\infty e^{-x\cosh t}\sinh\frac{t}{2}f(t)dt. \tag{1.15}$$

Proof. This follows from (1.9)–(1.10) and from Fubini's theorem [2]. ∎

The equations (1.12)–(1.15) are useful for the simplification and the calculation of different integrals containing $ReK_{1/2+i\beta}(x)$ and $ImK_{1/2+i\beta}(x)$.

For example, let $f(\beta)=e^{-\alpha\beta}$, then $F_C(t)=\sqrt{\frac{2}{\pi}}\frac{\alpha}{\alpha^2+t^2}$, $F_S(t)=\sqrt{\frac{2}{\pi}}\frac{t}{\alpha^2+t^2}$ and

$$\int_0^\infty ReK_{\frac{1}{2}+i\beta}(x)e^{-\alpha\beta}d\beta = \alpha\int_0^\infty(\alpha^2+t^2)^{-1}e^{-x\cosh t}\cosh\frac{t}{2}dt,$$
$$\int_0^\infty ReK_{\frac{1}{2}+i\beta}(x)\frac{1}{\alpha^2+\beta^2}d\beta = \frac{\pi}{2\alpha}\int_0^\infty e^{-\alpha t-x\cosh t}\cosh\frac{t}{2}dt,$$
$$\int_0^\infty ImK_{\frac{1}{2}+i\beta}(x)e^{-\alpha\beta}d\beta = \int_0^\infty t(\alpha^2+t^2)^{-1}e^{-x\cosh t}\sinh\frac{t}{2}dt,$$
$$\int_0^\infty ImK_{\frac{1}{2}+i\beta}(x)\frac{\beta}{\alpha^2+\beta^2}d\beta = \frac{\pi}{2}\int_0^\infty e^{-\alpha t-x\cosh t}\sinh\frac{t}{2}dt.$$

If $f(\beta)=\Gamma(\frac{1}{4}+\frac{i\beta}{2})\Gamma(\frac{1}{4}-\frac{i\beta}{2})$, then $F_C(t)=\frac{2\pi}{\sqrt{\cosh t}}$ and

$$\int_0^\infty ReK_{\frac{1}{2}+i\beta}(x)\Gamma(\frac{1}{4}+\frac{i\beta}{2})\Gamma(\frac{1}{4}-\frac{i\beta}{2})d\beta =$$
$$=\sqrt{2\pi}\pi\int_0^\infty e^{-x\cosh t}\frac{\cosh\frac{t}{2}}{\sqrt{\cosh t}}dt = \pi\sqrt{\pi}e^{-\frac{x}{2}}K_0(\frac{x}{2}).$$

If $f(\beta)=\frac{\sinh(2\pi\beta)}{\cosh(2\pi\beta)+\cos(2\pi\alpha)}$, $|Re\alpha|<\frac{1}{2}$, then $F_S(t)=\frac{\cosh(\alpha t)}{\sqrt{2\pi}\sinh\frac{t}{2}}$ and

$$\int_0^\infty \frac{ImK_{\frac{1}{2}+i\beta}(x)\sinh(2\pi\beta)}{\cosh(2\pi\beta)+\cos(2\pi\alpha)}d\beta=\frac{1}{2}\int_0^\infty e^{-x\cosh t}\cosh(\alpha t)dt=\frac{1}{2}K_\alpha(x).$$

REMARK 1.3. *All formulas of the present paragraph remain valid if x is changed to z lying in the right-hand half-plane.*

1.1. The Laplace transform of $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$.
The Laplace transform of $K_{i\beta}(x)$ is computed in [3]. We use the representation (1.1) for the evaluation of the Laplace transformation of $ReK_{\frac{1}{2}+i\beta}(x)$. We have

$$\begin{aligned}L\left[ReK_{\frac{1}{2}+i\beta}(x);\beta\right]&=\int_0^\infty \cos(\beta t)\cosh\frac{t}{2}\int_0^\infty e^{-(p+\cosh t)x}\,dx\,dt\\&=\int_0^\infty \frac{\cos(\beta t)\cosh\frac{t}{2}}{\cosh t+\cosh\alpha}dt\ (p=\cosh\alpha)\\&=\sqrt{\frac{\pi}{2}}F_C\left(\frac{\cosh\frac{t}{2}}{\cosh t+\cosh\alpha}\right)=\frac{\pi}{2}\frac{\cos(\alpha\beta)}{\cosh\frac{\alpha}{2}\cosh(\pi\beta)}.\end{aligned}$$

Equivalently, we can write

$$L^{-1}\left[\frac{\cos(\beta\cosh^{-1}p)}{\sqrt{\frac{p+1}{2}}}\right]=\left(\frac{\pi}{2}\right)^{-1}\cosh(\pi\beta)ReK_{\frac{1}{2}+i\beta}(x).$$

For the evaluation of the Laplace transform of $ImK_{\frac{1}{2}+i\beta}(x)$ we utilize the representation (1.2). We have

$$L\left[ImK_{\frac{1}{2}+i\beta}(x);p\right]=\sqrt{\frac{\pi}{2}}F_S\left(\frac{\sinh\frac{t}{2}}{\cosh t+\cosh\alpha}\right)=\frac{\pi}{2}\frac{\sin(\alpha\beta)}{\cosh(\pi\beta)\sinh\frac{\alpha}{2}},$$

or, equivalently,

$$L^{-1}\left[\frac{\sin(\beta\cosh^{-1}p)}{\sqrt{\frac{p-1}{2}}}\right]=\sqrt{\frac{\pi}{2}}\cosh(\pi\beta)ImK_{\frac{1}{2}+i\beta}(x).$$

We note that these equations can also be obtained directly from the formula for the Laplace transforms of $K_\nu(x)$ by separating real and imaginary parts.

1.2. The asymptotic behavior of $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$ for $x\to 0, x\to\infty$ and $\beta\to\infty$. For $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$ the following asymptotic formulas for $\beta\to\infty$ are valid [1],

$$\begin{aligned}ReK_{\frac{1}{2}+i\beta}(x)&\sim\left(\frac{\pi}{x}\right)^{\frac{1}{2}}e^{-\frac{\pi\beta}{2}}\cos\left(\beta\ln\beta-\beta-\beta\ln\frac{x}{2}\right),\\ImK_{\frac{1}{2}+i\beta}(x)&\sim\left(\frac{\pi}{x}\right)^{\frac{1}{2}}e^{-\frac{\pi\beta}{2}}\sin\left(\beta\ln\beta-\beta-\beta\ln\frac{x}{2}\right),\end{aligned}$$

where x is a fixed positive number.

It follows immediately from (1.3)–(1.6) that for $x \to 0$ we have

$$K_{\frac{1}{2}+i\beta}(x) \sim \frac{1}{2}\left(\frac{x}{2}\right)^{-\frac{1}{2}} \frac{\cos\left(\beta \ln \frac{x}{2}\right) - i \sin\left(\beta \ln \frac{x}{2}\right)}{\Gamma\left(\frac{1}{2} - i\beta\right)},$$

whence

$$\begin{aligned} ReK_{\frac{1}{2}+i\beta}(x) \sim \frac{1}{2}\left(\frac{x}{2}\right)^{-\frac{1}{2}} \Bigg(& Re\Gamma\left(\frac{1}{2}+i\beta\right)\cos\left(\beta \ln \frac{x}{2}\right) \\ & + Im\Gamma\left(\frac{1}{2}+i\beta\right)\sin\left(\beta \ln \frac{x}{2}\right)\Bigg), \\ ImK_{\frac{1}{2}+i\beta}(x) \sim \frac{1}{2}\left(\frac{x}{2}\right)^{-\frac{1}{2}} \Bigg(& Im\Gamma\left(\frac{1}{2}+i\beta\right)\cos\left(\beta \ln \frac{x}{2}\right) \\ & - Re\Gamma\left(\frac{1}{2}+i\beta\right)\sin\left(\beta \ln \frac{x}{2}\right)\Bigg), \end{aligned}$$

For large values x the following asymptotic expansion is valid [4]

$$K_{\frac{1}{2}+i\beta}(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \sum_{k=0}^{\infty} \left(\frac{1}{2}+i\beta, k\right)(2x)^{-k},$$

where

$$(\nu, k) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2k-1)^2)}{2^{2k} k!}.$$

In particular, therefore,

$$\begin{aligned} ReK_{\frac{1}{2}+i\beta}(x) &\sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left(1 - \frac{\beta^2}{2x} + \cdots\right), \\ ImK_{\frac{1}{2}+i\beta}(x) &\sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left(\frac{\beta}{2x} + \cdots\right) = \frac{\beta}{2x}\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}(1 + \cdots). \end{aligned}$$

1.3. The series expansions in powers of β. The solutions of problems in mathematical physics connected with the use of the Kontorovitch–Lebedev integral transforms are often expressed as integrals with respect to β of the functions $K_{i\beta}(x)$, $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$. Both the asymptotic expansions of these integrals for large values β, and the expansions of these functions in powers of β, are of interest for the analysis of the behavior of these integrals.

The expansions of these functions in powers of β are deduced from their integral representations (1.1)–(1.2). Substituting in them $\cos(\beta t)$ and $\sin(\beta t)$ by their series expansions and interchanging the order of the summation and integration, we obtain

$$\begin{aligned} K_{i\beta}(x) &= \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} \int_0^{\infty} t^{2k} e^{-x\cosh t} dt = \\ &= K_0(x) - \frac{\beta^2}{2!}\int_0^{\infty} t^2 e^{-x\cosh t} dt + \frac{\beta^4}{4!}\int_0^{\infty} t^4 e^{-x\cosh t} dt + \cdots, \end{aligned} \tag{1.16}$$

$$\begin{aligned} ReK_{\frac{1}{2}+i\beta}(x) &= \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} \int_0^{\infty} t^{2k} e^{-x\cosh t} \cosh \frac{t}{2} dt = \\ &= K_{\frac{1}{2}}(x) - \frac{\beta^2}{2!} \int_0^{\infty} t^2 e^{-x\cosh t} \cosh \frac{t}{2} dt \\ &\quad + \frac{\beta^4}{4!} \int_0^{\infty} t^4 e^{-x\cosh t} \cosh \frac{t}{2} dt + \cdots , \end{aligned} \tag{1.17}$$

$$\begin{aligned} ImK_{\frac{1}{2}+i\beta}(x) &= \sum_{k=0}^{\infty} \frac{\beta^{2k+1}}{(2k+1)!} \int_0^{\infty} t^{2k+1} e^{-x\cosh t} \sinh \frac{t}{2} dt = \\ &= \beta \int_0^{\infty} t e^{-x\cosh t} \sinh \frac{t}{2} dt - \frac{\beta^3}{3!} \int_0^{\infty} t^3 e^{-x\cosh t} \sinh \frac{t}{2} dt + \cdots . \end{aligned} \tag{1.18}$$

These functions are entire functions in the variable β, and therefore the series converge for all real values of β. From these expansions it is possible to obtain the series for the derivatives and for the integrals of these functions with respect to the variable β, which will converge for all real β also. Similar integrals for the spherical functions are stated in [5].

