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# On the distinguished role of the Mittag-Leffler and Wright functions in fractional calculus

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# 1. Introduction to the Mittag-Leffler functions

We note that the Mittag-Leffler functions are present in the Mathematics Subject Classification since the year 2000 under the number 33E12 under recommendation of Prof. Gorenflo.

A description of the most important properties of the functions (with relevant references up to the fifties) can be found in the third volume of the Bateman Project edited by Erdelyi et al (1955) in the chapter *XVIII* on *Miscellaneous Functions*. The treatises where great attention is devoted to is that by Djrbashian (1966), unfortunately in Russian. We also recommend the classical treatise on complex functions by Sansone & Gerretsen (1960). Nowadays the Mittag-Leffler functions are widely used in the framework of integral and differential equations of fractional order, as shown in the treatises on fractional calculus and its applications, see *e.g.* Samko, Kilbas & Marichev (1987-93), Podlubny (1999), West, Bologna & Grigolini (2003), Kilbas, Srivastava & Trujillo (2006), Magin (2006), Mathai and Haubold (2008), Mainardi (2010), Diethelm (2010).

As pioneering works of mathematical nature in the field of fractional integral and differential equations, we like to quote Hille & Tamarkin (1930) who have provided the solution of the Abel integral equation of the second kind in terms of a Mittag-Leffler function, and Barret (1954) who has expressed the general solution of the linear fractional differential equation with constant coefficients in terms of Mittag-Leffler functions.

As former applications in physics we like to quote the contributions by Cole (1933) in connection with nerve conduction, see also Davis (1936) and by Gross (1947) in connection with mechanical relaxation.

Subsequently, Caputo & Mainardi (1971a), (1971b) have shown that Mittag-Leffler functions are present whenever derivatives of fractional order are introduced in the constitutive equations of a linear viscoelastic body. Since then, several other authors have pointed out the relevance of these functions for fractional viscoelastic models, see *e.g.* Mainardi (1997), (2010).

## 2. The Mittag-Leffler functions: $E_\alpha(z)$ , $E_{\alpha,\beta}(z)$

The Mittag-Leffler functions, that we denote by $E_\alpha(z)$, $E_{\alpha,\beta}(z)$ are so named in honour of Gösta Mittag-Leffler, the eminent Swedish mathematician, who introduced and investigated these functions in a series of notes starting from 1903 in the framework of the theory of entire functions. The functions are defined by the series representations, convergent in the whole complex plane $\mathbb{C}$

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}\,, \quad \mathrm{Re}(\alpha) > 0\,; \tag{2.4}$$

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}\,, \quad \mathrm{Re}(\alpha) > 0\,,\ \beta \in \mathbb{C}\,; \tag{2.5}$$

Originally Mittag-Leffler assumed only the parameter $\alpha$ and assumed it as positive, but soon later the generalization with two complex parameters was considered by Wiman. In both cases the Mittag-Leffler functions are entire of order $1/\mathrm{Re}(\alpha)$. Generally $E_{\alpha,1}(z) = E_\alpha(z)$.

Using their series representations it is easy to recognize

$$\left\{ \begin{array}{ll} E_{1,1}(z) = E_1(z) = \mathrm{e}^z\,, & E_{1,2}(z) = \dfrac{\mathrm{e}^z - 1}{z}\,, \\ E_{2,1}(z^2) = \cosh(z)\,, & E_{2,1}(-z^2) = \cos(z)\,, \\ E_{2,2}(z^2) = \dfrac{\sinh(z)}{z}\,, & E_{2,2}(-z^2) = \dfrac{\sin(z)}{z}\,, \end{array} \right. \tag{2.6}$$

and more generally

$$\left\{ \begin{array}{l} E_{\alpha,\beta}(z) + E_{\alpha,\beta}(-z) = 2\, E_{2\alpha,\beta}(z^2)\,, \\ E_{\alpha,\beta}(z) - E_{\alpha,\beta}(-z) = 2z\, E_{2\alpha,\alpha+\beta}(z^2)\,. \end{array} \right. \tag{2.7}$$

Furthermore, for $\alpha = 1/2$,

$$E_{1/2}(\pm z^{1/2}) = \mathrm{e}^z \left[1 + \mathrm{erf}\,(\pm z^{1/2})\right] = \mathrm{e}^z\, \mathrm{erfc}\,(\mp z^{1/2})\,,$$

where where erf (erfc) denotes the (complementary) error function defined as

$$\mathrm{erf}\,(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2}\, du\,, \quad \mathrm{erfc}\,(z) := 1 - \mathrm{erf}\,(z)\,, \quad z \in \mathbb{C}\,.$$

