Global Asymptotics of the Charlier Polynomials via the Riemann-Hilbert Method

Chun-Hua Ou

Department of Mathematics and Statistics

Memorial University of Newfoundland

Joint work with Prof. R. Wong

Outline

- Background and previous work on Charlier polynomials: open question.
- Solution to the open question: Construction of RHP for Charlier;
- Reduce the problem from discrete RHP to continuous one;
- Deift's method is applied to simplify the RHP.
- MRS number and the potential function g.
- Main results and the estimate of zeros.
- Comparison with past ones.

Charlier polynomials

- the Charlier polynomials $C_n^{(a)}(x)$ is defined by

$$C_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k!(-a)^{n-k}, \qquad a > 0.$$

- Discrete orthogonality

$$\sum_{j=0}^{\infty} C_n^{(a)}(j) C_m^{(a)}(j) \frac{a^j}{j!} = e^a a^n n! \delta_{m,n} \tag{1}$$

with nodes at integers $j = 0, 1, 2, \cdots$.

- Three-term recurrence relation. Indeed, we have

$$C_{n+1}^{(a)}(x) = (x - n - a) C_n^{(a)}(x) - an C_{n-1}^{(a)}(x); \tag{2}$$

Asymptotica when n large

- Difficuly in deriving asymptotics: they do not satisfy a second-order linear differential equation in the independent variable x.
- Goh [1998] studied the asymptotics of $C_n^{(a)}(x)$ by dividing the positive $x-$axis into seven regions:

(1) $\{x : x = \beta n, 1 + \varepsilon \le \beta \le M\}$,

(2) $\{x : x = n + a + \alpha n^{1/2},\ 2a^{1/2} + \varepsilon \le \alpha \le M\}$,

(3) $\{x : x = n + a + 2\sqrt{na} + tn^{1/6}, t$ bounded$\}$,

(4) $\{x : x = n + a + \alpha n^{1/2},\ -2a^{1/2} + \varepsilon \le \alpha \le 2a^{1/2} - \varepsilon\}$,

(5) $\{x : x = n + a - 2a^{1/2}n^{1/2} + tn^{1/6}, t$ bounded$\}$,

(6) $\{x : x = n + a + \alpha n^{1/2},\ -M \le \alpha \le -2a^{1/2} - \varepsilon\}$,

(7) $\{x : x = \beta n, \varepsilon \le \beta \le 1 - \varepsilon\}$,

Goh's result

- In particular, for regions (3) and (5), he gave, respectively,

$$\frac{C_n^{(a)}(x)}{n!} = e^{3a/2}(an)^{-1/6}\left(\frac{n}{a}\right)^{(x-n)/2}\left(\mathrm{Ai}(ta^{-1/6}) + O(n^{-1/21})\right) \tag{3}$$

with $x = n + a + 2\sqrt{na} + tn^{1/6}$, and

$$\begin{aligned}\frac{C_n^{(a)}(x)}{n!} &= e^{3a/2}(an)^{-1/6}\left(\frac{n}{a}\right)^{(x-n)/2}(-1)^n \\ &\times \left\{\mathrm{Re}\left(e^{x\pi i+\pi i/6}\mathrm{H}_i(ta^{-1/6}e^{-\pi i/3})\right)\right. \\ &\left. - \sin \pi x \, \mathrm{H}_i(-ta^{-1/6}) + O(n^{-1/21})\right\}\end{aligned} \tag{4}$$

for $x = n + a - 2a^{1/2}n^{1/2} + tn^{1/6}$, where

$$\mathrm{H}_i(u) = \frac{1}{\pi}\int_0^\infty e^{uq-q^3/3}dq.$$

- There are certain portions of the positive x-axis that are not being

Bo and Wong's result

- Bo and Wong [1994]:for $0 < \varepsilon \le \beta \le M < \infty$,

$$\frac{C_n^{(a)}(n\beta)}{n!} \sim e^{a+nq} E^{-n(\beta-1)/2} \left\{ a_0 J_{n(\beta-1)}(2n\sqrt{E} + b_0\sqrt{E} J'_{n(\beta-1)}(2n\sqrt{E}) \right\}, \tag{5}$$

where $a_0 \sim \frac{1}{2}(\sqrt{\beta}+1),\ b_0 \sim 1/(\sqrt{\beta}+1)$,

$$q = (1-\beta)+\beta\log\beta + \left[\frac{\beta}{1-\beta} - \frac{1}{(1-\beta)e}\beta^{\beta/(\beta-1)}\right]\left(\frac{a}{n}\right) + o\left(\frac{1}{n}\right) \qquad \text{if } \beta \neq 1,$$

