Asymptotic expansions of the second and third Appell's functions for one large variable

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Introduction

Appell's function	Series Definition	Region of convergence
$F_1(a,b,b',c;x,y) =$	$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n$	$\max\{\lvert x\rvert,\lvert y\rvert\}<1$
$F_2(a,b,b',c,c';x,y) =$	$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m!n!}x^m y^n$	$\lvert x\rvert+\lvert y\rvert<1$
$F_3(a,a',b,b',c;x,y) =$	$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n$	$\max\{\lvert x\rvert,\lvert y\rvert\}<1$
$F_4(a,b,c,c';x,y) =$	$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n m!n!}x^m y^n$	$\sqrt{\lvert x\rvert}+\sqrt{\lvert y\rvert}<1$

Objective

Analyze the asymptotic behaviour of the Appell's functions for large values of x and/or y.

	Appell's function	Asymptotic expansions	Paper
KNOWN RESULTS	$F_1(a,b,b',c;x,y)$	large values of x and/or y	2004
	$F_2(a,b,b',c,c';x,y)$	large values of x and y	2010
	$F_3(a,b,b',c,c';x,y)$	large values of x and y	
	$F_4(a,b,b',c,c';x,y)$	large values of one variable	

Objective

Analyze the asymptotic behaviour of the Appell's functions for large values of x and/or y.

	Appell's function	Asymptotic expansions
UNKNOWN RESULTS	$F_2(a, b, b', c, c'; x, y)$	large values of one variable
	$F_3(a, b, b', c, c'; x, y)$	large values of one variable
	$F_4(a, b, b', c, c'; x, y)$	large values of x and y

Outline

1. Known results
2. The second Appell's function F_2 for one large variable
3. The third Appell's function F_3 for one large variable
4. Future work (F_4 for two large variables)

The first Appell's function F_1

C. FERREIRA AND J. L. LÓPEZ, Asymptotic expansions of the Appell function F_1, *Q. Appl. Math.*, **62** vol. 2 (2004), 235–257.

$$F_1(a,b,c,d;x,y) \equiv \frac{\Gamma(d)}{\Gamma(d-a)\Gamma(a)} \int_0^1 s^{a-1}(1-s)^{d-a-1}(1-sx)^{-b}(1-sy)^{-c} ds$$

where $\Re(a) > 0$, $\Re(d-a) > 0$, $x \notin [1,\infty)$ if $b \geq 1$, and $y \notin [1,\infty)$ if $c \geq 1$

Asymptotic method for **Mellin convolution integrals**

Asymptotic expansion of F_1 for one large variable

For $\Re a > 0$, $\Re(d-a) > 0$, $|\text{Arg}(z)| < \pi$ and $y - 1 \notin \mathbb{R}^+ \cup \{0\}$ if $\Re c \geq 1$

First Appell's function: $F_1(a, b, c, d; 1-z, y)$

	Asymptotic expansion of F_1 for one large variable
$1 + a - b \notin \mathbb{Z}$	$\sum_{k=0}^{n-1} \frac{\hat{B}_k}{z^{k+b}} + \sum_{k=0}^{n-\lfloor \Re(1-b+a) \rfloor - 1} \frac{\hat{A}_k}{z^{k+a}}$
$1 + a - b \in \mathbb{Z}$	$\sum_{k=0}^{b-a-1} \frac{\hat{A}_k}{z^{k+a}} + \sum_{k=0}^{n-1} \frac{\tilde{B}_k \log(z) - \tilde{C}_k}{z^{k+b}}$

Asymptotic expansion of F_1 for two large variables

For $\Re a > 0$, $\Re(d-a) > 0$, $|\mathrm{Arg}(xz)| < \pi$ and $|\mathrm{Arg}(yz)| < \pi$,