It's possible to rewrite the expansions (1.16)–(1.18) in terms of Laplace transforms as follows,

$$K_{i\beta}(x) = e^{-x} \sum_{k=0}^{\infty} (-1)^k L\left[\frac{\operatorname{arccosh}^{2k}(y+1)}{\sqrt{(y+1)^2-1}}; y \to x \right] \frac{\beta^{2k}}{(2k)!}, \tag{1.19}$$

$$ReK_{\frac{1}{2}+i\beta}(x) = e^{-x} \sum_{k=0}^{\infty} (-1)^k L\left[\frac{\operatorname{arccosh}^{2k}(y+1)}{\sqrt{2y}}; y \to x \right] \frac{\beta^{2k}}{(2k)!}, \tag{1.20}$$

$$ImK_{\frac{1}{2}+i\beta}(x) = e^{-x} \sum_{k=0}^{\infty} (-1)^k L\left[\frac{\operatorname{arccosh}^{2k+1}(y+1)}{\sqrt{2(y+2)}}; y \to x \right] \frac{\beta^{2k+1}}{(2k+1)!}, \tag{1.21}$$

This form of writing may be more convenient since it is possible to use numerical methods for evaluating Laplace transforms.

The expansions (1.19)–(1.21) are convenient for the calculation of the kernels of Kontorovitch-Lebedev integral transforms for small values β.

2. Inequalities for the MacDonald functions $K_{i\beta}(x)$, $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$. It follows from (1.1) that for all $\beta \in [0, \infty)$

$$|ReK_{\frac{1}{2}+i\beta}(x)| \le K_{\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x},$$

and it follows from (1.2) that for all $\beta \in [0, \infty)$

$$|ImK_{\frac{1}{2}+i\beta}(x)| \le \int_0^{\infty} e^{-x\cosh t} \sinh \frac{t}{2} dt = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{x} \left[1 - \phi((2x)^{\frac{1}{2}})\right] \le B \frac{e^{-x}}{x},$$

where B is some positive constant [6],[7].

In [8], for arbitrary $\nu = \sigma + i\beta$, $\sigma \ge 0$, the following inequality is derived

$$|I_\nu(x)| \le e^{\frac{\pi|\beta|}{2}} I_\sigma(x).$$

Taking advantage of the formula [8]

$$|K_\nu(x)| \le \left(C_1(x,\sigma) + C_2(x,\sigma)|\beta|^{\sigma-\frac{1}{2}}\right) e^{-\frac{\pi|\beta|}{2}},$$

we obtain that beginning with some $T, |\beta| > T$,

$$|K_{\frac{1}{2}+i\beta}(x)| \le C(x)e^{-\frac{\pi|\beta|}{2}}.$$

But this inequality is too rough and may be insufficient for conducting various proofs. To obtain more refined inequalities, we use [9]

$$|K_{i\beta}(x)| \le Ax^{-\frac{1}{4}}e^{-\frac{\pi|\beta|}{2}}, \tag{2.1}$$

where A is some positive constant, and the representations [1]

$$\begin{aligned} ReK_{\frac{1}{2}+i\beta}(x) &= \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \frac{e^{-x}}{\cosh(\pi\beta)} \\ &\quad + \frac{\beta\tanh(\pi\beta)}{(2\pi)^{\frac{1}{2}}} \int_0^x \left[\frac{e^{-(x-y)}}{\sqrt{x-y}} - \frac{e^{-(x+y)}}{x^{\frac{1}{2}}}\right] K_{i\beta}(y)dy \\ &\quad - \frac{\beta\tanh(\pi\beta)}{(2\pi)^{\frac{1}{2}}} \int_x^\infty \frac{e^{-(x+y)}}{x^{\frac{1}{2}}} \frac{K_{i\beta}(y)}{y}\,dy, \\ ImK_{\frac{1}{2}+i\beta}(x) &= \frac{\beta e^x}{(2\pi)^{\frac{1}{2}}} \int_x^\infty \frac{e^{-y}K_{i\beta}(y)}{y(y-x)^{\frac{1}{2}}}\,dy. \end{aligned} \tag{2.2}$$

LEMMA 2.1. *The following inequalities hold for $x > 0$*

$$|ReK_{\frac{1}{2}+i\beta}(x)| \le c|\beta|e^{-\frac{\pi|\beta|}{2}}x^{-\frac{3}{4}} + \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} e^{-x}e^{-\pi|\beta|},$$
$$|ImK_{\frac{1}{2}+i\beta}(x)| \le c_0|\beta|e^{-\frac{\pi|\beta|}{2}}x^{-\frac{3}{4}},$$

where c_0 and c are some positive constants.

Proof. We estimate the second additive term in (2.2), using the inequality (2.1),

$$\begin{aligned} &|\frac{\beta\tanh(\pi\beta)}{(2\pi)^{\frac{1}{2}}} \int_0^x \left[\frac{e^{-(x-y)}}{(x-y)^{\frac{1}{2}}} - \frac{e^{-(x+y)}}{x^{\frac{1}{2}}}\right] \frac{K_{i\beta}(y)}{y}dy| \\ &\le A|\beta|e^{-\frac{\pi|\beta|}{2}} \frac{e^{-x}}{\sqrt{x}} \int_0^x \frac{\frac{e^y}{\sqrt{1-\frac{y}{x}}} - e^y}{y} y^{-\frac{1}{4}}dy \le A|\beta|e^{-\frac{\pi|\beta|}{2}}x^{-\frac{3}{4}}. \end{aligned} \tag{2.3}$$

We next estimate the third additive term,

$$\begin{aligned} |\frac{\beta\tanh(\pi\beta)}{\sqrt{2\pi}} \int_x^\infty \frac{e^{-(x+y)}}{\sqrt{x}} \frac{K_{i\beta}(y)}{y}dy| &\le B|\beta|e^{-\frac{\pi|\beta|}{2}} \frac{e^{-x}}{\sqrt{x}} \int_x^\infty e^{-y}y^{-\frac{5}{4}}dy \\ &\le B|\beta|e^{-\frac{\pi|\beta|}{2}}e^{-2x}x^{-\frac{3}{4}}. \end{aligned} \tag{2.4}$$

Combining the first term and estimates (2.3) and (2.4), we obtain the required inequality.

Furthermore, we obtain

$$\begin{aligned} |ImK_{\frac{1}{2}+i\beta}(x)| &\le |\frac{\beta e^x}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-y}K_{i\beta}(y)}{y\sqrt{y-x}}dy| \\ &\le c_0|\beta|e^{-\frac{\pi|\beta|}{2}}e^x \int_x^{infty} \frac{e^{-y}y^{-\frac{5}{4}}}{\sqrt{y-x}}dy \le c_0|\beta|e^{-\frac{\pi|\beta|}{2}}x^{-\frac{3}{4}}. \end{aligned}$$

For future use, an analysis of the behavior of the modified Bessel function $K_{\sigma+i\beta}(x)$ for large values of β is necessary. $\square$

LEMMA 2.2. *For* $0 \le \sigma \le \frac{1}{2}$, $|\beta| \ge \beta_0 \ge 1$, $x \ge x_0 \ge 1$, *the following inequality holds,*

$$|K_{\sigma+i\beta}(x)| \le \left(c_1 x^{\sigma-\frac{1}{2}} + c_2 x^{\frac{1}{2}-\sigma}|\beta|^{\sigma-\frac{1}{2}}\right) e^{x-\frac{\pi|\beta|}{2}},$$

where $c_1 > 0$, $c_2 > 0$, c_1, c_2, β_0, x_0 *are some constants.*

Proof. We use the formula [10]

$$K_\mu(x) = \frac{\pi}{2\sin(\pi\mu)}(I_{-\mu}(x) - I_\mu(x)), \qquad \mu = \sigma + i\beta.$$

1. We first estimate $\sin(\pi\mu)$. It is possible to show that for $|\beta| \ge \beta_0^{(1)} > 0$, $\beta_0^{(1)}$ some constant, the following inequality is valid

$$a_1 e^{\pi|\beta|} \le |\sin(\pi\mu)| \le a_2 e^{\pi|\beta|}, \tag{2.5}$$

where $a_1 > 0$, $a_2 > 0$, a_1, a_2 are some constants.

2. We next estimate $I_\mu(x)$, $\mu = \sigma + i\beta$, $\sigma \ge 0$. The following inequality is derived for $\sigma \ge 0$ in [8]

$$|I_\mu(x)| \le I_0(x)\frac{(\frac{x}{2})^\sigma e^{\frac{\pi|\beta|}{2}}}{\Gamma(\sigma+1)}.$$

It follows from the asymptotics of $I_0(x)$ [10] for large x that for $x \ge x_0^{(1)} > 0$, $x_0^{(1)}$ some constant,

$$|I_\mu(x)| \le a_3 x^{\sigma-\frac{1}{2}} e^{x+\frac{\pi\ |\beta|}{2}}, \tag{2.6}$$

where $a_3 > 0,\ a_3$ some constant.

3. We finally estimate $I_{-\mu}(x)$, $\mu = \sigma + i\beta$, $\sigma \ge 0$. Proceeding analogously [8], we can rewrite $I_{-\mu}(x)$ in the form

$$I_{-\mu}(x) = \frac{(\frac{x}{2})^{-\mu}}{\Gamma(1-\mu)}\psi(x,\mu),$$

where

$$\psi(x,\mu) = 1 + \sum_{s=1}^{\infty} \frac{(\frac{x}{2})^{2s}}{s!\prod_{k=1}^{k=s}(-\mu+k)}.$$

Then $|k-\mu| = \sqrt{(k-\sigma)^2+\beta^2} \ge k-1$ for $0 \le \sigma \le \frac{1}{2}$, $k = 2, 3, \ldots$, and $\sqrt{(1-\sigma)^2+\beta^2} \ge \frac{1}{2},\ \beta$ arbitrary. Therefore, $\prod_{k=1}^{k=s}\sqrt{(k-\sigma)^2+\beta^2} \ge \frac{(s-1)!}{2}$. We obtain, after some calculations, that

$$\psi(x,\mu) \le 1 + 2\sum_{s=1}^{\infty}\frac{(\frac{x}{2})^{2s}}{s!(s-1)!} \le 2\left(1+\frac{x}{2}I_1(x)\right).$$

Using for $|\beta| \ge \beta_0^{(2)} \ge 1$, $0 \le \sigma \le \frac{1}{2}$, the expansion of the gamma-function from [9] and the asymptotics [10] for $I_1(x)$ we obtain that beginning with some $x_0^{(2)}$, $x \ge x_0^{(2)} \ge 1$, the following estimation holds,

$$|I_{-\mu}(x)| \le a_4 x^{\frac{1}{2}-\sigma}|\beta|^{\sigma-\frac{1}{2}} e^{x+\frac{\pi|\beta|}{2}}, \tag{2.7}$$

$a_4 > 0$, a_4 some constant.