and

$$q(\beta, n) = -\frac{a}{2n} \qquad \text{if } \beta = 1.$$

Furthermore,

$$E = \frac{a}{e}\beta^{\beta/(\beta-1)}\frac{1}{n}\left\{1 - \left[\frac{\beta+1}{(1-\beta)^2} - \frac{2}{e(1-\beta)^2}\beta^{\beta/(\beta-1)}\right]\left(\frac{a}{n}\right) + o\left(\frac{1}{n}\right)\right\} \quad \text{if } \beta \neq 1,$$

$$E = \frac{a}{n} + o\left(\frac{a}{n}\right) \qquad \text{if } \beta = 1.$$

Dunster Mark's result

- In 2001, Dunster observed that $C_n^{(a)}(x)$ satisfies a second-order linear differential equation (in the parameter a)

$$a\frac{d^2y}{da^2} + (1 + x - n - a)\frac{dy}{da} + ny = 0.$$

- By using known results for differential equations (a) without turning points, and (b)for intervals containing a double pole and a coalescing turning point
- six different subcases depending on the real parameters x and a.
- His results are restricted to $x \in (0, \infty)$.
- Open question: can we obtain the asymptotics of the Charlier polynomials in the whole complex plane?
- Solution: RHP approach.

Formulation of Riemann-Hilbert problems

- we first set

$$x_j = \left(j + \frac{1}{6}\right), \qquad j = 0, 1, 2...$$

and

$$\pi_n(z) = C_n^{(a)}\left(z - \frac{1}{6}\right). \tag{6}$$

Clearly, $\pi_n(z)$ is a monic polynomial and satisfies

$$\sum_{j=0}^{\infty} \pi_n(x_j) x_j^m w(x_j) = 0, \qquad m < n,$$

where the weight function is given by

$$w(x_j) = \frac{a^{x_j - \frac{1}{6}}}{(x_j - \frac{1}{6})!} = \frac{a^{x_j - \frac{1}{6}}}{\Gamma(x_j + \frac{5}{6})}.$$

Riemann-Hilbert Problem for Y

- a 2×2 matrix-value function Y with the properties:

(Y$_a$) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \{x_j : j = 0, 1, 2, ...,\}$;

(Y$_b$) at each node x_j, $Y(z)$ satisfies

$$\operatorname*{Res}_{z=x_j} Y(z) = \lim_{z \to x_j} Y(z) \begin{pmatrix} 0 & \dfrac{w(x_j)}{2i} \\ 0 & 0 \end{pmatrix};$$

(Y$_c$) as $z \to \infty$ (not near the node points)

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix};$$

Y

- **Lemma 1.** *The unique solution to the Riemann-Hilbert problem for Y is given by*

$$Y(z) = \begin{pmatrix} \pi_n(z) & \dfrac{1}{2i}\displaystyle\sum_{k=0}^{\infty} \dfrac{w(x_k)\pi_n(x_k)}{z - x_k} \\ \gamma_{n-1}\pi_{n-1}(z) & \dfrac{\gamma_{n-1}}{2i}\displaystyle\sum_{k=0}^{\infty} \dfrac{\pi_{n-1}(x_k)w(x_k)}{z - x_k} \end{pmatrix},$$

where $\gamma_n = 2i/e^a a^n n!$.

- Remove Saturated Regions: let $k_n \neq n$ be a positive integer in $[\alpha_n, \beta_n]$, where α_n and β_n are the Mhaskar-Rakhmanov-Saff (MRS) numbers and will be determined later.

- Set

$$H(z) := Y(z) \begin{pmatrix} \prod_{j=0}^{k_n-1}(z - x_j)^{-1} & 0 \\ 0 & \prod_{j=0}^{k_n-1}(z - x_j) \end{pmatrix}. \tag{7}$$

H

- (H_a) $H(z)$ is analytic for $z \in \mathbb{C} \setminus \{x_j : j = 0, 1, 2, ..., \}$;

  (H_b) the jump at the pole x_k is given by

$$\operatorname*{Res}_{z=x_k} H(z) = \lim_{z\to x_k} H(z) \begin{pmatrix} 0 & \dfrac{w(x_k)}{2i} \displaystyle\prod_{j=0}^{k_n-1} (x_k - x_j)^2 \\ 0 & 0 \end{pmatrix}$$

  for $k = k_n, k_n + 1, \cdots$, and

$$\operatorname*{Res}_{z=x_k} H(z) = \lim_{z\to x_k} H(z) \begin{pmatrix} 0 & 0 \\ \dfrac{2i}{w(x_k)} \displaystyle\prod_{j=0, j\neq k}^{k_n-1} (x_k - x_j)^{-2} & 0 \end{pmatrix}$$

  for $k = 0, \cdots, k_n - 1$;

  (H_c) as $z \to \infty$(not near the nodes),

$$H(z) = \left(I + O\left(\tfrac{1}{z}\right)\right) \begin{pmatrix} z^{n-k_n} & 0 \\ 0 & z^{-n+k_n} \end{pmatrix}$$