## Summation and integration formulas

Concerning summation we outline

$$E_\alpha(z) = \frac{1}{p} \sum_{h=0}^{p-1} E_{\alpha/p} \left( z^{1/p} e^{i2\pi h/p} \right) , \quad p \in \mathbb{N} , \qquad (F.38)$$

from which we derive the **duplication formula**, ,

$$E_\alpha(z) = \frac{1}{2} \left[ E_{\alpha/2}(+z^{1/2}) + E_{\alpha/2}(-z^{1/2}) \right] . \qquad (F.39)$$

As an example of this formula we can recover, for $\alpha = 2$ , the expressions of $\cosh z$ and $\cos z$ in terms of two exponential functions.

Concerning integration we outline another interesting **duplication formula** valid for $x > 0$ , $t > 0$ ,

$$E_{\alpha/2}(-t^{\alpha/2}) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \mathrm{e}^{-x^2/(4t)} \, E_\alpha(-x^\alpha) \, dx , \quad . \qquad (F.40)$$

## The Mittag-Leffler functions of rational order

Let us now consider the Mittag-Leffler functions of rational order $\alpha = p/q$ with $p\,,\ q \in \mathbb{N}$ , relatively prime.

$$\left(\frac{d}{dz}\right)^p E_p\left(z^p\right) = E_p\left(z^p\right), \qquad (F.41)$$

$$\frac{d^p}{dz^p} E_{p/q}\left(z^{p/q}\right) = E_{p/q}\left(z^{p/q}\right) + \sum_{k=1}^{q-1} \frac{z^{-k\,p/q}}{\Gamma(1-k\,p/q)}, \qquad (F.42)$$

$$E_{p/q}\left(z\right) = \frac{1}{p}\sum_{h=0}^{p-1} E_{1/q}\left(z^{1/p} e^{i2\pi h/p}\right), \qquad (F.43)$$

$$E_{1/q}\left(z^{1/q}\right) = \mathrm{e}^{z}\left[1 + \sum_{k=1}^{q-1}\frac{\gamma(1-k/q, z)}{\Gamma(1-k/q)}\right], \qquad (F.44)$$

where $\gamma(a, z)$ denotes the **incomplete gamma function**

$$\gamma(a, z) := \int_0^z \mathrm{e}^{-u}\, u^{a\,-\,1}\, du\,. \qquad (F.45)$$

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## Integral representation and asymptotics

Many of the most important properties of $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ follow from their integral representation for $z \in \mathbb{C}$,

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1}\, \mathrm{e}^{\,\zeta}}{\zeta^\alpha - z}\, d\zeta\,, \quad \alpha > 0\,, \qquad (F.7)$$

$$E_{\alpha.\beta}(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-\beta}\, \mathrm{e}^{\,\zeta}}{\zeta^\alpha - z}\, d\zeta\,, \quad \alpha > 0\,,\ \beta \in \mathbb{C}\,,\,, \qquad (F.10)$$

where the path of integration $Ha$ (the *Hankel path*) is a loop which starts and ends at $-\infty$ and encircles the circular disk $|\zeta| \le |z|^{1/\alpha}$ in the positive sense: $-\pi \le \arg\zeta \le \pi$ on $Ha$.

The most interesting properties of the Mittag-Leffler function are associated with its asymptotic developments as $z \to \infty$ in various sectors of the complex plane. For these details we refer (as usual) to the Chapter 18 of Vol. 3 of the Bateman Project and to the most recent contributions including the book by Paris & Kaminski (2001) and the papers by Paris (2002) and by Wong & Zou (2002).

## The auxiliary functions of the Mittag-Leffler type

In view of applications we introduce the following causal functions in time domain

$$e_\alpha(t;\lambda) := E_\alpha\left(-\lambda\, t^\alpha\right) \div \frac{s^{\alpha-1}}{s^\alpha+\lambda}\,, \tag{2.8}$$

$$e_{\alpha,\beta}(t;\lambda) := t^{\beta-1}\, E_{\alpha,\beta}\left(-\lambda\, t^\alpha\right) \div \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}\,, \tag{2.9}$$

$$e_{\alpha,\alpha}(t;\lambda) := t^{\alpha-1}\, E_{\alpha,\alpha}\left(-\lambda\, t^\alpha\right) = \frac{d}{dt}e_\alpha(-\lambda\, t^\alpha) = \div - \frac{\lambda}{s^\alpha+\lambda}\,.$$