Combining the estimations (2.5)–(2.7), we obtain that for $0 \le \sigma \le \frac{1}{2}$, $|\beta| \ge \beta_0 \ge 1$, $x \ge x_0 \ge 1$, $\beta_0 = \max(\beta_0^{(1)}, \beta_0^{(2)})$, $x_0 = \max(x_0^{(1)}, x_0^{(2)})$ the following inequality is valid,

$$|K_\mu(x) \le \frac{\pi}{2a_1} e^{-\pi|\beta|} |I_{-\mu}(x) - I_\mu(x)| \le \frac{\pi}{2a_1}\left(a_3 x^{\sigma-\frac{1}{2}} + a_4 x^{\frac{1}{2}-\sigma}|\beta|^{\sigma-\frac{1}{2}}\right) e^{x-\frac{\pi|\beta|}{2}}.$$

Denoting $c_1 = \frac{\pi}{2a_1}a_3$, $c_2 = \frac{\pi}{2a_1}a_4$, we obtain the statement of the lemma. □

The properties of the modified Kontorovitch–Lebedev integral transforms are considered in [11]–[14].

3. Tau method approximation for modified Bessel function of imaginary order. Several approaches for the evaluation of the modified Bessel functions are elaborated in [15]–[19]. The Tau method [20] realization, with minimal residue choice for the determination of the polynomial approximations of the solutions of the second order differential equations with polynomial coefficients [16] of the following form

$$(a_0 y^2 + a_5 y)v''(y) + (a_1 y + a_2)v'(y) + a_3 v(y) = 0,\ v(0) = a_4, \quad y \in [0,1],$$

is supposed. An n-th approximation of the solution is sought in the form of the n-th degree polynomial $v_n(y)$, which is the solution of the equation

$$(b_0 y^2 + b_5 y)v(y) = \int_0^y (b_1 x + b_2 y + b_3)v(x)dx + b_4 y + \tau_{n+2}T^*_{n+2}[(1-\alpha_{n+2})y + \alpha_{n+2}],$$

where the coefficients a_i, $i = 0,\dots,5$, may be expressed by coefficients b_i, $i = 0,\dots,5$, $\alpha_{n+2} = \sin^2 \frac{\pi}{4(n+2)}$ — the leftmost root of the shifted Chebyshev polynomial of the $n+2$-th degree $T^*_{n+2}(y)$ in the interval $[0,1]$, τ_{n+2} — undefined coefficient.

The problem about determination of the polynomial $P_n(y) = \sum_{k=0}^n p_k y^k$, which is the least deviated from zero on the interval $[0,1]$ among all n-th degree polynomials, satisfying the pair of linear correlations on the coefficients $p_0 = 0$, $\sum_{i=1}^n c_i^{(n)} p_i = 1$ was considered. The following theorem is proved [16]:

THEOREM 3.1. *If the sequence of numbers $c_i^{(n)}$, $i = 1,\dots,n$, is alternating, then the polynomial $\tau_n T^*_n[(1-\alpha_n)y + \alpha_n]$ is the polynomial least deviating from zero in the uniform metric on $[0,1]$ among all polynomials of degree n, satisfying the indicated pair of linear relations.*

On the basis of this theorem it's shown (as suggested by us) in the Tau method residue, in a number of significant cases, is a minimal in the uniform metric on $[0,1]$, among all possible polynomial residues permitting the Volterra integral equations solution.

We have the following differential equation with polynomial coefficients for the approximation and computing of the second kind modified Bessel function $K_{i\beta}(x)$:

$$y^2 v''(y) + 2(y+1)v'(y) + (1/4 + \beta^2)v(y) = 0,$$

$$v(0) = 1,$$

and the Volterra integral equation

$$y^2 v(y) = \int_0^y \left[\left(\frac{9}{4} + \beta^2\right)x - \left(\frac{1}{4} + \beta^2\right)y - 2\right] v(x)dx + 2y.$$

We obtain the following recurrence formulas for the coefficients of canonical polynomials $Q_m(y) = \sum_{k=0}^{m} q_{km} y^k$ in this case:

$$q_{00} = \frac{2}{\frac{1}{4} + \beta^2}, \quad q_{0k} = -\frac{2(k+2)}{k^2 + k + \frac{1}{4} + \beta^2} q_{0k-1}, \quad k = 1, \dots$$

The minimality of the residue suggested by us follows from the Theorem 3.1 as $\frac{q_{0m}}{|q_{0m}|} = (-1)^m$, $m = 0, 1, \dots$

The advantages of this modification, as compared with usual and other tau-methods, is shown.

4. Tau method approximation for modified Bessel function of complex order. A new numerical scheme of the Tau method application is proposed for the solution of the second order linear differential equations systems, with the second order polynomial coefficients of the following kind:

$$(a_0^{(j)} y^2 + a_1^{(j)} y) v_j''(y) + \sum_{i=1}^{k} [(a_{3i-1}^{(j)} y - a_{3i}^{(j)}) v_i'(y) + a_{3i+1}^{(j)} v_i(y)] = 0,$$
$$v_j(0) = a_{3k+2}^{(j)}, \quad j = 1, \dots, k, \quad y \in [0, 1],$$

in the unknown vector-function $v(y) = (v_1(y), \dots, v_k(y))$. It is assumed to have only one solution. Integrating twice and carrying an addition in the right part in the kind of the vector-polynomial $P_n(y)$, we derive for the determination of the n-th approximation of the solution $v(y) = (v_1(y), \dots, v_k(y))$ the system of Volterra integral equations with polynomial kernels

$$(b_0^{(j)} y^2 + b_1^{(j)} y) v_j(y) = \int_0^y [\sum_{i=1}^{k} (b_{3i-1}^{(j)} x + b_{3i}^{(j)} y + b_{3i+1}^{(j)}) v_i(x)] dx + P_{jn+2}(y),$$
$$j = 1, \dots, k,$$

where the coefficients $b_i^{(j)}$ and $a_i^{(j)}, i = 0, \dots, 3k+2$ and $j = 1, \dots, k$, are connected in a definite way and $P_{jn+2}(y)$, $j = 1, \dots, k$, are $n+2$-th degree polynomials. The different variables of the vector residue choice and its minimization are analyzed. The recurrence formulas for the canonical vector-polynomials coefficients convenient for the calculations are given.

Consider the system of two second order differential equations ($k = 2$) in more detail. This case is of particular interest for differential equations with complex coefficients.

The scheme of the integral form of the Tau Method described in this paper can be used for deriving polynomial approximations of hypergeometric and confluent hypergeometric functions of the first kind with complex parameters.

The modified Kontorovich–Lebedev integral transforms [11] with kernels $ReK_{\frac{1}{2}+i\beta}(x) = \frac{K_{\frac{1}{2}+i\beta}(x) + K_{\frac{1}{2}-i\beta}(x)}{2}$ and $ImK_{\frac{1}{2}+i\beta}(x) = \frac{K_{\frac{1}{2}+i\beta}(x) - K_{\frac{1}{2}-i\beta}(x)}{2i}$, where $K_s(x)$ is MacDonald's function, is of great importance in solving some problems of mathematical physics, in particular mixed boundary value problems for the HELMHOLTZ equation in wedge and cone domains. We find it necessary to compute $ReK_{\frac{1}{2}+i\beta}(x)$ and $ImK_{\frac{1}{2}+i\beta}(x)$ to use this transform in practice [21]. These functions also occur in solving some classes of dual integral equations with kernels which contain MacDonald's function of imaginary index $K_{i\beta}(x)$ [11]. Therefore, now we consider the second kind modified Bessel function $K_{\frac{1}{2}+i\beta}(x)$ in more detail.

We have a system of two second order differential equations

$$y^2 v_1'' + 2(y+1)v_1' + \beta^2 v_1 + \beta v_2 = 0,$$

$$y^2 v_2'' + 2(y+1)v_2' - \beta v_1 + \beta^2 v_2 = 0,$$

$$v_1(0) = 1, v_2(0) = 0,$$

or the system of Volterra integral equations

$$y^2 v_1(y) = \int_0^y ((2+\beta^2)x - (2+\beta^2 y))v_1(x)dx + \beta \int_0^y (x-y)v_2(x)dx + 2y,$$

$$y^2 v_2(y) = \beta \int_0^y (y-x)v_1(x)dx + \int_0^y ((2+\beta^2)x - (2+\beta^2 y))v_2(x)dx,$$

$$K_{\frac{1}{2}+i\beta}(x) = (\frac{\pi}{2x})^{\frac{1}{2}} e^{-x}(v_1(\frac{1}{x}) + iv_2(\frac{1}{x}), \ x \geq 1.$$

The following formulas for the coefficients of canonical vector-polynomials are derived [16]

$$q_{1m}^{(1)} = \frac{(\beta^2 + m(m+1))(m+1)(m+2)}{(\beta^2 + m(m+1))^2 + \beta^2}, \qquad q_{2m}^{(1)} = \frac{\beta(m+2)(m+1)}{(\beta^2 + m(m+1))^2 + \beta^2},$$

$$q_{1m}^{(2)} = -q_{2m}^{(1)}, \qquad q_{2m}^{(2)} = q_{1m}^{(1)},$$

$$q_{1i}^{(j)} = -\frac{2(i+1)((\beta^2 + i(i+1))q_{1i+1}^{(j)} - \beta q_{2i+1}^{(j)})}{(\beta^2 + i(i+1))^2 + \beta^2},$$

$$q_{2i}^{(j)} = -\frac{2(i+1)(\beta q_{1i+1}^{(j)} + (\beta^2 + i(i+1))q_{2i+1}^{(j)})}{(\beta^2 + i(i+1))^2 + \beta^2},$$

$$i = m-1, \ldots, 0, \quad j = 1, 2.$$

By means of computations is shown that the choice of the residue in the form $P_{jn+2}(y) = \tau_{jn+2} T_{n+2}[(1-\alpha_{n+2})y + \alpha_{n+2}]$, $j = 1, 2$, is optimal as compared with other known variants in this case too.