Removes the poles of H

- Let

$$R(z) := H(z)\begin{pmatrix} 1 & 0 \\ \pm\dfrac{2e^{\pm i\pi(z+\frac{1}{3})}}{\Gamma(\frac{1}{6}-z)\Gamma(\frac{5}{6}+z)w(z)\prod_{j=0}^{k_n-1}(z-x_j)^2} & 1 \end{pmatrix} \tag{8}$$

for $z \in \Omega_{\pm}^{\Delta}$,

$$R(z) := H(z)\begin{pmatrix} 1 & \mp\dfrac{1}{2}\, e^{\pm i\pi(z+\frac{1}{3})}\,\Gamma(\dfrac{1}{6}-z)\Gamma(\dfrac{5}{6}+z)w(z)\displaystyle\prod_{j=0}^{k_n-1}(z-x_j)^2 \\ 0 & 1 \end{pmatrix} \tag{9}$$

for $z \in \Omega_{\pm}^{\nabla}$, and

$$R(z) := H(z) \tag{10}$$

for all other $z \in \mathbb{C} \setminus (\Omega_{\pm}^{\Delta} \cup \Omega_{\pm}^{\nabla} \cup \Sigma)$, where $\Sigma = (0,\infty) \cup \Sigma_{\pm}^{\Delta} \cup \Sigma_{\pm}^{\nabla}$. The domains $\Omega_{\pm}^{\Delta}$, $\Omega_{\pm}^{\nabla}$ and the contour Σ are depicted in Figure 1.