A function $f(t)$ defined in $\mathbb{R}^+$ is **completely monotone** (CM) if $(-1)^n\, f^n(t) \geq 0$. The function $\mathrm{e}^{-t}$ is the prototype of a CM function. For a Bernstein theorem a generic CM function reads

$$f(t) = \int_0^\infty \mathrm{e}^{-rt}\, K(r)\, \mathrm{d}r\,,\;\; K(r) \geq 0\,. \tag{2.11}$$

We have for $\lambda > 0$

$$e_{\alpha,\beta}(t;\lambda) := t^{\beta-1}\, E_{\alpha,\beta}\left(-\lambda t^\alpha\right) \text{ CM iff } 0 < \alpha \leq \beta \leq 1\,. \tag{2.12}$$

Using the Laplace transform we can prove, following Gorenflo and Mainardi (1997) that for $0 < \alpha < 1$ (with $\lambda = 1$)

$$E_\alpha(-t^\alpha) \simeq \begin{cases} 1 - \dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \cdots & t \to 0^+ , \\ \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} - \dfrac{t^{-2\alpha}}{\Gamma(1-2\alpha)} \cdots & t \to +\infty , \end{cases} \tag{2.13}$$

and

$$E_\alpha(-t^\alpha) = \int_0^\infty \mathrm{e}^{-rt} K_\alpha(r)\, \mathrm{d}r \tag{2.14}$$

with

$$K_\alpha(r) = \frac{1}{\pi} \frac{r^{\alpha-1} \sin(\alpha\pi)}{r^{2\alpha} + 2\, r^\alpha \cos(\alpha\pi) + 1} > 0 \,.$$

In the following we will outline the key role of the auxiliary functions in the treatment of integral and differential equations of fractional order, including the Abel integral equation of the second kind and the differential equations for fractional relaxation and oscillation.

# 3. Abel integral equation of the second kind

Let us now consider the Abel equation of the second kind

$$u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = f(t)\,,\ \alpha > 0\,,\ \lambda \in \mathbb{C}\,. \quad (3.1)$$

In terms of the fractional integral operator such equation reads

$$(1 + \lambda\, J^\alpha)\; u(t) = f(t)\,, \quad (3.2)$$

and consequently can be formally solved as follows:

$$u(t) = (1 + \lambda J^\alpha)^{-1}\; f(t) = \left(1 + \sum_{n=1}^{\infty} (-\lambda)^n\, J^{\alpha n}\right)\; f(t)\,. \quad (3.3)$$

Recalling the definition of the fractional integral the formal solution reads

$$u(t) = f(t) + \left(\sum_{n=1}^{\infty} (-\lambda)^n \frac{t_+^{\alpha n-1}}{\Gamma(\alpha n)}\right) * f(t)\,. \quad (3.4)$$

Recalling the definition of the function,

$$e_\alpha(t;\lambda) := E_\alpha(-\lambda\, t^\alpha) = \sum_{n=0}^{\infty} \frac{(-\lambda\, t^\alpha)^n}{\Gamma(\alpha n + 1)}\,, \tag{3.5}$$

where $E_\alpha$ denotes the Mittag-Leffler function of order $\alpha$, we note that

$$\sum_{n=1}^{\infty} (-\lambda)^n \frac{t_+^{\alpha n - 1}}{\Gamma(\alpha n)} = \frac{d}{dt} E_\alpha(-\lambda t^\alpha) = e'_\alpha(t;\lambda)\,, \quad t > 0\,. \tag{3.6}$$

Finally, the solution reads

$$u(t) = f(t) + e'_\alpha(t;\lambda) \, * \, f(t)\,. \tag{3.7}$$

Of course the above formal proof can be made rigorous. Simply observe that because of the rapid growth of the gamma function the infinite series in (3.4) and (3.6) are uniformly convergent in every bounded interval of the variable $t$ so that term-wise integrations and differentiations are allowed.

However, we prefer to use the alternative technique of Laplace transforms, which will allow us to obtain the solution in different forms, including the result (3.7).