The applications for the numerical solution of boundary value problems in wedge domains are given in [22],[23].

REFERENCES

[1] N. N. Lebedev and I. P. Skalskaya, *Some integral transforms related to Kontorovitch–Lebedev transforms*, (in Russian), in The Questions of the Mathematical Physics, pp. 68–79, Nauka, Leningrad, 1976.

[2] N. Viener, *Fourier Integral and Some Its Applications*, (in Russian), Fizmatgiz, Moscow, 1963.

[3] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972.

[4] A. F. NIKIFOROV AND V. B. UVAROV, *The Special Functions of the Mathematical Physics*, (in Russian), Nauka, Moscow, 1975.

[5] M. I. ZHURINA AND L. N. KARMAZINA, *The Tables and Formulas for the Spherical Functions* $P^m_{-\frac{1}{2}+i\tau}(z)$, (in Russian), VC AN SSSR, Moscow, 1962.

[6] A. P. PRUDNIKOV, Y. A. BRYCHKOV, AND O. I. MARICHEV, *The Integrals and Series*, (in Russian), Nauka, Moscow, 1981.

[7] A. P. PRUDNIKOV, Y. A. BRYCHKOV, AND O. I. MARICHEV, *The Integrals and Series, Special Functions*, (in Russian), Nauka, Moscow, 1983.

[8] A. G. GRINBERG, *The Selected Questions of the Mathematical Theory of the Electric and Magnetic Phenomenons*, (in Russian), Publishing House of AN SSSR, Moscow-Leningrad, 1948.

[9] N. N. LEBEDEV, *Some Integral Transforms of the Mathematical Physics*, PhD Thesis, (in Russian), Ioffe Physics-Technical Institute, Leningrad, 1950.

[10] N. N. LEBEDEV, *The Special Functions and Their Applications*, 2nd edition, revised and complemented, (in Russian), Fizmatgiz, Moscow-Leningrad, 1963.

[11] N. N. LEBEDEV AND I. P. SKALSKAYA, *The dual integral equations connected with the Kontorovitch–Lebedev transform*, (in Russian), Prikl. Matem. and Mechan., 38 (1974), pp. 1090–1097.

[12] V. B. PORUCHIKOV AND J.M. RAPPOPORT, *Inversion formulas for modified Kontorovitch–Lebedev transforms*, (in Russian), Diff. Uravn., 20 (1984), pp. 542–546.

[13] J. M. RAPPOPORT, *Some properties of modified Kontorovitch-Lebedev integral transforms*, (in Russian), Diff. Uravn., 21 (1985), pp. 724–727.

[14] J. M. RAPPOPORT, *Some results for modified Kontorovitch-Lebedev integral transforms*, in Proceedings of the 7th International Colloquium on Finite or Infinite Dimensional Complex Analysis, Marcel Dekker, 2000, pp. 473–477.

[15] U. T. EHRENMARK, *The numerical inversion of two classes of Kontorovich-Lebedev transforms by direct quadrature*, J. Comput. Appl. Math., 61 (1995), pp. 43–72.

[16] J. M. RAPPOPORT, *The canonical vector-polynomials at computation of the Bessel functions of the complex order*, Comput. Math. Appl., 41 (2001), pp. 399–406.

[17] A. GIL, J. SEGURA AND N. M. TEMME, *Evaluation of the modified Bessel function of the third kind of imaginary orders*, J. Comput. Phys., 175 (2002), pp. 398–411.

[18] J. M. RAPPOPORT, *Some numerical quadrature algorithms for the computation of the MacDonald function*, in Proceedings of the Third ISAAC Congress, Progress in Analysis, vol. 2, World Scientific Publishing, 2003, pp. 1223–1230.

[19] B. R. FABIJONAS, D. L. LOZIER, AND J. M. RAPPOPORT, *Algorithms and codes for the MacDonald function: Recent progress and comparisons*, Journ. Comput. Appl. Math., 161 (2003), pp. 179–192.

[20] J. H. FREILICH AND E. L. ORTIZ, *Numerical solution of systems of ordinary differential equations with the Tau Method*, Math. Comp., 39 (1982), pp. 467–479.

[21] J. M. RAPPOPORT, *Tables of modified Bessel functions* $K_{\frac{1}{2}+i\beta}(x)$, (in Russian), Nauka, Moscow, 1979.

[22] J. M. RAPPOPORT, *Tau method and numerical solution of some mixed boundary value problems*, in Proceedings of the Conference on Computational and Mathematical Methods on Science and Engineering CMMSE-2004, Angstrom Laboratory, 2004, pp. 236–242.

[23] J. M. RAPPOPORT, *Dual integral equations for some mixed boundary value problems*, in Advances in Analysis, Proceedings of the 4th ISAAC Congress, World Scientific Publishing, 2005, pp. 167–176,

Tau method and numerical solution of some mixed boundary value problems

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Abstract

The new realization of the LANCZOS Tau Method with minimal residue is proposed for the numerical solution of the second order differential equations with polynomial coefficients. The computational scheme of Tau method [1] is extended for the systems of hypergeometric type differential equations. The programs of evaluation are prepared and tables of the modified BESSEL functions $K_{1/2+i\beta}(x)$ are published. A Tau Method computational scheme is applied to the approximate solution of a system of differential equations related to the differential equation of hypergeometric type. Various vector perturbations are discussed. Our choice of the perturbation term is a shifted CHEBYSHEV polynomial with a special form of selected transition and normalization. The minimality conditions for the perturbation term are found for one equation. They are sufficiently simple for the verification in number of important cases.
The new applications of modified integral KONTOROVITCH–LEBEDEV transforms [2] for the solution of some problems of mathematical physics are given. The algorithm of numerical solution of some mixed boundary value problems for the Helmholtz equation in wedge domains is developed.

Key words: LANCZOS *Tau method,* BESSEL *functions,* KONTOROVITCH–LEBEDEV *integral transforms,* CHEBYSHEV *polynomials, dual integral equation*
MSC 2000: 65D20, 33F05, 33C10, 41A10, 65L10, 65R10

1 Tau method

The questions of the approximation of the solutions of the linear differential equations with polynomial coefficients by means of polynomials coefficients and construction approximations of the KONTOROVITCH–LEBEDEV integral transforms kernels are considered. The Tau method realization with minimal residue choice for the determination of the polynomial approximations of the solutions of the second order differential equations with polynomial coefficients [3] of the following form

$$(a_0y^2 + a_5y)v''(y) + (a_1y + a_2)v'(y) + a_3v(y) = 0, v(0) = a_4, y \in [0,1],$$

is supposed. By its using n-th approximation of the solution is seeked in the form of the n-th degree polynomial $v_n(y)$, which is the solution of the equation

$$(b_0y^2+b_5y)v(y) = \int_0^y (b_1x+b_2y+b_3)v(x)dx+b_4y+\tau_{n+2}T^*_{n+2}[(1-\alpha_{n+2})y+\alpha_{n+2}],$$

where coefficients a_i, i=0,...,5, may be expressed by coefficients b_i, i=0,...,5, $\alpha_{n+2} = \sin^2(\pi/(4(n+2)))$- the most left root of the shifted Chebyshev polynomial of the $n+2$ -th degree $T^*_{n+2}(y)$ in the interval $[0,1]$, τ_{n+2} - undefined coefficient.

The problem about determination of the polynomial $P_n(y) = \sum_{k=0}^n p_k y^k$, which is the least deviated from zero on the interval $[0,1]$ among all n-th degree polynomials, satisfying the pair of linear correlations on the coefficients $p_0 = 0, \sum_{i=1}^n c_i^{(n)} p_i = 1$ was considered. The following theorem is proved:

Theorem 1. If the sequence of numbers $c_i^{(n)}, i = 1, \ldots, n$, is alternating then the polynomial $\tau_n T^*_n[(1-\alpha_n)y + \alpha_n]$ is the polynomial least deviating from zero in the uniform metric on $[0,1]$ among all polynomials of degree n, satisfying the indicated pair of linear relations.

On the basis of this theorem it's shown that suggested by us in the Tau method residue, in the number of significant cases, is the minimal in the uniform metric on $[0,1]$ among all possible polynomial residues permitting the Volterra integral equations solution in the kind of polynomial also.

On the example of computing the second kind modified BESSEL function $K_{i\beta}(x)$ this modification's advantages are shown as compared with usual and other tau-method and approximation method's variants.

The new numerical scheme of the Tau method application is proposed for the solution of the second order linear differential equations systems with the second order polynomial coefficients of the following kind:

$$(a_0^{(j)}y^2 + a_1^{(j)}y)v_j''(y) + \sum_{i=1}^k [(a_{3i-1}^{(j)}y - a_{3i}^{(j)})v_i'(y) + a_{3i+1}^{(j)}v_i(y)] = 0,$$

$$v_j(0) = a_{3k+2}^{(j)},\ j = 1, \ldots, k,\ y \in [0,1],$$

in the unknown vector-function $v(y) = (v_1(y), \ldots, v_k(y))$. It is assumed to have only one solution. Integrating twice and carrying an addition in the right part in the kind of the vector-polynomial $P_n(y)$, we derive for the determination of the n-th approximation of the solution $v(y) = (v_1(y), \ldots, v_k(y))$ the system of Volterra integral equations with polynomial kernels

$$(b_0^{(j)}y^2 + b_1^{(j)}y)v_j(y) = \int_0^y [\sum_{i=1}^k (b_{3i-1}^{(j)}x + b_{3i}^{(j)}y + b_{3i+1}^{(j)})v_i(x)]dx + P_{jn+2}(y),$$

$$j = 1, \ldots, k,$$

where coefficients $b_i^{(j)}$ and $a_i^{(j)}$, $i = 0, \ldots, 3k+2$ and $j = 1, \ldots, k$, are connected in definite way and $P_{jn+2}(y), j = 1, \ldots, k, - n+2$-th degree polynomials. The different variables of the vector residue choice and its minimization are analyzed. The recurrent formulas for the canonical vector-polynomials coefficients convenient for the calculations are given.

Consider the system of two second order differential equations ($k = 2$) in more detail. This case is of particular interest for differential equations with complex coefficients.

The scheme of the integral form of the Tau Method described in this paper can be used for deriving polynomial approximations of hypergeometric and confluent hypergeometric functions of the first kind with complex parameters.