First Appell's function: $F_1(a, b, c, d; 1-xz, 1-yz)$

	Asymptotic expansion of F_1 for two large variables
$1+a-b-c \notin \mathbb{Z}$	$\displaystyle\sum_{k=0}^{n-1} \frac{\hat{B}_k}{z^{k+b+c}} + \sum_{k=0}^{n-\lfloor \Re(1+a-b-c) \rfloor -1} \frac{\hat{A}_k}{z^{k+a}}$
$1+a-b-c \in \mathbb{Z}$	$\displaystyle\sum_{k=0}^{b+c-a-1} \frac{\hat{A}_k}{z^{k+a}} + \sum_{k=0}^{n-1} \frac{\tilde{B}_k \log(z) - \tilde{C}_k}{z^{k+b+c}}$

The second Appell's function F_2

E. García and J. L. López, The Appell's function F_2 for large values of its variables, *Q. Appl. Math.*, **68** vol. 4 (2010), 701–712.

$$F_2(a,b,b',c,c';x,y) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-s} s^{a-1} \; {}_1F_1(b,c;xs)\,{}_1F_1(b',c';ys)ds$$

where $a > 0$ and $\Re(x+y) < 1$

Asymptotic method for **Mellin convolution integrals**

Asymptotic expansion of F_2 for two large variables

For $\Re x < 0$, $\Re y < 0$, $a > 0$ and $b + b' > 0$, $F_2(a, b, b', c, c'; x, y)$

Asymptotic expansion of F_2 for two large variables	
$b + b' - a \notin \mathbb{Z}$	$\displaystyle\sum_{k=0}^{n-1} \frac{\hat{B}_k(x/y)}{x^{k+b+b'}} + \sum_{k=0}^{m-1} \frac{\hat{A}_k(x/y)}{x^{k+a}}$
$b + b' - a \in \mathbb{N}$	$\displaystyle\sum_{k=0}^{b+b'-a-1} \frac{\hat{A}_k(x/y)}{x^{k+a}} + \sum_{k=0}^{n-1} \frac{\tilde{C}_k(x/y) - \tilde{B}_k(x/y)\log(-x)}{x^{k+b+b'}}$
$a - b - b' \in \mathbb{N}$	$\displaystyle\sum_{k=0}^{a-b-b'-1} \frac{\hat{B}_k(x/y)}{x^{k+b+b'}} + \sum_{k=0}^{m-1} \frac{\tilde{C}_{k+a-b-b'}(x/y) - \tilde{B}_{k+a-b-b'}(x/y)\log(-x)}{x^{k+a}}$

The third Appell's function F_3

A. ERDELYI, *Higher transcendental functions*, Vol I, McGraw-Hill, New York, 1953. Formula $5.11(10)$

$$F_3(a,a',b,b',c;x,y)=\sum\frac{\Gamma(c)\Gamma(\rho-\lambda)\Gamma(\sigma-\nu)}{\Gamma(\rho)\Gamma(\sigma)\Gamma(c-\lambda-\nu)}(-x)^{-\lambda}(1-y)^{-\nu}$$
$$\times F_2(a+a'+1-c,a,a',a+1-b,a'+1-b';1/x,1/y)$$

where the sum consists of four terms in which

	1st	2nd	3rd	4th
λ	a	a	b	b
ν	a'	b'	a'	b'
ρ	b	b	a	a
σ	b'	a'	b'	a'

The fourth Appell's function F_4

A. Erdelyi, *Higher transcendental functions*, Vol I, McGraw-Hill, New York, 1953. Formula $5.11(9)$

$$
\begin{aligned}
F_4(a,&b,c,c';x,y)\\
&= \frac{\Gamma(c')\Gamma(b-a)}{\Gamma(c'-a)\Gamma(b)}(-c)^{-a}F_4(a,a+1-c',c,a+1-b;x/y,1/y)\\
&\quad + \frac{\Gamma(c')\Gamma(a-b)}{\Gamma(c'-b)\Gamma(a)}(-c)^{-b}F_4(b+1-c',b,c,b+1-a;x/y,1/y)
\end{aligned}
$$

1. Known results
2. The second Appell's function F_2 for one large variable
3. The third Appell's function F_3 for one large variable
4. Future work