Applying the Laplace transform to (3.1) we obtain

$$\left[1+\frac{\lambda}{s^\alpha}\right]\widetilde{u}(s)=\widetilde{f}(s) \Longrightarrow \widetilde{u}(s)=\frac{s^\alpha}{s^\alpha+\lambda}\,\widetilde{f}(s)\,. \tag{3.8}$$

Now, let us proceed to obtain the inverse Laplace transform of (3.8) using the following Laplace transform pair

$$e_\alpha(t;\lambda):=E_\alpha(-\lambda\, t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha+\lambda}\,. \tag{3.9}$$

We can choose three different ways to get the inverse Laplace transforms from (3.8), according to the standard rules. Writing (3.8) as

$$\widetilde{u}(s)=s\left[\frac{s^{\alpha-1}}{s^\alpha+\lambda}\,\widetilde{f}(s)\right]\,, \tag{3.10a}$$

we obtain

$$u(t)=\frac{d}{dt}\int_0^t f(t-\tau)\,e_\alpha(\tau;\lambda)\,d\tau\,. \tag{3.11a}$$

If we write (3.8) as

$$\widetilde{u}(s) = \frac{s^{\alpha-1}}{s^\alpha+\lambda}\,[s\,\widetilde{f}(s) - f(0^+)] + f(0^+)\,\frac{s^{\alpha-1}}{s^\alpha+\lambda}\,, \qquad (3.10b)$$

we obtain

$$u(t) = \int_0^t f'(t-\tau)\,e_\alpha(\tau;\lambda)\,d\tau + f(0^+)\,e_\alpha(t;\lambda)\,. \qquad (3.11b)$$

We also note that, $e_\alpha(t;\lambda)$ being a function differentiable with respect to $t$ with $e_\alpha(0^+;\lambda) = E_\alpha(0^+) = 1$, there exists another possibility to re-write (3.8), namely

$$\widetilde{u}(s) = \left[s\,\frac{s^{\alpha-1}}{s^\alpha+\lambda} - 1\right]\,\widetilde{f}(s) + \widetilde{f}(s)\,. \qquad (3.10c)$$

Then we obtain

$$u(t) = \int_0^t f(t-\tau)\,e'_\alpha(\tau;\lambda)\,d\tau + f(t)\,, \qquad (3.11c)$$

in agreement with (3.7). We see that the way $b)$ is more restrictive than the ways $a)$ and $c)$ since it requires that $f(t)$ be differentiable with $\mathcal{L}$-transformable derivative.

# 4. Fractional relaxation and oscillation

Generally speaking, we consider the following differential equation of fractional order $\alpha > 0$, for $t \geq 0$:

$$D_*^\alpha\, u(t) = D^\alpha \left( u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}\, u^{(k)}(0^+) \right) = -u(t) + q(t)\,, \quad (4.1)$$

where $u = u(t)$ is the field variable and $q(t)$ is a given function, continuous for $t \geq 0$. Here $m$ is a positive integer uniquely defined by $m-1 < \alpha \leq m$, which provides the number of the prescribed initial values $u^{(k)}(0^+) = c_k\,,\ k = 0, 1, 2, \ldots, m-1$.

In particular, we consider in detail the cases
(a) **fractional relaxation** $0 < \alpha \leq 1$,
(b) **fractional oscillation**$1 < \alpha \leq 2$.

The application of the Laplace transform yields

$$\widetilde{u}(s) = \sum_{k=0}^{m-1} c_k \, \frac{s^{\alpha-k-1}}{s^\alpha + 1} + \frac{1}{s^\alpha + 1} \, \widetilde{q}(s) \,. \tag{4.2}$$

Then, putting for $k = 0, 1, \ldots, m-1$,

$$u_k(t) := J^k e_\alpha(t) \div \frac{s^{\alpha-k-1}}{s^\alpha + 1} \,, \; e_\alpha(t) := E_\alpha(-t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha + 1} \,, \tag{4.3}$$

and using $u_0(0^+) = 1$, we find,

$$u(t) = \sum_{k=0}^{m-1} c_k \, u_k(t) - \int_0^t q(t-\tau) \, u_0'(\tau) \, d\tau \,. \tag{4.4}$$

In particular, the formula (4.4) encompasses the solutions for $\alpha = 1\,,\, 2\,,$ since

$$\alpha = 1\,, \quad u_0(t) = e_1(t) = \exp\,(-t)\,,$$

$$\alpha = 2\,, \quad u_0(t) = e_2(t) = \cos t\,,\; u_1(t) = J^1 e_2(t) = \sin t\,.$$

When $\alpha$ is not integer, namely for $m-1 < \alpha < m$, we note that $m-1$ represents the integer part of $\alpha$ (usually denoted by $[\alpha]$) and $m$ the number of initial conditions necessary and sufficient to ensure the uniqueness of the solution $u(t)$. Thus the $m$ functions $u_k(t) = J^k e_\alpha(t)$ with $k = 0, 1, \ldots, m-1$ represent those particular solutions of the *homogeneous* equation which satisfy the initial conditions $u_k^{(h)}(0^+) = \delta_{k\,h}\,,\quad h, k = 0, 1, \ldots, m-1\,,$ and therefore they represent the **fundamental solutions** of the fractional equation (4.1), in analogy with the case $\alpha = m$. Furthermore, the function $u_\delta(t) = -u_0'(t) = -e_\alpha'(t)$ represents the **impulse-response solution**.