Figure

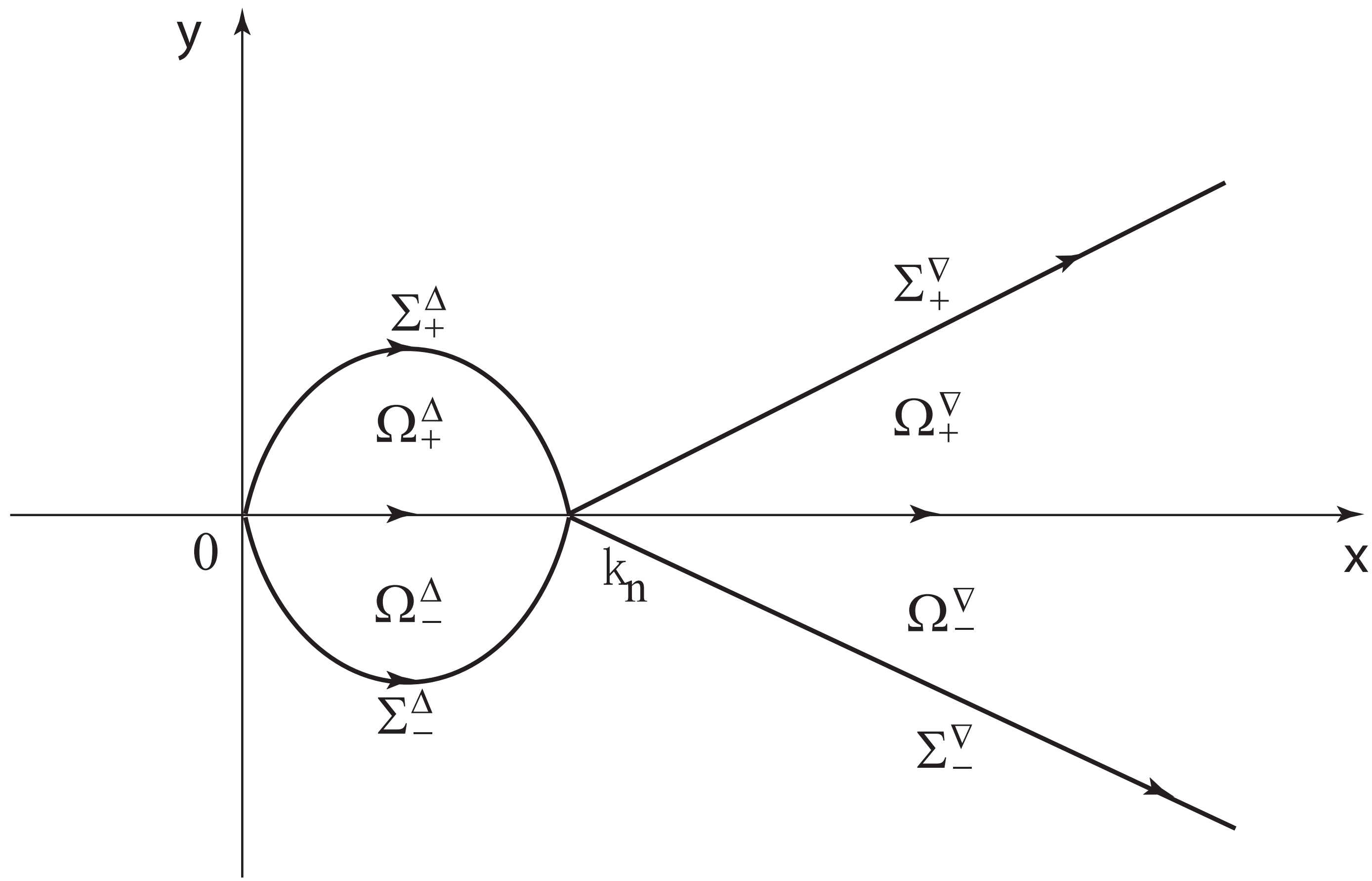

Figure 1: The domains $\Omega_\pm^\Delta$, $\Omega_\pm^\nabla$ and the contour Σ. The contour Σ is dependent on n we may choose Σ so that z/k_n is indendent of n for $z \in \Sigma$.

RHP for R

(R_a) $R(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$;

(R_b) $R(z)$ satisfies the following jump conditions on the curve Σ: for $x \in (0, k_n)$,

$$R_+(x) = R_-(x) \begin{pmatrix} 1 & 0 \\ -r_{1,n}(x) & 1 \end{pmatrix},$$

where

$$r_{1,n}(x) = \frac{-4\cos(\pi(x+\frac{1}{3}))}{\Gamma(\frac{1}{6}-x)\Gamma(\frac{5}{6}+x)w(x)\prod_{j=0}^{k_n-1}(x-x_j)^2};$$

for $x \in (k_n, \infty)$,

$$R_+(x) = R_-(x) \begin{pmatrix} 1 & r_{2,n}(x) \\ 0 & 1 \end{pmatrix},$$

where

$$r_{2,n}(x) = -\cos(\pi(x+\frac{1}{3}))\,\Gamma\left(\frac{1}{6}-x\right)\Gamma\left(\frac{5}{6}+x\right)w(x)\prod_{j=0}^{k_n-1}(x-x_j)^2; \qquad (11)$$

for $z \in \Sigma_\pm^\Delta$,

g function and the function ϕ.

Definition 1. *The logarithmic potential g-functions are defined by*

$$\tilde{g}_n(z) := \int_{\alpha_n}^{\beta_n} \log(z-s)\mu_n(s)ds, \qquad z \in \mathbb{C} \setminus [\alpha_n, \infty) \tag{12}$$

and

$$g_n(z) := \int_{\alpha_n}^{\beta_n} \log(z-s)\mu_n(s)ds, \qquad z \in \mathbb{C} \setminus (-\infty, \beta_n]. \tag{13}$$

Definition 2. *The so-called ϕ-functions are defined by*

$$\tilde{\phi}_n(z) := -\int_{\alpha_n}^{z} \tilde{v}_n(s)ds \tag{14}$$

for $z \in \mathbb{C} \setminus ((-\infty, 0] \cup [\alpha_n, \infty))$, and

$$\phi_n(z) := \int_{\beta_n}^{z} v_n(s)ds \tag{15}$$

for $z \in \mathbb{C} \setminus (-\infty, \beta_n]$.

Here, $\tilde{v}_n(z)$ and $v_n(z)$ are the analytic continuations of the density function $\mu_n(x)$ (up to a constant multiple) and satisfy

$$\frac{n}{n-k_n}\tilde{v}_{n,\pm}(x) = \pm\pi i\mu_n(x) \qquad \text{for } x \in (\alpha_n, k_n) \tag{16}$$

and

$$\frac{n}{n-k_n}v_{n,\pm}(x) = \pm\pi i\mu_n(x) \qquad \text{for } x \in (k_n, \beta_n). \tag{17}$$

$$-(n-k_n)\big[(\tilde{g}_n)_+(x)+(\tilde{g}_n)_-(x)\big]+\log r_{1,n}(x)=\text{constant} \tag{18}$$

for $x\in(\alpha_n,k_n)$, and

$$(n-k_n)\big[(g_n)_+(x)+(g_n)_-(x)\big]+\log r_{2,n}(x)=\text{constant} \tag{19}$$

for $x\in(k_n,\beta_n)$.

Taking derivatives on both sides of (18) and (19) gives

$$(\tilde{g}'_{n,+}(x)+\tilde{g}'_{n,-}(x))=\frac{1}{n-k_n}\frac{r'_{1,n}(x)}{r_{1,n}(x)} \qquad \text{for } x\in(\alpha_n,k_n), \tag{20}$$

and

$$(g'_{n,+}(x)+g'_{n,-}(x))=\frac{-1}{n-k_n}\frac{r'_{2,n}(x)}{r_{2,n}(x)} \qquad \text{for } x\in(k_n,\beta_n). \tag{21}$$