Starting point

$$F_2(a,b,b',c,c';x,y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')} \times \int_0^1 du \int_0^1 \frac{u^{b-1}v^{b'-1}(1-u)^{c-b-1}(1-v)^{c'-b'-1}}{(1-ux-vy)^a} dv,$$

where $\Re c > \Re b > 0$, $\Re c' > \Re b' > 0$, $x, y \in \mathbb{C}$

Objective: asymptotic expansions of $F_2(a,b,b',c,c';x,y)$ for **large negative values** of $\mathbf{x}$ and fixed a, b, b', c, c', y

- After the change of variables $\widetilde{x} := -1/x$, $u \to \widetilde{x}u$ and $\widetilde{x} \to 0^+$

$$F_2(a, b, b', c, c'; -1/\widetilde{x}, y) = \widetilde{x}^b \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')} \times \int_0^{1/\widetilde{x}} du \int_0^1 \frac{u^{b-1}v^{b'-1}(1-xu)^{c-b-1}(1-v)^{c'-b'-1}}{(1+u-vy)^a} dv,$$

with $\Re c > \Re b > 0$, $\Re c' > \Re b' > 0$ and $y \notin [1, +\infty)$ if $\Re(a) \geq 1$

$$F_2(a,b,b',c,c';-1/\widetilde{x},y)=\widetilde{x}^b\frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}$$
$$\times\int_0^1 v^{b'-1}(1-v)^{c'-b'-1}dv\int_0^\infty h(\widetilde{x}u)f_v(u)du,$$

with $\widetilde{x}\to 0^+$,

$$h(u)=(1-u)^{c-b-1}\chi_{[0,1)}(u)$$

and

$$f_v(u)=\frac{u^{b-1}}{(1+u-vy)^a}=\frac{u^{b-a-1}}{\left(1+\frac{1-vy}{u}\right)^a}$$

Asymptotic method

J. L. López, Asymptotic expansions of Mellin convolution integrals, *SIAM Rev.*, **50** n. 2 (2008), 275–293.

$$\int_0^\infty h(xt)f(t)dt$$

Functions $f(t)$ and $h(t)$ are locally integrable on $[0,\infty)$ and

	$f(t)$	$h(t)$
$t=\infty$	$\sum_{k=0}^{n-1}\frac{a_k}{t^{\alpha_k}}+\mathcal{O}(t^{-\alpha_n})$	$\mathcal{O}(t^{-\beta})$, $\beta\in\mathbb{R}$
$t=0^+$	$\mathcal{O}(t^{-\alpha})$, $\alpha\in\mathbb{R}$	$\sum_{k=0}^{m-1} b_k t^{\beta_k}+\mathcal{O}(t^{\beta_m})$

$$h(u) = (1-u)^{c-b-1}\chi_{[0,1)}(u), \quad f_v(u) = \frac{u^{b-a-1}}{\left(1+\frac{1-vy}{u}\right)^a}$$

1. Functions $f_v(t)$ and $h(t)$ are locally integrable on $[0,\infty)$

2. $f_v(t)$
   - $t=\infty$:
   $$f_v(t) = \sum_{k=0}^{n-1} \frac{A_k}{t^{k+1+a-b}} + f_{n,v}(t), \quad A_k := \frac{(-1)^k(1-vy)^k(a)_k}{k!}$$
   - $t=0^+$: $f_v(t) = \mathcal{O}(t^{-(1-b)})$

3. $h(t)$
   - $t=0^+$: $h(t) = \sum_{k=0}^{m-1} B_k t^k + h_m(t), \quad B_k := \frac{(1+b-c)_k}{k!}$
   - $t=\infty$: $h(t) = \mathcal{O}(t^{-(1+b-c)})$

The non-logarithmic case

J. L. López, Asymptotic expansions of Mellin convolution integrals, *SIAM Rev.*, **50** n. 2 (2008), 275–293.

When $b - a \notin \mathbb{Z}$:

$$\int_0^\infty h(\tilde{x}u) f_v(u) du = \sum_{k=0}^{n-1} A_k M[h; b-a-k] \tilde{x}^{k+a-b} + \sum_{j=0}^{m-1} B_j M[f; j+1] \tilde{x}^j + \int_0^\infty f_{n,v}(u) h_m(\tilde{x}u) du,$$

where $n = m + \lfloor b - a \rfloor$ and $M[g; z]$ is the Mellin transform of a function $g \in L_{\text{Loc}}(0, \infty)$: $M[g; z] = \int_0^\infty g(t) t^{z-1} dt$