Now we derive the relevant properties of the basic functions $e_\alpha(t)$ directly from their Laplace representation for $0 < \alpha \le 2$,

$$e_\alpha(t) = \frac{1}{2\pi i} \int_{Br} \mathrm{e}^{\,st} \, \frac{s^{\alpha-1}}{s^\alpha + 1} \, ds\,, \tag{4.5}$$

without detouring on the general theory of Mittag-Leffler functions in the complex plane. Here $Br$ denotes a Bromwich path, *i.e.* a line $\mathrm{Re}(s) = \sigma > 0$ and $\mathrm{Im}(s)$ running from $-\infty$ to $+\infty$.

For transparency reasons, we separately discuss the cases (a) $0 < \alpha < 1$ and (b) $1 < \alpha < 2$, recalling that in the limiting cases $\alpha = 1\,,\,2\,,$ we know $e_\alpha(t)$ as elementary function, namely $e_1(t) = \mathrm{e}^{-t}$ and $e_2(t) = \cos\, t\,.$

For $\alpha$ not integer the power function $s^\alpha$ is uniquely defined as $s^\alpha = |s|^\alpha\, \mathrm{e}^{i \arg s}$, with $-\pi < \arg s < \pi$, that is in the complex $s$-plane cut along the negative real axis.

The essential step consists in decomposing $e_\alpha(t)$ into two parts according to $e_\alpha(t) = f_\alpha(t) + g_\alpha(t)$, as indicated below. In case (a) the function $f_\alpha(t)$, in case (b) the function $-f_\alpha(t)$ is **completely monotone**; in both cases $f_\alpha(t)$ tends to zero as $t$ tends to infinity, from above in case (a), from below in case (b). The other part, $g_\alpha(t)$, is identically vanishing in case (a), but of **oscillatory** character with exponentially decreasing amplitude in case (b).

For the **oscillatory part** we obtain via the residue theorem of complex analysis

$$g_\alpha(t) = \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)\right], \text{ if } 1 < \alpha < 2. \quad (4.6)$$

We note that this function exhibits oscillations with **circular frequency**

$$\omega(\alpha) = \sin(\pi/\alpha)$$

and with an **exponentially decaying amplitude** with rate

$$\lambda(\alpha) = |\cos(\pi/\alpha)| = -\cos(\pi/\alpha)\,.$$

For the **monotonic part** we obtain

$$f_\alpha(t) := \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha(r)\, dr\,, \tag{4.7}$$

with

$$\begin{aligned} K_\alpha(r) &= -\frac{1}{\pi}\,\mathrm{Im}\left( \frac{s^{\alpha-1}}{s^\alpha+1}\Big|_{s\,=\,r\,\mathrm{e}^{i\pi}} \right) \\ &= \frac{1}{\pi}\,\frac{r^{\alpha-1}\,\sin\,(\alpha\pi)}{r^{2\alpha}+2\,r^\alpha\,\cos\,(\alpha\pi)+1}\,. \end{aligned} \tag{4.8}$$

This function $K_\alpha(r)$ vanishes identically if $\alpha$ is an integer, it is positive for all $r$ if $0 < \alpha < 1$, negative for all $r$ if $1 < \alpha < 2$. In fact in (4.8) the denominator is, for $\alpha$ not integer, always positive being $> (r^\alpha - 1)^2 \geq 0$.

Hence $f_\alpha(t)$ has the aforementioned monotonicity properties, decreasing towards zero in case (a), increasing towards zero in case (b).

We note that, in order to satisfy the initial condition $e_\alpha(0^+) = 1$, we find

$$\int_0^\infty K_\alpha(r)\, dr = 1 \quad \text{if} \quad 0 < \alpha \le 1\,,$$

$$\int_0^\infty K_\alpha(r)\, dr = 1 - 2/\alpha \quad \text{if} \quad 1 < \alpha \le 2\,.$$

In Figs. 1a and 1b we display the plots of $K_\alpha(r)$, that we denote as the **basic spectral function**, for some values of $\alpha$ in the intervals (a) $0 < \alpha < 1$, (b) $1 < \alpha < 2$.