The modified Kontorovitch - Lebedev integral transforms [2] with kernels $\mathrm{ReK}_{1/2+i\beta}(x) = (K_{1/2+i\beta}(x) + K_{1/2-i\beta}(x))/2$ and $\mathrm{ImK}_{1/2+i\beta}(x) = (K_{1/2+i\beta}(x) - K_{1/2-i\beta}(x))/2i$, where $K_s(x)$ is MacDonald's function, is of great importance in solving some problems of mathematical physics, in particular mixed boundary value problems for the Helmholtz equation in wedge and cone domains. We find it necessary to compute $\mathrm{ReK}_{1/2+i\beta}(x)$ and $\mathrm{ImK}_{1/2+i\beta}(x)$ to use this transform in practice. These functions also occur in solving some classes of dual integral equations with kernels which contain MacDonald's function of imaginary index $K_{i\beta}(x)$ [4]. Therefore, now we consider the second kind modified Bessel function $K_{1/2+i\beta}(x)$ in more detail.

We have a system of two second order differential equations

$$\begin{gathered} y^2v_1'' + 2(y+1)v_1' + \beta^2 v_1 + \beta v_2 = 0, \\ y^2v_2'' + 2(y+1)v_2' - \beta v_1 + \beta^2 v_2 = 0, \\ v_1(0) = 1, v_2(0) = 0, \end{gathered}$$

or the system of Volterra integral equations

$$y^2v_1(y) = \int_0^y ((2+\beta^2)x - (2+\beta^2 y))v_1(x)dx + \beta\int_0^y (x-y)v_2(x)dx + 2y,$$

$$y^2 v_2(y) = \beta \int_0^y (y-x) v_1(x) dx + \int_0^y ((2+\beta^2)x - (2+\beta^2 y)) v_2(x) dx,$$

$$K_{1/2+i\beta}(x) = (\pi/(2x))^{1/2} e^{-x} (v_1(1/x) + i v_2(1/x)), x \geq 1.$$

By means of computations is shown that the choice of the residue in the form $P_{jn+2}(y) = \tau_{jn+2} T_{n+2}[(1-\alpha_{n+2})y + \alpha_{n+2}], j = 1, 2,$ is optimal as compared with other known variants in this case too.

2 Mixed boundary value problems

The definition of two pairs of direct and inverse modified KONTOROVITCH–LEBEDEV integral transforms [2] are cited

$$F_+(\tau) = \int_0^\infty f(x) Re K_{1/2+i\tau}(x) dx, 0 \leq \tau \leq \infty,$$

$$f(x) = (4/\pi^2) \int_0^\infty ch(\pi\tau) F_+(\tau) Re K_{1/2+i\tau}(x) d\tau, 0 < x < \infty,$$

and

$$F_-(\tau) = \int_0^\infty f(x) Im K_{1/2+i\tau}(x) dx, 0 \leq \tau \leq \infty,$$

$$f(x) = (4/\pi^2) \int_0^\infty ch(\pi\tau) F_-(\tau) Im K_{1/2+i\tau}(x) d\tau, 0 < x < \infty.$$

The sufficient conditions of the existence of these transforms and the validity of the inversion formulas are given.

It's shown that the inversion formulas of the modified KONTOROVITCH–LEBEDEV integral transforms can be deduced from the inversion formulas of the "usual" KONTOROVITCH–LEBEDEV transforms and the corresponding theorem is proven.

For the case of nonnegative finite functions with restricted variation the conditions of present theorem are reduced to one condition, which is necessary and sufficient then.

The verification of the solution of singular integral equations of the form

$$\phi(x) = f(x) + \lambda \int_0^\infty (K_1(x+y) \pm K_0(x+y)) \phi(y) dy, 0 < x < \infty,$$

where $f(x)$ - given function, λ - parameter, complying with the condition $\lambda < 1/\pi$, is given by means of the modified KONTOROVITCH–LEBEDEV transforms. The proof of the PARSEVAL equalities for these transforms is conducted.

The problem of the evaluation of the modified KONTOROVITCH–LEBEDEV transforms is greatly simplified by means of their decomposition into the form of compositions of more simple integral transforms, in particular Fourier and Laplace transforms. The expression of the modified Kontorovitch–Lebedev integral transforms over the general Meyer integral transforms of special index and argument is given.

The dual integral equations with Macdonald's function of the imaginary order $K_{i\tau}(x)$ in the kernel of the following form were introduced by Lebedev and Skalskaya [2]

$$\int_0^\infty M(\tau)\tau\tanh(\alpha\tau)K_{i\tau}(kr)d\tau = rg(r), 0 < r < a,$$

$$\int_0^\infty M(\tau)K_{i\tau}(kr)d\tau = f(r), r > a,$$

where $g(r)$ and $f(r)$ - given functions. They showed [2] that solutions of this equations may be determined in the form of single quadratures from solutions of second kind Fredholm integral equations with symmetric kernel containing MacDonald's function of the complex order $K_{1/2+i\tau}(x)$.

$$M(\tau) = \frac{2sqrt(2)\sinh(\pi\tau)}{\pi sqrt(\pi)\sinh(\alpha\tau)}\int_0^\infty \psi(t)ReK_{1/2+i\tau}(kt)dt,$$

$$\psi(t) = h(t) - \int_0^\infty K(s,t)\psi(s)ds, a \le t < \infty,$$

where $ReK_{1/2+i\tau}(z)$ - real part of MacDonald's function of complex order $1/2 + i\tau$. In the case $g(r) = 0$

$$h(t) = -\frac{sqrt(k)\exp(kt)}{\pi}\frac{d}{dt}\int_0^\infty \frac{\exp(-kr)f(r)}{sqrt(r-t)}dr$$

$$K(s,t) = \frac{4}{\pi}\int_0^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)}ReK_{1/2+i\tau}(ks)ReK_{1/2+i\tau}(kt)d\tau.$$

The numerical solution is conducted. The economical methods of the evaluation of kernels of the integral equations based on Gauss quadrature formulas on Lagerre polynomial's knots are proposed. The procedures of the preliminary transformation of integrals and extraction of the singularity in the integrand are used for the increase of accuracy and speed of algorithms. The cases of dual integral equations admitting complete analytical solution are considered. Observed examples demonstrate the efficiency of this approach in the numerical solution of the mixed boundary value problems of elasticity and combustion in the wedge domains.

The application of the integral Kontorovitch–Lebedev transforms and dual integral equations to the solution of the mixed boundary value problems are considered. The diffusion and elastic problems reduced to the solution of the proper mixed boundary value problem for the HELMHOLTZ equation

$$\Delta u - k^2 u = 0, \quad \frac{\partial u}{\partial \eta}|_{\varphi=\pm\alpha, 0<r<a}(r) = g(r), \quad u|_{\varphi=\pm\alpha, r>a}(r) = f(r),$$

$$u|_{r \Rightarrow 0} \text{— restricted}, \quad u|_{r \Rightarrow \infty} \text{— restricted}.$$

The solution of the problem as derived by LEBEDEV is determined by the next way in the form of the integral KONTOROVITCH–LEBEDEV transform

$$u(r, \varphi) = \int_0^\infty M(\tau) \frac{\cosh \varphi\tau}{\cosh \alpha\tau} K_{i\tau}(kr) d\tau,$$

where $M(\tau)$ is the solution of dual integral equation.

It is shown that the above-mentioned problems solution for the HELMHOLTZ equation are present in the form of single quadrature from HELMHOLTZ equation are present in the form of single quadrature from FREDHOLM integral equation type. The dimension of the problem is lowered on unit by this, what is the essential advantage of this method. The examples permitting the complete analitical solution of the problem are given.

The numerical solution of the mixed boundary value problems and received dual integral equations is carried out. It consists of numerical solution of the second kind FREDHOLM integral equation with symmetric kernels and the followed taking of quadratures from their solution. The estimation of error is given. The control calculations results give the precision for the solution in 6-7 digits after comma. The considered examples demonstrate the efficiency of the dual integral method in the solution of the mixed boundary value problems for the HELMHOLTZ equation in the wedge domains.

References

[1] J. H. FREILICH, E. L. ORTIZ, "Numerical solution of systems of ordinary differential equations with the Tau Method", *Math. Comp.* **39** (1982) 467–479.

[2] N. N. LEBEDEV, I. P. SKALSKAYA, "The dual integral equations connected with Kontorovitch-Lebedev transform", *Prikl. Matem. and Mechan.* **38**, no.6 (1974) 1090–1097. (in Russian).

[3] J. M. RAPPOPORT, "Canonical vector polynomials for the computation of complex order Bessel functions with the Tau method", *Comp. Math. Appl.* **41** (2001) 399–406.

[4] B. R. FABIJONAS, D. W. LOZIER, J. M. RAPPOPORT, "Algorithms and codes for the Macdonald function: Recent progress and comparisons", *Journ. Comput. Appl. Math.* **161** (2003) 179–192.

DUAL INTEGRAL EQUATIONS METHOD FOR SOME MIXED BOUNDARY VALUE PROBLEMS

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Abstract The new applications of modified KONTOROVITCH–LEBEDEV integral transforms for the solution of some problems of mathematical physics are given. The algorithm of numerical solution of some mixed boundary value problems for the HELMHOLTZ equation in wedge domains by means of dual integral equations method is developed.

Keywords: Dual integral equations, boundary value problems, KONTOROVITCH–LEBEDEV integral transform, modified BESSEL function

1. Introduction

The method of dual integral equations [1,4,10] is one of the effective approaches for the solution of boundary value problems of the mathematical physics. The dual integral equations may be reduced to FREDHOLM integral equations or infinite systems of linear algebraic equations. The special emphasis is done in this paper on dual integral equations with modified Bessel function of imaginary index $K_{i\tau}(x)$ in the kernel. The problems of computation of this function are considered in [2,3,7]. But the theory and applications of this type of dual integral equations are elaborated quite unsufficiently till this time. The paper presents important results for the numerical solution of these types of problems.

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2. The Application of Dual Integral Equations Method for Some Mixed Boundary Value Problems

The modified KONTOROVITCH–LEBEDEV integral transforms [5,8] with kernels $ReK_{1/2+i\beta}(x) = (K_{1/2+i\beta}(x)+K_{1/2-i\beta}(x))/2$ and $ImK_{1/2+i\beta}(x) = (K_{1/2+i\beta}(x) - K_{1/2-i\beta}(x))/2i$, where $K_s(x)$ is MACDONALD'S function, are of great importance in solving some problems of mathematical physics, in particular mixed boundary value problems for the HELMHOLTZ equation in wedge and cone domains. It's necessary to compute $ReK_{1/2+i\beta}(x)$ and $ImK_{1/2+i\beta}(x)$ [6,7,9] to use this transforms in practice. These functions also occur in solving some classes of dual integral equations with kernels which contain MACDONALD'S function of imaginary index $K_{i\beta}(x)$. Therefore the computation of the second kind modified Bessel function $K_{1/2+i\beta}(x)$ is considered in more detail [6,7,9].