The non-logarithmic case

When $b - a \notin \mathbb{Z}$:

$$\int_0^\infty h(\tilde{x}u) f_v(u) du = \sum_{k=0}^{n-1} A_k \frac{\Gamma(c-b)\Gamma(b-a-k)}{\Gamma(c-a-k)} \tilde{x}^{k+a-b}$$
$$+ \sum_{j=0}^{m-1} B_k (1-vy)^{j+b-a} \frac{\Gamma(a-b-j)\Gamma(j+b)}{\Gamma(a)} \tilde{x}^j + \int_0^\infty f_{n,v}(u) h_m(\tilde{x}u) du,$$

where $\int_0^\infty f_{n,v}(u) h_m(\tilde{x}u) du = \mathcal{O}(\tilde{x}^{m+\lfloor b-a \rfloor + a - b})$

The non-logarithmic case

When $b - a \notin \mathbb{Z}$:

$$F_2(a,b,b',c,c';x,y) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{n-1} \frac{\widehat{A}_k}{x^{k+a}} + \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{m-1} \frac{\widehat{B}_k}{x^{k+b}} + R_m(x)$$

where

$$\widehat{A}_k = \frac{(-1)^a (a)_k}{k!} \frac{\Gamma(b-a-k)}{\Gamma(c-a-k)} {}_2F_1(b', -k, c'; y),$$

$$\widehat{B}_k = \frac{(-1)^b (b)_k}{k!} \frac{\Gamma(a-b-k)}{\Gamma(c-b-k)} {}_2F_1(b', a-b-k, c'; y),$$

and $R_m(x) = \mathcal{O}(x^{-m-a-\lfloor b-a \rfloor})$ when $x \to -\infty$

The logarithmic case I

J. L. López, Asymptotic expansions of Mellin convolution integrals, *SIAM Rev.*, **50** n. 2 (2008), 275–293.

When $b-a\in\mathbb{N}$:

$$\int_0^\infty h(\widetilde{x}u)f_v(u)du=\sum_{j=0}^{b-a-1}A_jM[h;b-a-j]\widetilde{x}^{j+a-b}$$

$$+\sum_{j=0}^{m-1}\widetilde{x}^j\left\{\lim_{z\to 0}\left[A_{j+b-a}M[h;z-j]\ +B_jM[f;z+j+1]\right]-A_{j+b-a}B_j\log\widetilde{x}\right\}$$

$$+\int_0^\infty f_{n,v}(u)h_m(\widetilde{x}u)du,$$

The logarithmic case I

When $b-a\in\mathbb{N}$:

$$\int_0^\infty h(\widetilde{x}u)f_v(u)du=\sum_{j=0}^{b-a-1}A_j\frac{\Gamma(c-b)\Gamma(b-a-j)}{\Gamma(c-a-j)}\widetilde{x}^{j+a-b}$$

$$+\sum_{j=0}^{m-1}\widetilde{x}^j\left\{\frac{(-1)^{b-a}(1-vy)^{j+b-a}(b)_j}{(j+b-a)!j!}\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(a)\Gamma(c-b-j)}\left[\psi(j+b-a+1)\right.\right.$$

$$\left.\left.+\psi(j+1)-\psi(j+b)-\psi(c-b-j)-\log(1-vy)\right]-A_{j+b-a}B_j\log\widetilde{x}\right\}$$

$$+\int_0^\infty f_{n,v}(u)h_m(\widetilde{x}u)du.$$

where $\int_0^\infty f_{n,v}(u)h_m(\widetilde{x}u)du=\mathcal{O}(\widetilde{x}^m\log\widetilde{x})$