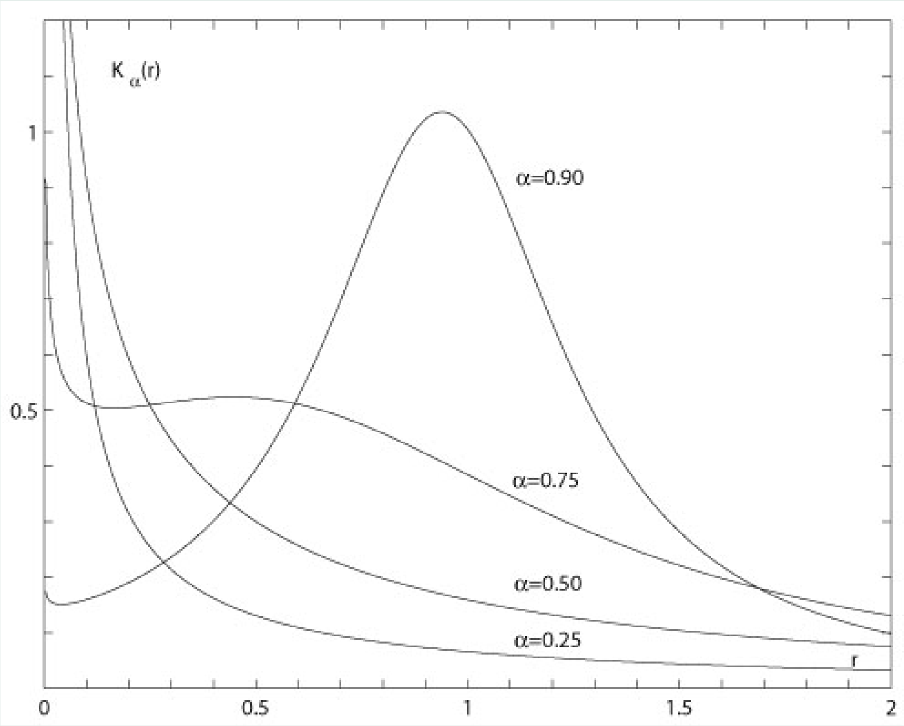


**Fig. 1a** – Plots of the *basic spectral function* $K_\alpha(r)$ for $0 < \alpha < 1$ :
$\alpha = 0.25\,,\quad \alpha = 0.50\,,\quad \alpha = 0.75\,,\quad \alpha = 0.90\,.$

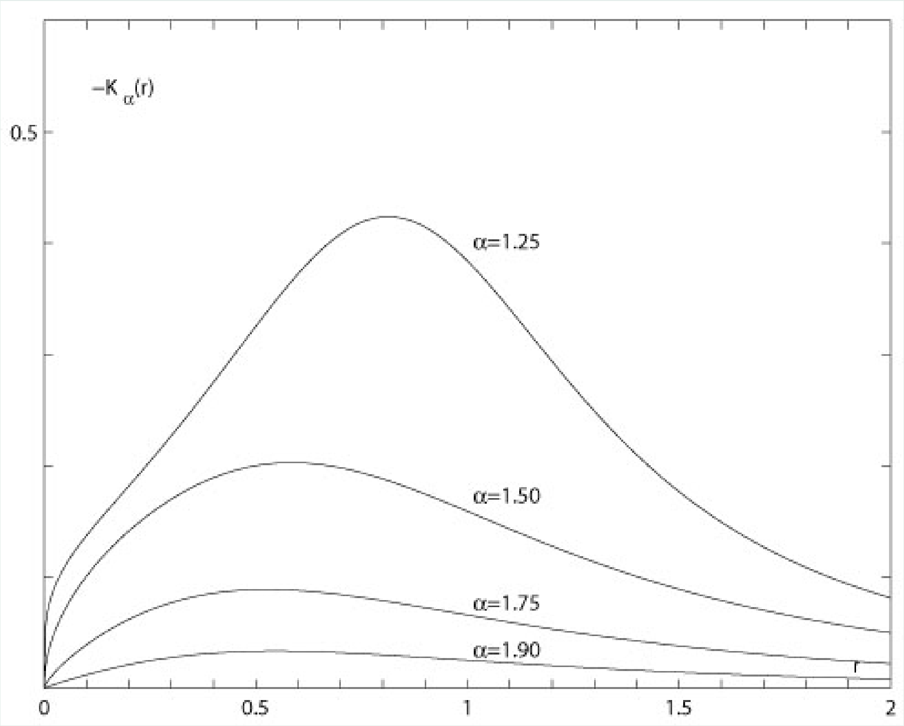


**Fig. 1b** – Plots of the *basic spectral function* $-K_\alpha(r)$ for $1 < \alpha < 2$ :
$\alpha = 1.25\,, \quad \alpha = 1.50\,, \quad \alpha = 1.75\,, \quad \alpha = 1.90\,.$

In addition to the basic fundamental solutions, $u_0(t) = e_\alpha(t)$, we need to compute the impulse-response solutions $u_\delta(t) = -D^1\, e_\alpha(t)$ for cases (a) and (b) and, only in case (b), the second fundamental solution $u_1(t) = J^1\, e_\alpha(t)$.