The definition of two pairs of direct and inverse modified KONTOROVITCH–LEBEDEV integral transforms [5] are cited

$$F_+(\tau) = \int_0^\infty f(x)ReK_{1/2+i\tau}(x)dx, 0 \le \tau \le \infty,$$

$$f(x) = (4/\pi^2)\int_0^\infty ch(\pi\tau)F_+(\tau)ReK_{1/2+i\tau}(x)d\tau, 0 < x < \infty,$$

and

$$F_-(\tau) = \int_0^\infty f(x)ImK_{1/2+i\tau}(x)dx, 0 \le \tau \le \infty,$$

$$f(x) = (4/\pi^2)\int_0^\infty ch(\pi\tau)F_-(\tau)ImK_{1/2+i\tau}(x)d\tau, 0 < x < \infty.$$

The sufficient conditions of the existence of these transforms and the validity of the inversion formulas are given.

It's shown that the inversion formulas of the modified KONTOROVITCH–LEBEDEV integral transforms can be deduced from the inversion formulas of the "usual" KONTOROVITCH–LEBEDEV transforms and the corresponding theorem is proven. For the case of nonnegative finite functions with restricted variation the conditions of present theorem are reduced to one condition, which is necessary and sufficient then.

The verification of the solution of singular integral equations of the form

$$\phi(x) = f(x) + \lambda\int_0^\infty (K_1(x+y)\overset{+}{-}K_0(x+y))\phi(y)dy, 0 < x < \infty,$$

where $f(x)$ - given function, λ - parameter, complying with the condition $\lambda < 1/\pi$, is given by means of the modified KONTOROVITCH–LEBEDEV transforms. The proof of the PARSEVAL equalities for these transforms is conducted.

The problem of the evaluation of the modified KONTOROVITCH–LEBEDEV transforms is greatly simplified by means of their decomposition into the form of compositions of more simple integral transforms, in particular FOURIER and LAPLACE transforms. The expression of the modified KONTOROVITCH–LEBEDEV integral transforms over the general MEYER integral transforms of special index and argument is given.

The dual integral equations with MACDONALD'S function of the imaginary order $K_{i\tau}(x)$ in the kernel of the following form were introduced by LEBEDEV and SKALSKAYA [4]

$$\int_0^\infty M(\tau)\tau\tanh(\alpha\tau)K_{i\tau}(kr)d\tau = rg(r), 0 < r < a,$$

$$\int_0^\infty M(\tau)K_{i\tau}(kr)d\tau = f(r), r > a,$$

where $g(r)$ and $f(r)$ - given functions. They showed [4] that solutions of these equations may be determined in the form of single quadratures from solutions of second kind FREDHOLM integral equations with symmetric kernel containing MACDONALD'S function of the complex order $K_{1/2+i\tau}(x)$

$$M(\tau) = \frac{2\sqrt{2}\sinh(\pi\tau)}{\pi\sqrt{\pi}\sinh(\alpha\tau)}\int_0^\infty \psi(t)ReK_{1/2+i\tau}(kt)dt,$$

$$\psi(t) = h(t) - \int_0^\infty K(s,t)\psi(s)ds,\ a \le t < \infty,$$

where $ReK_{1/2+i\tau}(z)$ - real part of MACDONALD'S function of complex order $1/2 + i\tau$. In the case $g(r) = 0$

$$h(t) = -\frac{\sqrt{k}e^{kt}}{\pi}\frac{d}{dt}\int_0^\infty \frac{e^{-kr}f(r)}{\sqrt{r-t}}dr,$$

$$K(s,t) = \frac{4}{\pi}\int_0^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)}ReK_{1/2+i\tau}(ks)ReK_{1/2+i\tau}(kt)d\tau.$$

The numerical solution is conducted. The economical methods of the evaluation of kernels of the integral equations based on GAUSS quadrature formulas on LAGUERRE polynomial's knots are proposed. The procedures of the preliminary transformation of integrals and extraction of the singularity in the integrand are used for the increase of accuracy and speed of algorithms. The cases of dual integral equations admitting complete analytical solution are considered. Observed examples demonstrate the efficiency of this approach in the numerical solution of the mixed boundary value problems of elasticity and combustion in the wedge domains [1].

The application of the Kontorovitch–Lebedev integral transforms and dual integral equations to the solution of the mixed boundary value problems are considered. The diffusion and elastic problems reduced to the solution of the proper mixed boundary value problems for the Helmholtz equation.

The mixed boundary value problems for the Helmholtz equation [4]

$$\Delta u - k^2 u = 0 \tag{1}$$

are arised in some fields of mathematical physics.

The solution of this type of problems in the wedge domains is determined by the next way in the form of the Kontorovitch–Lebedev integral transform [4]

$$u(r,\varphi) = \int_0^\infty M(\tau)\frac{\cosh\varphi\tau}{\cosh\alpha\tau}K_{i\tau}(kr)d\tau,$$

where $M(\tau)$ is the solution of dual integral equation.

It is shown that the above-mentioned problems solution for the Helmholtz equation are present in the form of single quadrature from the solution of Fredholm integral equation type. The dimension of the problem is lowered on unit by this, what is the essential advantage of this method. The examples permitting the complete analitical solution of the problem are given.

The numerical solution of the mixed boundary value problems and received dual integral equations is carried out. It consists of numerical solution of the second kind Fredholm integral equation with symmetric kernels and the followed taking of quadratures from their solution. The estimation of error is given. The control calculations results give the precision for the solution in 6-7 digits after comma. The considered examples demonstrate the efficiency of the dual integral method in the solution of the mixed boundary value problems for the Helmholtz equation in the wedge domains.

Let's use the following notations here and further: r, φ - polar coordinates of the point; α - angle of the sectorial domain; u - desired function; η - normal to the boundary.

The numerical solution of some boundary value problems for the equation of the form (1) in arbitrary sectorial domains is considered in our work under the assumption that the function $u|_\Gamma$ is known on the part of the boundary and the normal derivative $\frac{\partial u}{\partial \eta}|_\Gamma$ is known on the other part of the boundary. The KONTOROVITCH–LEBEDEV integral transforms [4] and dual integral equations method [4,10] are used for the searching of the solution.

Let's consider the symmetric case for the simplicity of the calculations

$$\begin{cases} \Delta u - k^2 u = 0, \\ \dfrac{\partial u}{\partial \eta}|_{\varphi=\pm\alpha}(r) = g(r), \quad 0 < r < a, \\ u|_{\varphi=\pm\alpha}(r) = f(r), \quad r > a, \\ u|_{r\to 0} \text{ - restricted,} \\ u|_{r\to\infty} \text{ - restricted.} \end{cases} \tag{2}$$

The solution of (2) is determined by the following way in the form of KONTOROVITCH-LEBEDEV integral transforms [4]

$$u(r,\varphi) = \int_0^\infty M(\tau)\frac{\cosh\varphi\tau}{\cosh\alpha\tau}K_{i\tau}(kr)d\tau, \tag{3}$$

where $M(\tau)$ is the solution of dual integral equation

$$\begin{gathered} \int_0^\infty M(\tau)\tau\tanh(\alpha\tau)K_{i\tau}(kr)d\tau = rg(r), \ 0 < r < a, \\ \int_0^\infty M(\tau)K_{i\tau}(kr)d\tau = f(r), \ r > a, \end{gathered} \tag{4}$$

where $g(r)$ and $f(r)$ - given functions and $K_\nu(z)$ - modified BESSEL function (MACDONALD function) of imaginary order.

The dimension of the problem is lowered on unit by this approach as it can be seen easily.

The dual integral equations of this type were considered in [4]. It was shown in [4] that the solutions of these equations may be determined in the form of single quadratures from auxiliary functions satisfying to the second kind FREDHOLM integral equations with symmetric kernel containing MACDONALD'S function of the complex order $K_{1/2+i\tau}(x)$.

The general case is reduced to the case $g(r) = 0$ as it follows from [4]. Let's consider this case for the simplicity further in this paper.

Let's denote

$$\begin{gathered} h(t) = -\frac{\sqrt{k}e^{kt}}{\pi}\frac{d}{dt}\int_0^\infty \frac{e^{-kr}f(r)}{\sqrt{r-t}}dr, \\ K(s,t) = \frac{4}{\pi}\int_0^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)} ReK_{1/2+i\tau}(ks)ReK_{1/2+i\tau}(kt)d\tau, \end{gathered} \tag{5}$$

where $ReK_{1/2+i\tau}(z)$ - real part of MACDONALD'S function of complex order $1/2 + i\tau$.

Then we obtain the following procedure for the determination of $M(\tau)$ on the basis of [4]

$$M(\tau) = \frac{2\sqrt{2}\sinh(\pi\tau)\cosh(\alpha\tau)}{\pi\sqrt{\pi}\sinh(\alpha\tau)} \int_a^\infty \psi(t) Re K_{1/2+i\tau}(kt)dt, \tag{6}$$

where $\psi(t)$ - solution of the integral FREDHOLM equation of the second kind

$$\psi(t) = h(t) - \frac{k}{\pi}\int_a^\infty K(s,t)\psi(s)ds, a \leq t < \infty. \tag{7}$$

It's useful under the decision of boundary value problems to find the solution u on the boundary of sectorial domain

$$u|_\Gamma(r) = \int_0^\infty M(\tau)K_{i\tau}(kr)d\tau. \tag{8}$$

Substituting expression (6) for $M(\tau)$ in (8) and transposing the order of the integration we obtain

$$u|_\Gamma(r) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_a^\infty \psi(t)G_r(t)dt, \tag{9}$$

where

$$G_r(t) = \int_0^\infty \frac{\sinh(\pi\tau)\cosh(\alpha\tau)}{\sinh(\alpha\tau)} K_{i\tau}(kr) Re K_{1/2+i\tau}(kt)d\tau. \tag{10}$$

So the numerical solution of the boundary value problem (2) consists from the numerical solution of integral FREDHOLM equation of the second kind with symmetric kernel and from the consequent taking of the quadratures from its solution.

Let's truncate the integral equation (7) by the following way

$$\psi(t) = h(t) - \frac{k}{\pi}\int_a^b K(s,t)\psi(s)ds, a \leq t \leq b. \tag{11}$$

The conducted estimations show that we don't obtain any loss of accuracy in the bounds $10^{-7} - 10^{-8}$ under the truncation of the integral equation (7) for $b \geq 10$ in view of fast decrease of the kernels $K(s,t)$ for $s,t \to \infty$.