The logarithmic case I

When $b-a\in\mathbb{N}$:

$$F_2(a,b,b',c,c';x,y)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{j=0}^{b-a-1}\frac{\widehat{A}_j}{x^{j+a}}+\frac{\Gamma(c)}{\Gamma(a)}\sum_{j=0}^{m-1}\frac{[\widetilde{C}_j+\widetilde{B}_j\log(-x)]}{x^{j+b}}+R_m(x)$$

$$\widetilde{B}_j:=\frac{(-1)^{j+2b-a}}{j!(j+b-a)!}\frac{(b)_j}{\Gamma(c-b-j)}{}_2F_1(b',a-b-j,c';y),$$

$$\widetilde{C}_j:=\widetilde{B}_j\Big\{[\psi(j+b-a+1)+\psi(j+1)-\psi(j+b)-\psi(c-b-j)]$$
$$-\frac{1}{{}_2F_1(b',a-b-j,c';y)}\frac{d}{dz}{}_2F_1'(b',-z,c';y)\Big|_{z=j+b-a}\Big\}$$

and $R_m(x)=\mathcal{O}(x^{-m-b}\log(-x))$ when $x\to-\infty$.

The logarithmic case II

When $a-b\in\mathbb{N}$:

$$F_2(a,b,b',c,c';x,y)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{j=0}^{a-b-1}\frac{\widehat{B}_j}{x^{j+b}}+\frac{\Gamma(c)}{\Gamma(b)}\sum_{j=0}^{n-1}\frac{[\widetilde{C}_j+\widetilde{B}_j\log(-x)]}{x^{j+a}}+R_m(x)$$

$$\widetilde{B}_j:=\frac{(-1)^{j+2a-b}(a)_j}{j!(j+a-b)!\Gamma(c-a-j)}{}_2F_1(b',-j,c';y),$$

$$\widetilde{C}_j:=\widetilde{B}_j\left\{[\psi(1+j+a-b)+\psi(1+j)-\psi(a+j)-\psi(c-a-j)]\right.$$
$$\left.-\frac{1}{{}_2F_1(b',-j,c';y)}\frac{d}{dz}{}_2F_1'(b',-z,c';y)\Big|_{z=j}\right\}$$

and $R_m(x)=\mathcal{O}(x^{-m-b}\log(-x))$ when $x\to-\infty$.

The non-logarithmic case: $b - a \notin \mathbb{Z}$

$a = 1/2,\ b = b' = 1,\ c = c' = 1.5,\ y = 0.5$

m	$x = -10$	$x = -100$	$x = -1000$
1	0.004246	0.000123	3.7e−6
2	0.000180	5.3e−7	1.6e−9
3	9.6e−6	2.8e−9	8.8e−13

$a = 5,\ b = b' = 0.5,\ c = c' = 1,\ y = -0.75 + i$

n	$x = -10$	$x = -100$	$x = -1000$
1	0.009286	0.000879	0.000087
2	0.000620	5.1e−6	5.0e−8
3	0.000116	1.0e−7	9.6e−11

The logarithmic case I: $b - a \in \mathbb{N}$

$a = 1,\ b = b' = 2,\ c = c' = 2.5,\ y = 0.5$

n	$x = -10$	$x = -100$	$x = -1000$
1	0.003193	0.000051	7.2e−7
2	0.000142	2.3e−7	3.3e−10
3	7.5e−6	1.2e−9	1.9e−13

$a = 0.5,\ b = b' = 2.5,\ c = c' = 3.75,\ y = -1.5$

n	$x = -10$	$x = -100$	$x = -1000$
1	0.013396	0.000016	1.9e−8
2	0.000773	1.0e−7	1.3e−11
3	0.000081	6.4e−10	1.4e−14

1. Known results
2. The second Appell's function F_2 for one large variable
3. The third Appell's function F_3 for one large variable
4. Future work

Starting point

$$F_3(a,a',b,b',c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \times \int_0^1 du \int_0^{1-u} \frac{u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}}{(1-ux)^a(1-vy)^{a'}} dv,$$

where $\Re(c-b-b') > \Re b > 0$, $\Re b' > 0$, $x, y \in \mathbb{C}$

Objective: asymptotic expansions of $F_3(a,a',b,b',c;x,y)$ for **large negative values** of **y** and fixed a, a', b, b', c, x