For this purpose we note that in general it turns out that

$$J^k\, f_\alpha(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^k(r)\, dr\,, \tag{4.9}$$

with

$$K_\alpha^k(r) := (-1)^k\, r^{-k}\, K_\alpha(r) = \frac{(-1)^k}{\pi}\, \frac{r^{\alpha-1-k}\, \sin(\alpha\pi)}{r^{2\alpha} + 2\, r^\alpha\, \cos(\alpha\pi) + 1}\,, \tag{4.10}$$

where $K_\alpha(r) = K_\alpha^0(r)$, and

$$J^k g_\alpha(t) = \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) - k\frac{\pi}{\alpha}\right]. \tag{4.11}$$

For the impulse-response solution we note that the effect of the differential operator $D^1$ is the same as that of the virtual operator $J^{-1}$.

In conclusion the solutions for the **fractional relaxation** are :

(a) $0 < \alpha < 1$ ,

$$u(t) = c_0\, u_0(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau \,, \tag{4.12a}$$

where

$$\begin{cases} u_0(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^0(r)\, dr \,, \\ u_\delta(t) = -\int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^{-1}(r)\, dr \,, \end{cases} \tag{4.13a}$$

with

$$u_0(0^+) = 1\,,\; u_\delta(0^+) = \infty\,,$$

and for $t \to \infty$

$$u_0(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,,\; u_1(t) \sim \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\,. \tag{4.14a}$$

In conclusion the solutions for the **fractional oscillation** are :

(b) $1 < \alpha < 2\,,$

$$u(t) = c_0\, u_0(t) + c_1\, u_1(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \qquad (4.12b)$$

$$\begin{cases} u_0(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^0(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right)\right], \\ u_1(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^1(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right], \\ u_\delta(t) = -\displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^{-1}(r)\, dr - \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\right], \end{cases} \qquad (4.13b)$$

with

$$u_0(0^+) = 1,\; u_0'(0^+) = 0,\;\; u_1(0^+) = 0,\;\; u_1'(0^+) = 1,$$
$$u_\delta(0^+) = 0,\; u_\delta'(0^+) = +\infty,$$

and for $t \to \infty$

$$u_0(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)},\; u_1(t) \sim \frac{t^{1-\alpha}}{\Gamma(2-\alpha)},\; u_\delta(t) \sim -\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}. \qquad (4.14b)$$

In Figs. 2a and 2b we display the plots of the basic fundamental solution for the following cases, respectively :
(a) $\alpha = 0.25\,,\ 0.50\,,\ 0.75\,,\ 1\,,$
(b) $\alpha = 1.25\,,\ 1.50\,,\ 1.75\,,\ 2\,,$
obtained from the first formula in (4.13a) and (4.13b), respectively.

We now want to point out that in both the cases (a) and (b) (in which $\alpha$ is just not integer) *i.e.* for **fractional relaxation** and **fractional oscillation**, all the fundamental and impulse-response solutions exhibit an **algebraic decay** as $t \to \infty$ , as discussed above.

This **algebraic decay** is the most important effect of the non-integer derivative in our equations, which dramatically differs from the **exponential decay** present in the ordinary relaxation and damped-oscillation phenomena.