- Define a new function

$$G_n(z) = \frac{1}{\pi i}\int_{\alpha_n}^{\beta_n} \frac{\mu_n(s)}{s-z}ds = \frac{i}{\pi} g_n'(z) \qquad \text{for } z \in \mathbb{C}\backslash[\alpha_n, \beta_n]. \tag{22}$$

Then we have $\mathrm{Re}\,(G_n)_\pm(x) = \pm\mu_n(x)$ for for $x \in (\alpha_n, \beta_n)$.
Define

$$h_n(x) = \begin{cases} h_{1,n}(x) = \dfrac{1}{n-k_n}\dfrac{r_{1,n}'(x)}{r_{1,n}(x)}, & x \in (\alpha_n, k_n), \\ \\ h_{2,n}(x) = \dfrac{-1}{n-k_n}\dfrac{r_{2,n}'(x)}{r_{2,n}(x)}, & x \in (k_n, \beta_n). \end{cases} \tag{23}$$

- α_n and β_n are determined by

$$\int_{\alpha_n}^{\beta_n} \frac{h_n(s)}{\sqrt{(s-\alpha_n)(\beta_n - s)}}\, ds = 0 \tag{24}$$

and

$$\frac{1}{2\pi}\int_{\alpha_n}^{\beta_n} \frac{s h_n(s)}{\sqrt{(s-\alpha_n)(\beta_n - s)}}\, ds = 1, \tag{25}$$

-

$$\begin{cases} \alpha_n = n - 2\sqrt{an} + a + \frac{1}{3} - \dfrac{4\sqrt{a}\log 2 + c_n \log 2}{\pi\sqrt{(c_n + 2\sqrt{a})(2\sqrt{a} - c_n)}} + o(1) \\ \beta_n = n + 2\sqrt{an} + a + \frac{1}{3} + \dfrac{4\sqrt{a}\log 2 - \log 2\, c_n}{\pi\sqrt{(c_n + 2\sqrt{a})(2\sqrt{a} - c_n)}} + o(1). \end{cases} \tag{26}$$

- $$\mu_{0,n}(x) = \frac{1}{n\pi}\left(\arctan\sqrt{\frac{\alpha_n(\beta_n - x)}{\beta_n(x-\alpha_n)}} + \arctan\sqrt{\frac{\beta_n - x}{x-\alpha_n}}\right), \quad x \in [\alpha_n, \beta_n]. \tag{27}$$

- $$\begin{aligned} (n-k_n)l_n &\sim -a\log(na) + \left(\pi i - 2n - 2a + 4\sqrt{na} + \log\sqrt{\frac{\pi}{2a}}\right) \\ &\quad +(1+\log a)(n+a) + (n+a)\log n + 2a - 4\sqrt{na} \qquad (28) \\ &= \pi i + \log\sqrt{\frac{\pi}{2a}} - n + a + n\log a + n\log n. \end{aligned}$$

- $$\begin{aligned} (n-k_n)\tilde{l}_n &= (n-k_n)\left(l_n + 2\pi i\right) \\ &= (n-k_n)l_n + 2(n-k_n)\pi i. \end{aligned}$$

$$(n-k_n)\tilde{l}_n \sim 2(n-k_n)\pi i + \pi i + \log\sqrt{\frac{\pi}{2a}} - n + a + n\log a + n\log n. \tag{29}$$

Construction of the Parametrix

- $$Q(z) := 2\sqrt{\pi}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}^{-1} \left(\frac{f(z)^{\frac{1}{4}}}{a_n(z)}\right)^{\sigma_3} \begin{pmatrix} \mathrm{Ai}(f(z)) & -\omega^2\,\mathrm{Ai}(\omega^2 f(z)) \\ \mathrm{Ai}'(f(z)) & -\omega\,\mathrm{Ai}'(\omega^2 f(z)) \end{pmatrix} \text{ for } z \in \mathrm{II},$$
$$Q(z) := 2\sqrt{\pi}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}^{-1} \left(\frac{f(z)^{\frac{1}{4}}}{a_n(z)}\right)^{\sigma_3} \begin{pmatrix} \mathrm{Ai}(f(z)) & \omega\,\mathrm{Ai}(\omega f(z)) \\ \mathrm{Ai}'(f(z)) & \omega^2\,\mathrm{Ai}'(\omega f(z)) \end{pmatrix} \text{ for } z \in \mathrm{IV},$$
$$Q(z) := 2\sqrt{\pi}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}^{-1} \left(\tilde{f}(z)^{\frac{1}{4}} a_n(z)\right)^{\sigma_3} \begin{pmatrix} \omega\,\mathrm{Ai}(\omega\tilde{f}(z)) & \mathrm{Ai}(\tilde{f}(z)) \\ \omega^2\,\mathrm{Ai}'(\omega\tilde{f}(z)) & \mathrm{Ai}'(\tilde{f}(z)) \end{pmatrix},\ z \in \mathrm{I},$$
$$Q(z) := 2\sqrt{\pi}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}^{-1} \left(\tilde{f}(z)^{\frac{1}{4}} a_n(z)\right)^{\sigma_3} \begin{pmatrix} -\omega^2\,\mathrm{Ai}(\omega^2\tilde{f}(z)) & \mathrm{Ai}(\tilde{f}(z)) \\ -\omega\,\mathrm{Ai}'(\omega^2\tilde{f}(z)) & \mathrm{Ai}'(\tilde{f}(z)) \end{pmatrix}, \text{ for}$$
$z \in \mathrm{III}$, where $f(z)$ and $\tilde{f}(z)$ are defined by
$f(z) := \left[\frac{3}{2}n\phi_n(z)\right]^{2/3}$, $\quad \tilde{f}(z) := \left[\frac{3}{2}n\tilde{\phi}_n(z)\right]^{2/3}$, and $a_n(z)$ is given by
$a_n(z) := \frac{(z-\beta_n)^{1/4}}{(z-\alpha_n)^{1/4}}$ which is analytic in $\mathbb{C} \setminus [\alpha_n, \beta_n]$. The function $a_n(z)$ is introduced to ensure that $f^{\frac{1}{4}}(z)/a_n(z)$ is analytic for $z \geq k_n$, with no jumps on Γ.