- After the change of variables $\widetilde{y} := -1/y$, $u \to \widetilde{y}u$ and $\widetilde{y} \to 0^+$

$$F_3(a, a', b, b', c; x, -1/\widetilde{y}) = \frac{\widetilde{y}^{c-b}}{(1-x)^a} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^{1/\widetilde{y}} \frac{(1-\widetilde{y}u)^{b-1}}{\left(1+\frac{x\widetilde{y}}{1-x}u\right)^a} u^{c-b-1} {}_2F_1(a', b', c-b; -u)du,$$

with $\Re(c-b-b') > \Re b > 0$, $\Re b' > 0$ and $x \notin [1, \infty)$ if $\Re(a) \geq 1$

$$F_3(a,a',b,b',c;x,-1/\widetilde{y}) = \frac{\widetilde{y}^{c-b}}{(1-x)^a}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty h_x(\widetilde{y}u)f(u)du$$

with $\widetilde{y} \to 0^+$,

$$h_x(u) = \frac{(1-u)^{b-1}}{\left(1+\frac{x}{1-x}u\right)^a}\chi_{[0,1)}(u)$$

and

$$f(u) = u^{c-b-1}{}_2F_1(a',b',c-b;-u)$$

Replace ${}_2F_1$ [Abramowitz, 5.3.7+15.1.1] and, for $\Re(c-b-a')>0$ and $a'\neq b'$, we have $f(u)=C_1 f_1(u)+C_2 f_2(u)$:

$$\int_0^\infty h_x(\widetilde{y}u)f(u)du = C_1\int_0^\infty h_x(\widetilde{y}u)f_1(u)du + C_2\int_0^\infty h_x(\widetilde{y}u)f_2(u)du,$$

$$f_1(u)=u^{c-b-1-a'}\,{}_2F_1(a',1-c+b+a',1-b'+a';-1/u)$$

$$f_2(u)=u^{c-b-1-b'}\,{}_2F_1(b',1-c+b+b',1-a'+b';-1/u)$$

and

$$C_1=\frac{\Gamma(c-b)\Gamma(b'-a')}{\Gamma(b')\Gamma(c-b-a')},\quad C_2=\frac{\Gamma(c-b)\Gamma(a'-b')}{\Gamma(a')\Gamma(c-b-b')}$$

1. Functions $f_1(u)$, $f_2(u)$, $h_x(u)$ are locally integrable on $[0,\infty)$
2. $f_1(t), f_2(t)$
   - $t=\infty$: $f_1(t)=\sum_{k=0}^{n-1}\frac{A_k^1}{t^{k+1-c+b+a'}}+f_{1n}(t)$,
     $A_k^1 := \frac{(-1)^k(a')_k(1-c+b+a')_k}{k!(1-b'+a')_k}$
   - $t=\infty$: $f_2(t)=\sum_{k=0}^{n-1}\frac{A_k^2}{t^{k+1-c+b+b'}}+f_{2n}(t)$,
     $A_k^2 := \frac{(-1)^k(b')_k(1-c+b+b')_k}{k!(1-a'+b')_k}$
3. $h_x(t)$
   - $t=0^+$: $h_x(t)=\sum_{k=0}^{m-1}B_k t^k+h_{xm}(t)$,
     $B_k := \frac{(-1)^k(a)_k}{k!}\left(\frac{x}{1-x}\right)^k {}_2F_1\left(1-b,-k,1-a-k;\frac{x-1}{x}\right)$

Asymptotic expansions of F_3

	$-c+b+a' \notin \mathbb{Z}$	$-c+b+a' \in \mathbb{Z}$
$-c+b+b' \notin \mathbb{Z}$	I	II
$-c+b+b' \in \mathbb{Z}$	III	IV

Case I

For $\Re(c-b-b') > \Re b > 0$, $\Re b' > 0$, $\Re(c-b-a') > 0$, $a' \neq b'$ and $x \notin [1,\infty)$ if $\Re(a) \geq 1$