The method of mechanical quadratures with the use of combined SIMPSON formula with instant integration step is one of the most convenient methods of numerical solution of FREDHOLM integral equation of the second kind.

The application of combined SIMPSON formula possible in the case of fixed step Δt and odd number of steps leads to the linear inhomogeneous system of algebraic equations

$$\psi_i + \frac{\Delta t}{3}\frac{k}{\pi}\sum_{j=1}^{N} A_j K_{ij}\psi_j = h_i, i = 1, ..., N, \tag{12}$$

where

$$N = \frac{b-a}{\Delta t} + 1,$$
$$t_j = a + (j-1)\Delta t, j = 1, ..., N,$$
$$K_{ij} = K(s_i, t_j), h_i = h(t_i), i, j = 1, ..., N,$$
$$\psi_i \text{ - approximate values } \psi(t_i), i = 1, ..., N,$$
$$A_j = \begin{cases} 1 \text{ for } j = 1, j = N, \\ 4 \text{ for } j = 2, 4, 6, ..., N-1, \\ 2 \text{ for } j = 3, 5, 7, ..., N-2. \end{cases}$$

It's convenient to use the GAUSS elimination method for the solution (12). The speed and operating memory of the mainframe computer BESM-6 made possible to use under the calculations up to $N \approx 150$ knots.

The solution of the system (12) gives values $\psi_1, ..., \psi_n$. The approximate solution of the integral equation (7) upon the whole of interval $[a, b]$ is found by means of interpolation over this values $\psi_i,\ i = 1, ..., N$. For the analytical expression of the approximate solution we take the following magnitude

$$\psi_n(t) = h(t) - \frac{\Delta t}{3}\frac{k}{\pi}\sum_{j=1}^{n} A_j K(t, t_j)\psi_j, \tag{13}$$

having the values $\psi_1, ..., \psi_N$ in the points of interpolation. We obtain more biggest accuracy in this case with respect to the comparison with linear or quadratic interpolation.

Further the solution of dual equation was computed by the formulas (6) with the use of codes and routines for the computation $ReK_{1/2+i\tau}(x)$ [6,7,9].

The error estimation of the numerical solution scheme (12) - (13) may be carried on.

Integrals (5), (10) may be expressed through known functions for special values of the angle α, in particular for $\alpha = \frac{\pi}{n}, n = 1, 2, ...$

We compute the truncated integrals in fact under the computations of integrals (5), (10) on the computer: the integration is carried on over the some interval $[0, B]$. In view of this fact it's important to choose by the correct way the truncation interval $[0, B]$, ensuring the computation of stated integrals with necesssary precision without the expenditure of unnecessary computer time. The estimations of the error

$$K^B(s,t) = \frac{4}{\pi}\int_B^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)} ReK_{1/2+i\tau}(ks) ReK_{1/2+i\tau}(kt) d\tau,$$

$$G_r^B(t) = \int_B^\infty \frac{\sinh(\pi\tau)\cosh(\alpha\tau)}{\sinh(\alpha\tau)} K_{i\tau}(kr) ReK_{1/2+i\tau}(kt) d\tau,$$

arising from the truncation are useful for this purpose. On the basis of inequalities from [4,8] for functions $K_{i\tau}(x)$ and $ReK_{1/2+i\tau}(x)$ we obtain

$$K^B(s,t) \le c^2 k^{-3/2}(st)^{-3/4}\frac{e^{-2\alpha B}}{2\alpha}(B^2 + \frac{B}{\alpha} + \frac{1}{2\alpha^2}), \tag{14}$$

$$G_r^B(t) \le \frac{Ac}{\alpha} k^{-1} r^{-1/4} t^{-3/4} e^{-2\alpha B}, \tag{15}$$

where A and c - some positive constants having the multiplicity of a unit. As it can be seen from (14) and (15) it's necessary to take the extending interval $[0, B]$ for the decreasing values of angle α.

Let's consider the example admitting the complete analytical solution of the problem (2)

$$f(r) = \frac{\sqrt{\pi}}{k\sqrt{2}}(e^{-kr} + e^{kr}[1 - \Phi(\sqrt{2k(r+a)})]),$$

$$g(r) = 0, \alpha = \frac{\pi}{4}.$$

Then we obtain on the basis of relevant calculations [4] that

$$h(t) = e^{-kt} + \frac{1}{\pi}e^{-ka}K_0(k(t+a)),$$

$$K(s,t) = K_0(k(s+t)) + K_1(k(s+t)), \tag{16}$$

and $\psi(t) = e^{-kt}$.

Substituting (16) in (9) and performing some calculations we obtain for $r < a$

$$u|_\Gamma(r) = \frac{\sqrt{\pi}}{k\sqrt{2}}(e^{-kr}(1 - \phi(\sqrt{2k(a-r)})) + e^{kr}(1 - \phi(\sqrt{2k(a+r)})))$$

and for $r > a$

$$u|_\Gamma(r) = \frac{\sqrt{\pi}}{k\sqrt{2}}(e^{-kr} + e^{kr}(1 - \phi(\sqrt{2k(a+r)})))$$

(verification of the conditions of the problem).

We obtained the precision in 7-8 significant digits under the solution of dual integral equation (computation of the values $(\cosh \frac{\pi\tau}{2})^{-1}M(\tau)$) so for $a = 1.0, k = 1$ $(\cosh \frac{3\pi}{2})^{-1}M(3) = .9288253_{10} - 01$.

We obtained the precision in 6-7 digits after comma under the calculation of values $u|_\Gamma(r)$ so for $a = 1.0, k = 1$ $u|_\Gamma(2) = .1745444_{10} + 00$.

The different preliminary procedures of the separation of singularity or transformation of the integral into the integral without the singularity are useful for the computation of integral (9).

Let's consider the calculations for the case $\alpha = \frac{\pi}{2}$ in detail [4]. Let's introduce the functions

$$G_{1r}(t) = \frac{\pi\sqrt{\pi}}{2\sqrt{2}} \frac{e^{-k(r+t)}}{\sqrt{k(r+t)}}$$

and

$$G_{2r}(t) = \begin{cases} 0, \ t \le r \\ \dfrac{\pi\sqrt{\pi}}{2\sqrt{2}} \dfrac{e^{-k(t-r)}}{\sqrt{k(t-r)}}, \ t > r. \end{cases}$$

Then $G_r(t) = G_{1r}(t) + G_{2r}(t)$ and

$$u|_\Gamma(r) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_a^b \psi(t) G_{1r}(t) dt + \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_{max(a,r)}^b \psi(t) G_{2r}(t) dt. \quad (17)$$

The formula (17) is more convenient for the application of procedures of numerical integration then formula (9).

Here the function $G_{1r}(t)$ hasn't singularities for all $r, t > 0$ and the function $G_{2r}(t)$ hasn't singularities for $t \ge a, r < a$. For $r \ge a$ integrand of the second integral in (17) has singularity for $t = r$ and the integral itself is equal to

$$\int_a^b \psi(t) \frac{e^{-k(t-r)}}{\sqrt{k(t-r)}} dt.$$

Let's make the change of variables $t_1 = \sqrt{\frac{t-r}{b-r}}$ and introduce the function

$$g(t_1) = \frac{\sqrt{b-r}}{k} \psi((b-r)t_1^2 + r) e^{-k(b-r)t_1^2}.$$

Then the second integral in (17) is equal to $2\int_0^1 g(t_1)dt_1$, where the latter integral doesn't contain any singularities in the integrand.

It's strongly efficient to use the procedures of numerical integration for the transformed integral. The accuracy of computations is increased and the computer time is shorten by this approach.

3. Summary

The dual integral equations with MACDONALD'S function of the imaginary order $K_{i\tau}(x)$ in the kernel are considered. The solutions of these equations and proper mixed boundary value problems are determined in the form of single quadratures from solutions of second kind FREDHOLM integral equations. The numerical solution is conducted and the problems of the computational methodology are discussed. Examples demonstrate the efficiency of the dual integral method in the numerical solution of the mixed boundary value problems of elasticity, combustion and electrostatics in the wedge domains.

References

[1] Alexandrov, V.M., Pozharskii D.A. *Nonclassical spatial problems of the mechanics of contact interactions of elastic bodies.* Moscow, Factorial, 1998.-288pp. (in Russian).

[2] Ehrenmark, U.T. The numerical inversion of two classes of KONTOROVICH–LEBEDEV transform of direct quadrature. *Journ. Comp. Appl. Math.*, 61 (1995), 43–72.

[3] Fabijonas, B.R., Lozier, D.W., and Rappoport J.M. Algorithms and codes for the Macdonald function: recent progress and comparisons. *Journ. Comp. Appl. Math.*, 161(2003), 179–192.

[4] Lebedev, N.N., and Skalskaya, I.P. Dual integral equations connected with KONTOROVICH–LEBEDEV integral transform. *Prikl. Matem. Mechan.*, 38(1974), N 6, 1090–1097 (in Russian).

[5] Lebedev, N.N., and Skalskaya, I.P. Some integral transforms related KONTOROVICH–LEBEDEV transform. *The Problems of Mathematical Physics.* Leningrad: Nauka, Leningr. otd., 1976, 68–79 (in Russian).

[6] Rappoport, J.M. *Tables of Modified* BESSEL *Functions* $K_{1/2+i\tau}(x)$. Moscow: Nauka, 1979.-338pp. (in Russian).

[7] Rappoport, J.M. *The programs and some methods of the* MACDONALD *function computation.* Akad. Nauk SSSR, OVM, Moscow, 1991.-16pp. (in Russian).

[8] Rappoport, J.M. Some results for modified KONTOROVICH–LEBEDEV transforms. *Proceedings of the 7th International Colloquium on Finite or Infinite Dimensional Complex Analysis.* Marcel Dekker Inc., 2000, pp.473–477.

[9] Rappoport, J.M. The canonical vector-polynomials at computation of the BESSEL functions of the complex order. *Computers and Mathematics with Applications*, 2001, vol.41, No.3/4, pp.399–406.

[10] Ufliand, I.S. *Method of dual equations in the mathematical physics problems.* Leningrad: Nauka, Leningr. otd., 1977. - 220pp. (in Russian).