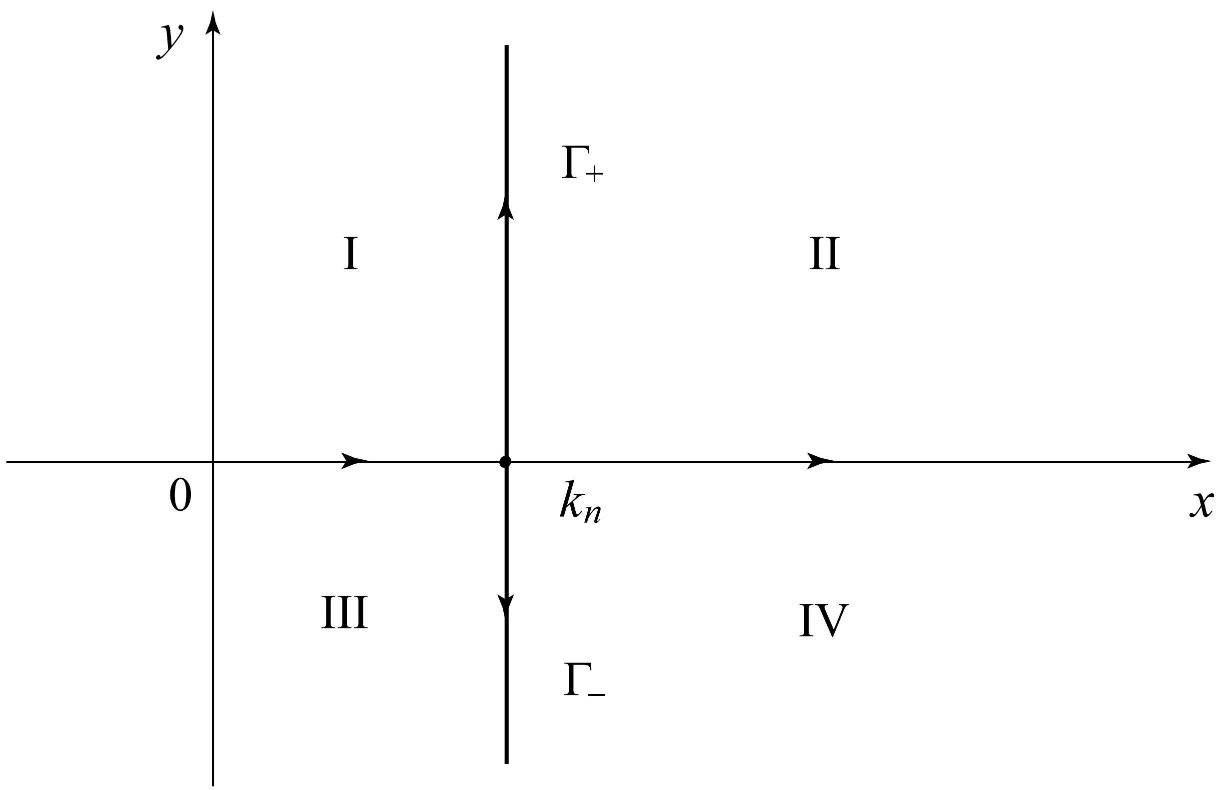

Figure 2: The domains I, II, III and IV

Construction of the Parametrix

- Approximate solution to the RHP for R is

$$\tilde{R}(z) := \begin{cases} e^{\frac{1}{2}(n-k_n)l_n\sigma_3} Q(z) r_{2,n}(z)^{-\frac{1}{2}\sigma_3}, & z \in \mathrm{II} \cup \mathrm{IV}, \\ e^{\frac{1}{2}(n-k_n)\tilde{l}_n\sigma_3} Q(z) r_{1,n}(z)^{\frac{1}{2}\sigma_3}, & z \in \mathrm{I} \cup \mathrm{III}. \end{cases} \tag{30}$$