When $\alpha_1 = -c+b+a' \notin \mathbb{Z}$ and $\alpha_2 = -c+b+b' \notin \mathbb{Z}$:

$$F_3(a,a',b,b',c;x,y) = \frac{\Gamma(c)\pi}{(1-x)^a\Gamma(a')\Gamma(b')\sin(\pi(a'-b'))}\left\{\sum_{k=0}^{n_1-1}\frac{\widehat{A}_k^1}{y^{k+a'}} - \sum_{k=0}^{n_2-1}\frac{\widehat{A}_k^2}{y^{k+b'}} + \frac{(-1)^{c-b}}{\pi\Gamma(b)}\sum_{k=0}^{m-1}\frac{\widehat{B}_k^1-\widehat{B}_k^2}{y^{k+c-b}}\right\} + R_m(y)$$

where for $i, j = 1, 2$ and $j \neq i$

$$\widehat{A}_k^i = \frac{(-1)^{k+1+\alpha_i+c-b}\Gamma(k+\alpha_i+c-b)}{k!\Gamma(-k-\alpha_i+b)\Gamma(k+1+\alpha_i-\alpha_j)} {}_2F_1\left(a, -k-\alpha_i, -k-\alpha_i+b; \frac{x}{x-1}\right),$$

$$\widehat{B}_k^i = (a)_k\Gamma(c-b+k)\left(\frac{x}{1-x}\right)^k {}_2F_1\left(1-b, -k, 1-a-k; \frac{x-1}{x}\right)$$
$$\times \frac{\sin(\pi\alpha_i)\Gamma(\alpha_i-k)}{\Gamma(k+1-\alpha_j)},$$

and $R_m(x) = \mathcal{O}(y^{-m-d-\lfloor c-b-d \rfloor})$, $d = \max\{a', b'\}$ when $y \to -\infty$.

Case II

For $\Re(c-b-b') > \Re b > 0$, $\Re b' > 0$, $\Re(c-b-a') > 0$, $a' \neq b'$ and $x \notin [1,\infty)$ if $\Re(a) \geq 1$

When $\alpha_1 = -c+b+a' \in \mathbb{Z}^-$ and $\alpha_2 = -c+b+b' \notin \mathbb{Z}$:

$$F_3(a,a',b,b',c;x,y) = \frac{\Gamma(c)\pi}{(1-x)^a\Gamma(a')\Gamma(b')\sin(\pi(a'-b'))}$$
$$\times\left\{\sum_{k=0}^{c-b-a'-1}\frac{\widehat{A}_k^1}{y^{k+a'}} - \sum_{k=0}^{n_2-1}\frac{\widehat{A}_k^2}{y^{k+b'}}\frac{(-1)^{c-b}}{\pi\Gamma(b)} - \sum_{k=0}^{m-1}\frac{\widehat{B}_k^2}{y^{k+c-b}}\right\} + R_m(y)$$

and $R_m(x) = \mathcal{O}(y^{-m-b'-\lfloor c-b-b'\rfloor})$ when $y\to-\infty$.

Case III

For $\Re(c-b-b')>\Re b>0$, $\Re b'>0$, $\Re(c-b-a')>0$, $a'\neq b'$ and $x\notin[1,\infty)$ if $\Re(a)\geq 1$

When $\alpha_1=-c+b+a'\notin\mathbb{Z}$ and $\alpha_2=-c+b+b'\in\mathbb{Z}^-$:

$$F_3(a,a',b,b',c;x,y)=\frac{\Gamma(c)\pi}{(1-x)^a\Gamma(a')\Gamma(b')\sin(\pi(a'-b'))}$$
$$\times\left\{\sum_{k=0}^{n_1-1}\frac{\widehat{A}_k^1}{y^{k+a'}}-\sum_{k=0}^{c-b-b'-1}\frac{\widehat{A}_k^2}{y^{k+b'}}+\frac{(-1)^{c-b}}{\pi\Gamma(b)}\sum_{k=0}^{m-1}\frac{\widehat{B}_k^1}{y^{k+c-b}}\right\}+R_m(y),$$

and $R_m(x)=\mathcal{O}(y^{-m-a'-\lfloor c-b-a'\rfloor})$ when $y\to-\infty$.