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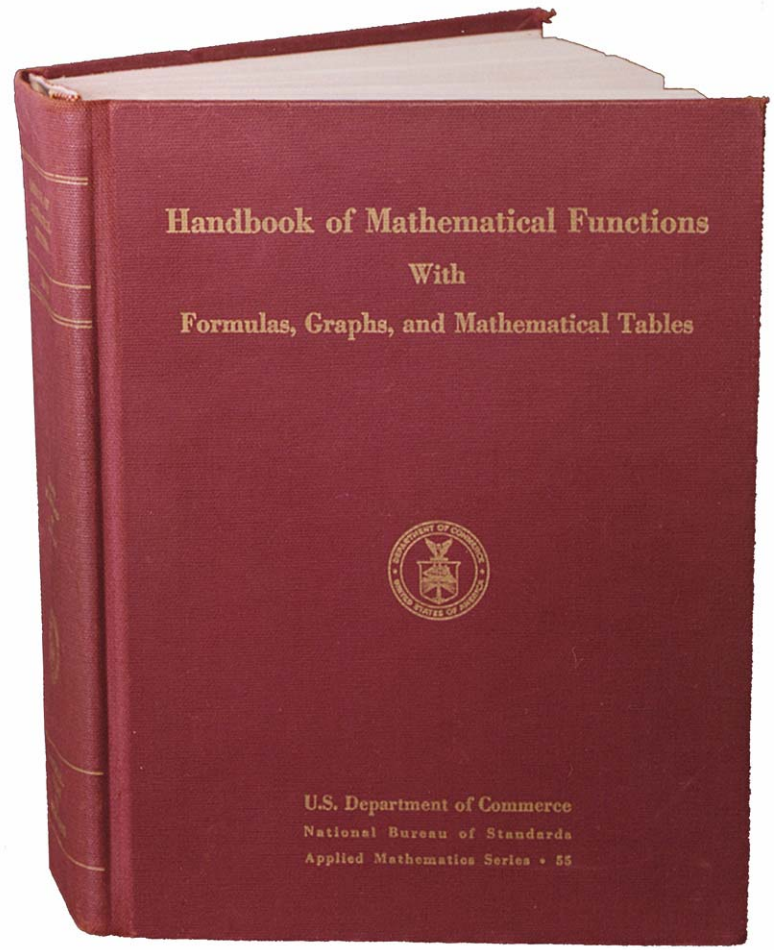
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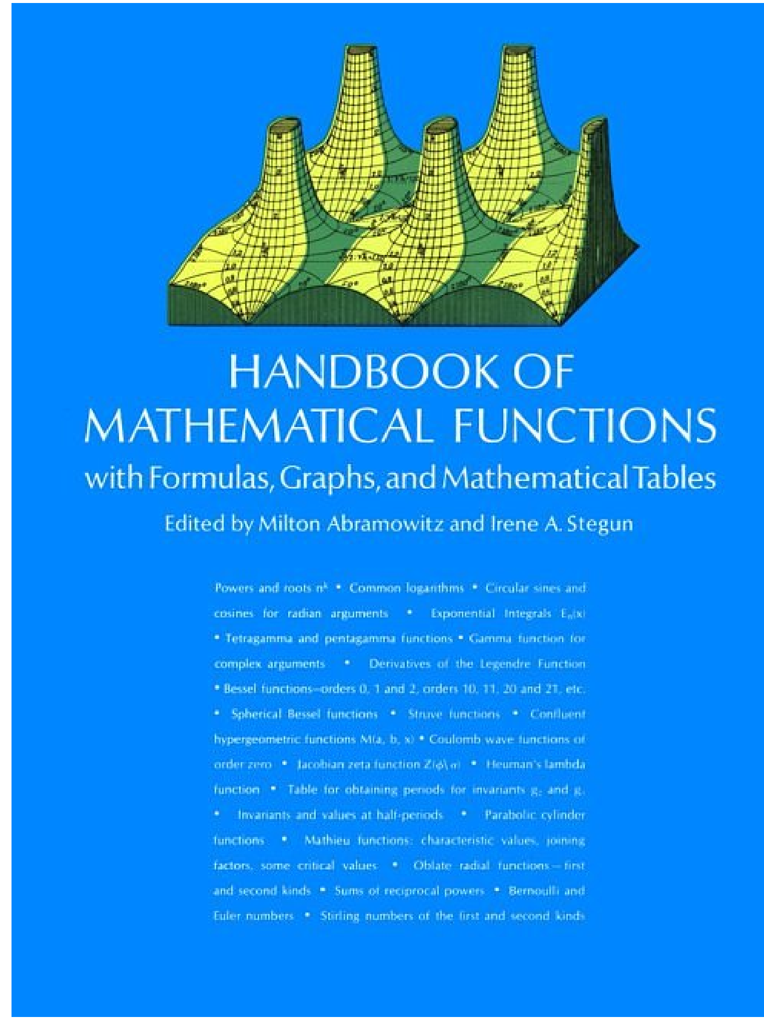
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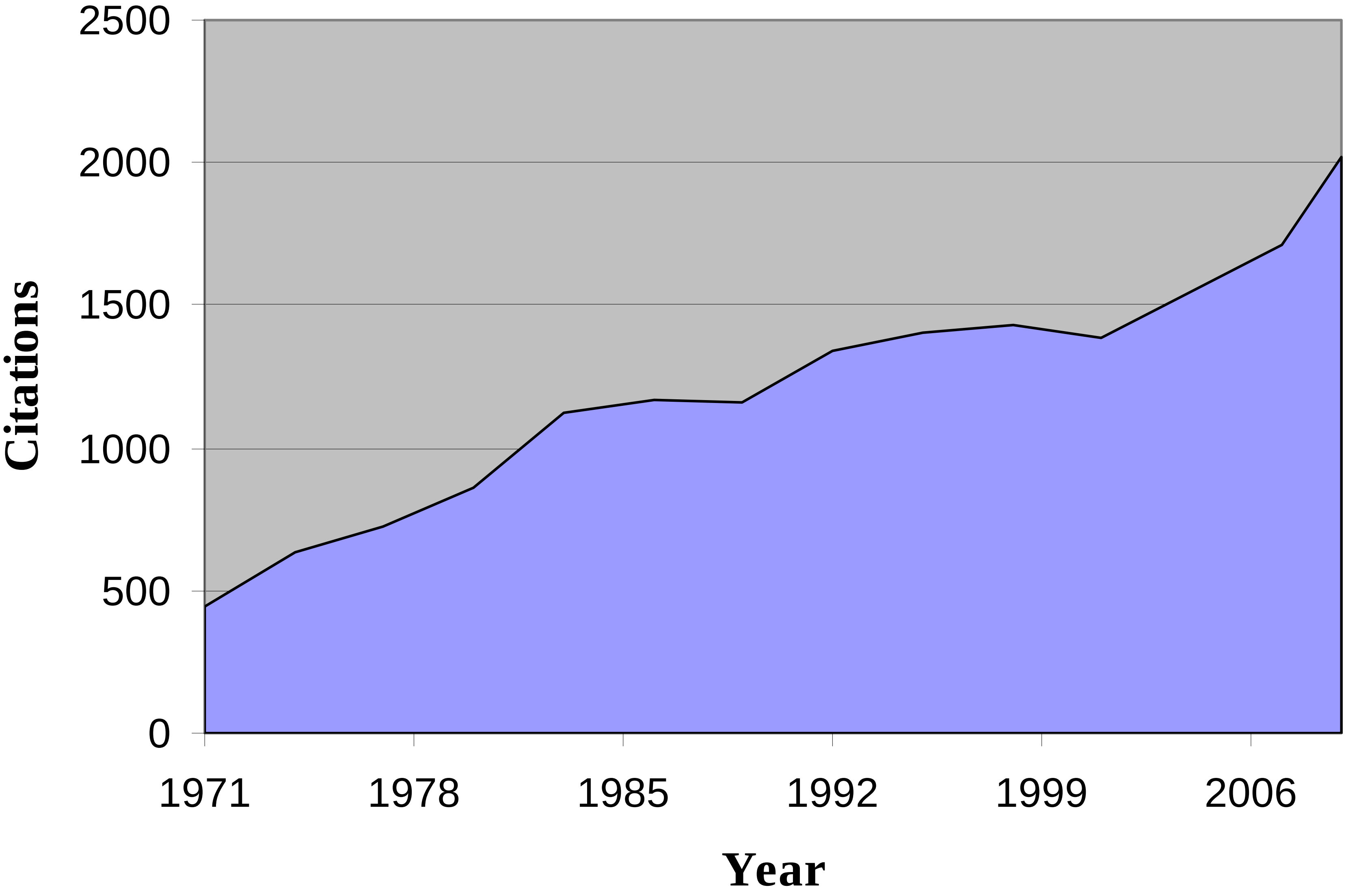

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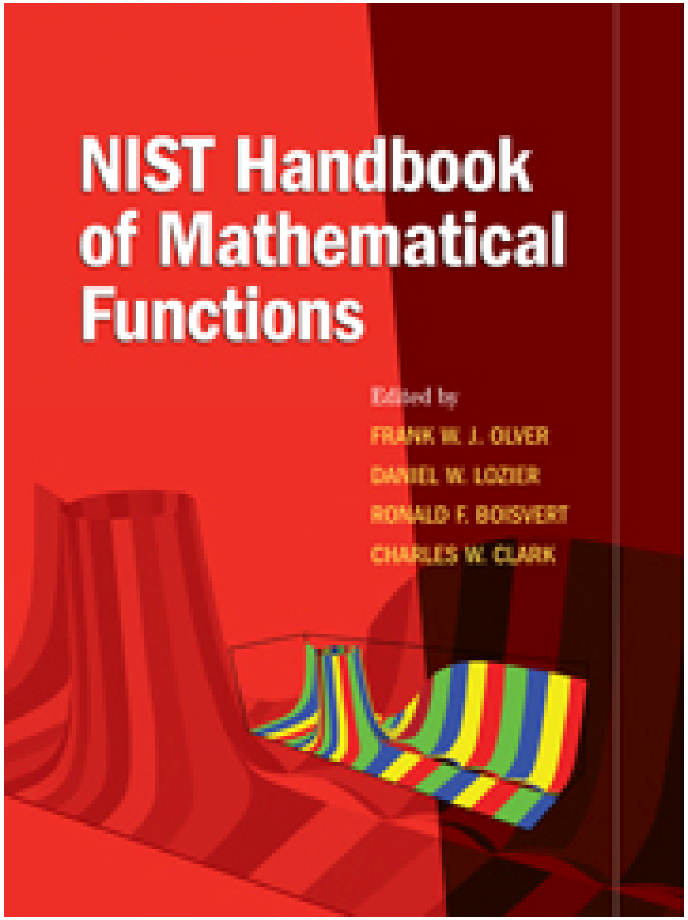

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