Indeed, one can verify that

$$\begin{aligned} \tilde{R}_+(x) &= \tilde{R}_-(x) \begin{pmatrix} 1 & 0 \\ -r_{1,n}(x) & 1 \end{pmatrix}, \qquad x \in (0, k_n), \\ \tilde{R}_+(x) &= \tilde{R}_-(x) \begin{pmatrix} 1 & r_{2,n}(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in (k_n, \infty), \end{aligned}$$

and that $\tilde{R}(z)$ satisfies the large -z behavior given in (R_c).

Construction of the Parametrix

- $S(z) := e^{-\frac{1}{2}(n-k_n)l_n\sigma_3} R(z)(\tilde{R}(z))^{-1} e^{\frac{1}{2}(n-k_n)l_n\sigma_3}$.

(S_a) $S(z)$ is analytic for $z \in \mathbb{C} \setminus (\Gamma \cup \Sigma)$;

(S_b) for $z \in (-\infty, 0] \cup \Gamma$,

$$S_+(z) = S_-(z) J_S(z),$$

where

$$J_S(z) := e^{-\frac{1}{2}(n-k_n)l_n\sigma_3} \tilde{R}_-(z)(\tilde{R}_+(z))^{-1} e^{\frac{1}{2}(n-k_n)l_n\sigma_3}; \tag{31}$$

for $z \in \Sigma_\pm^\nabla$, $S_+(z) = S_-(z) J_S(z)$, where

$$J_S(z) := e^{-\frac{1}{2}(n-k_n)l_n\sigma_3} \tilde{R}(z) \begin{pmatrix} 1 & r_\pm^\nabla \\ 0 & 1 \end{pmatrix} (\tilde{R}(z))^{-1} e^{\frac{1}{2}(n-k_n)l_n\sigma_3};$$

for $z \in \Sigma_\pm^\Delta$, $S_+(z) = S_-(z) J_S(z)$, where

$$J_S(z) := e^{-\frac{1}{2}(n-k_n)l_n\sigma_3} \tilde{R}(z) \begin{pmatrix} 1 & 0 \\ r_\pm^\Delta & 1 \end{pmatrix} (\tilde{R}(z))^{-1} e^{\frac{1}{2}(n-k_n)l_n\sigma_3};$$

(S_c) for $z \in \mathbb{C} \setminus (-\infty, 0] \cup \Gamma$, $S(z) = I + O\left(\frac{1}{z}\right)$ as $z \to \infty$

main result

- $\pi_n(z) = [-w(z)]^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_n)l_n} \left[\mathrm{Ai}(f(z))\mathbf{A}(z,n) + \mathrm{Ai}'(f(z))\mathbf{B}(z,n) \right]$, for $\mathrm{Re}\, z \geq \alpha_n + \delta\sqrt{n}$, where $\mathbf{A}(z,n)$ and $\mathbf{B}(z,n)$ are given by $\mathbf{A}(z,n) = \frac{f(z)^{\frac{1}{4}}}{a_n(z)}(S_{11}(z) - iS_{12}(z))$, $\qquad \mathbf{B}(z,n) = \frac{a_n(z)}{f(z)^{\frac{1}{4}}}(-S_{11}(z) - iS_{12}(z))$, and $\delta < 2\sqrt{a}$ is a small positive number.
-

$$\pi_n(z) = [-w(z)]^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_n)\tilde{l}_n + k_n \pi i}$$

$$\Big\{ \left[\cos(\pi(z+\frac{1}{3}))\,\mathrm{Bi}(\tilde{f}(z)) + \sin(\pi(z+\frac{1}{3}))\,\mathrm{Ai}(\tilde{f}(z))\right]\tilde{\mathbf{A}}(z,n)$$

$$+ \left[\cos(\pi(z+\frac{1}{3}))\,\mathrm{Bi}'(\tilde{f}(z)) + \sin(\pi(z+\frac{1}{3}))\,\mathrm{Ai}'(\tilde{f}(z))\right]\tilde{\mathbf{B}}(z,n)$$

for $\mathrm{Re}\, z < \beta_n - \delta\sqrt{n}$, $0 \leq |\arg(z)| < \pi$;