Case I: $-c+b+a', -c+b+b' \notin \mathbb{Z}$

$a=0.75,\ a'=1.2,\ b=1,\ b'=0.3\ ,\ c=2.5,\ x=0.1$

m	$y=-10$	$y=-50$	$y=-100$
1	0.00569	2.9e-4	8.2e-5
2	9.6-5	1.e-6	1.4e-7
3	5.e-6	1.e-8	7.6e-10

$a=2.5,\ a'=0.2,\ b=1.5,\ b'=0.3\ ,\ c=3.5,\ x=0.75+i$

m	$y=-10$	$y=-100$	$y=-1000$
1	0.001629	1.8e-5	1.9e-7
2	0.00012	1.4e-7	1.45e-10
3	1.e-5	1.2e-9	1.2e-13

Case II: $-c+b+a' \in \mathbb{Z}^-, -c+b+b' \notin \mathbb{Z}$

$a=1.2,\ a'=3,\ b=1,\ b'=1.4\ ,\ c=5,\ x=-1.5$

m	$y=-10$	$y=-50$	$y=-100$
1	0.00422	2.2e-5	2.5e-6
2	0.00039	4.5e-7	2.9e-8
3	2.8e-5	7.3e-9	2.4e-9

$a=0.7-2i,\ a'=1,\ b=1.2,\ b'=1.4+i\ ,\ c=4.2,\ x=-0.5$

m	$y=-10$	$y=-50$	$y=-100$
1	0.0023	4.4e-5	8.6e-6
2	0.0001	3.7e-7	3.6e-8
3	5.6e-6	3.7e-9	1.7e-10

Case III: $-c+b+a' \notin \mathbb{Z}, -c+b+b' \in \mathbb{Z}^-$

$a = 2.3,\ a' = 1+i,\ b = 1.2,\ b' = 1\ ,\ c = 4.2,\ x = -2.5$

m	$y=-10$	$y=-50$	$y=-100$
1	0.00037	1.8e-6	2.e-7
2	2.e-5	2.2e-8	1.2e-9
3	1.2e-6	2.8e-10	6.e-11

$a = 0.5+2i,\ a' = 1.5,\ b = 1,\ b' = 2\ ,\ c = 5,\ x = 0.5-i$

m	$y=-10$	$y=-50$	$y=-100$
1	0.06	0.00027	2.9e-5
2	0.0043	4.e-6	2.e-7
3	0.0003	5.6e-8	1.7e-9

1. Known results
2. The second Appell's function F_2 for one large variable
3. The third Appell's function F_3 for one large variable
4. Future work

Starting point

$$F_4(a,b,c,c';\tilde{x}(1-y),y(1-\tilde{x})) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)}$$
$$\times \int_0^1 du \int_0^1 dv \frac{u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}}{(1-u\tilde{x})^{c+c'-a-1}(1-vy)^{c+c'-b-1}(1-u\tilde{x}-vy)^{a+b-c-c'+1}}$$

where $\Re c > \Re a > 0$, $\Re c' > \Re b > 0$.

Objective: asymptotic expansions of $F_4(a,b,c,c';\tilde{x}(1-y),y(1-\tilde{x}))$ for **large values** of $\widetilde{x}$)

- After the change of variables $x := -1/\tilde{x}$ and $u \to xu$

$$F_4(a,b,c,c';\tilde{x}(1-y),y(1-\tilde{x})) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)}x^a$$

$$\times \int_0^{1/x} du \int_0^1 dv \frac{u^{a-1}v^{b-1}(1-xu)^{c-a-1}(1-v)^{c'-b-1}}{(1+u)^{c+c'-a-1}(1-vy)^{c+c'-b-1}(1+u-vy)^{a+b-c-c'+1}}$$

with $\Re c > \Re a > 0$, $\Re c' > \Re b > 0$

$$F_4(a,b,c,c';\tilde{x}(1-y),y(1-\tilde{x})) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)} x^a$$
$$\int_0^1 \frac{v^{b-1}(1-v)^{c'-b-1}}{(1-vy)^{c+c'-b-1}} dv \int_0^\infty h(xu) f_v(u) du$$

with

$$h(u) = (1-u)^{c-a-1} \chi_{[0,1)}(u)$$

and

$$f_v(u) = \frac{u^{a-1}}{(1+u)^{c+c'-a-1}(1+u-vy)^{a+b-c-c'+1}}$$