main result

- and

$$\pi_n(z) = [-w(z)]^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_n)\tilde{l}_n + k_n \pi i}$$
$$\Big\{ \big[\cos(\pi(z+\frac{1}{3}))\,\mathrm{Bi}(\tilde{f}(z)) + \sin(\pi(z+\frac{1}{3}))\,\mathrm{Ai}(\tilde{f}(z))\big]\,\tilde{\mathbf{A}}(z,n).$$
$$+ \big[\cos(\pi(z+\frac{1}{3}))\,\mathrm{Bi}'(\tilde{f}(z)) + \sin(\pi(z+\frac{1}{3}))\,\mathrm{Ai}'(\tilde{f}(z))\big]\,\tilde{\mathbf{B}}(z,n) \Big\}$$
$$- [-w(z)]^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_n)\tilde{l}_n + k_n \pi i} e^{\pm \pi i (z-\frac{1}{6})} \big[\mathrm{Ai}(\tilde{f}(z))\tilde{\mathbf{A}}(z,n) + \mathrm{Ai}'(\tilde{f}(z))\tilde{\mathbf{B}}(z,n)\big], \tag{32}$$

for $\mathrm{Re}\, z < \beta_n - \delta\sqrt{n},\ 0 < |\arg(z)| \le \pi$. We take positive(negative) sign in $e^{\pm\pi i(z-\frac{1}{6})}$ when z is above(below) the negative real line. The functions $\tilde{\mathbf{A}}(z,n)$ and $\tilde{\mathbf{B}}(z,n)$ are given by $\tilde{\mathbf{A}}(z,n) = \tilde{f}(z)^{\frac{1}{4}} a_n(z)(S_{11}(z) + iS_{12}(z)),\ \ \tilde{\mathbf{B}}(z,n) = \frac{1}{\tilde{f}(z)^{\frac{1}{4}} a_n(z)}(S_{11}(z) - iS_{12}(z))$.
Here in this theorem the matrix $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ tends to the identity matrix as $n \to \infty$ for z in the relevant regions.

Large zeros

Set

$$\bar{c}_n := \frac{\beta_n - \alpha_n}{2}, \ \bar{d}_n := \frac{\beta_n + \alpha_n}{2},$$

and denote the roots of $\pi_n(z)$ by

$$z_{n,n} < \cdots < z_{2,n} < z_{1,n}.$$

We also denote the zeros of the Airy function Ai (τ) by

$$\cdots < -\tau_2 < -\tau_1 < 0.$$

Theorem: For any fixed $k \in \mathbb{N} = \{1, 2, 3, \cdots, \}$, we have

$$\frac{z_{k,n} - \bar{d}_n}{\bar{c}_n} = 1 - \frac{\tau_k}{2(an)^{1/3}} + O(\frac{1}{\sqrt{n}}), \qquad \text{as } n \to \infty.$$

Small Zeros

Theorem: For any fixed $k \in \mathbb{N} = \{1, 2, 3, ..., \}$, we have the estimate of the small zeros of $C_n^{(a)}(z)$ as

$$z_{n-k+1,n} = k - 1 + \text{an exponentially small term}, \qquad \text{as } n \to \infty.$$

Comparison our main results with earlier ones in two regions: (1) $x = n + a + 2\sqrt{na} + tn^{1/6}$, t bounded; (2) $x = n + a - 2\sqrt{na} + tn^{1/6}$:

(1):

$$\frac{\pi_n(x+\frac{1}{6})}{n!} \sim e^{3a/2}\left(\frac{n}{a}\right)^{\frac{1}{2}(x-n)}(an)^{-1/6}\,\mathrm{Ai}(ta^{-1/6})(ta^{-1/6}),$$

(2):

$$\begin{aligned}\frac{C_n^{(a)}(z)}{n!} &= \frac{\pi_n(z+\frac{1}{6})}{n!} \sim e^{3a/2}\left(\frac{n}{a}\right)^{(x-n)/2}(an)^{-1/6}(-1)^n \\ &\quad \times 2\,\mathrm{Re}\left[e^{i\pi z+i\pi/3}\,\mathrm{Ai}(e^{\pi i/3}ta^{-1/6})\right],\end{aligned}$$

Both of them do agree with past results!

Summary

- The polynomials considered in Baik et al.[2007] are orthogonal on a finite number of nodes, but Charlier polynomials are orthogonal on an infinite number of nodes.
- We do not make any scale change in z, since we do not know a priorily what scale will work best.
- To remove all the poles x_k of the matrix Y and to make $R_+(x)$ and $R_-(x)$ continuous on the positive real axis $0 < x < \infty$, we use the Gamma function $\Gamma(\frac{1}{6} - z)\Gamma(\frac{5}{6} + z)$ instead of the finite product $\prod_{k=1}^{N-1}(z - x_k)$.
- The computation of equilibrium measure and the MRS numbers α_n and β_n is also done in a different manner.
- We need only three asymptotic expansions to describe the behavior of $C_n^{(a)}(z)$ for z in the whole complex plane, whereas seven such formulas are used in Baik et al.[2007] for the same purpose; see Theorems 2.7, 2.9, 2.10, 2.11, 2.13, 2.15, 2.16 in the same reference.