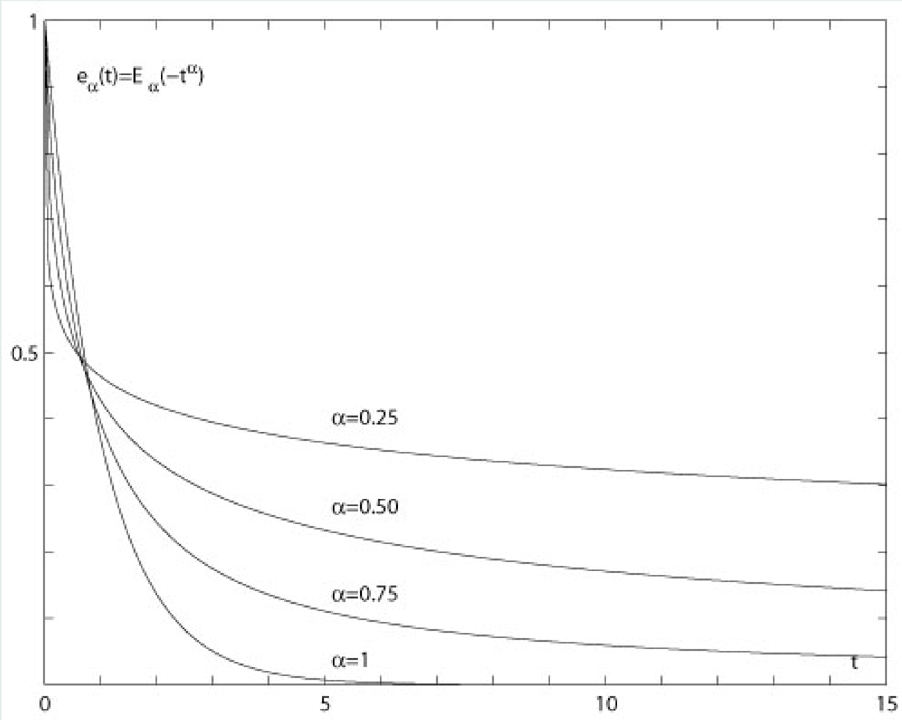


**Fig. 2a** – Plots of the **basic fundamental solution** $u_0(t) = e_\alpha(t)$
$\alpha = 0.25\,, \quad \alpha = 0.50\,, \quad \alpha = 0.75\,, \quad \alpha = 1\,.$

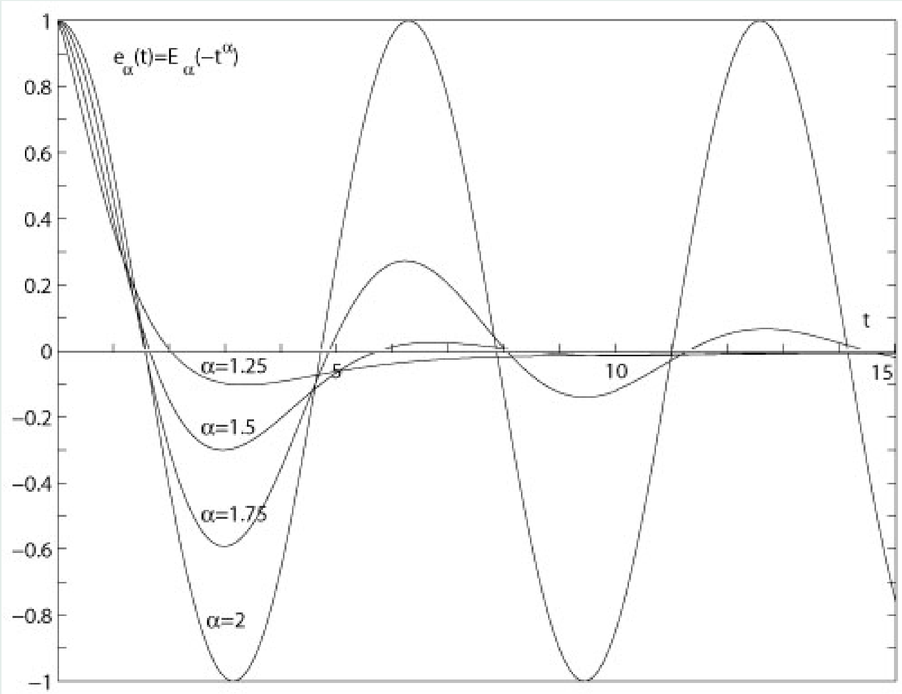


**Fig. 2b** – Plots of the **basic fundamental solution** $u_0(t) = e_\alpha(t)$:
$\alpha = 1.25\,,\quad \alpha = 1.50\,,\quad \alpha = 1.75\,,\quad \alpha = 2\,.$

We would like to remark the difference between fractional relaxation governed by the Mittag-Leffler type function $E_\alpha(-at^\alpha)$ and stretched relaxation governed by a stretched exponential function $\exp\left(-bt^\alpha\right)$ with $\alpha\,,a\,,\,b>0$ for $t\geq 0$. A common behaviour is achieved only in a restricted range $0\leq t\ll 1$ where we can have

$$E_\alpha(-at^\alpha)\simeq 1-\frac{a}{\Gamma(\alpha+1)}\,t^\alpha=1-b\,t^\alpha\simeq \mathrm{e}^{-b\,t^\alpha}\,,\;b=\frac{a}{\Gamma(\alpha+1)}\,.$$

In Figs. 3a, 3b, 3c we compare, for $\alpha=0.25\,,\,0.50\,,\,0.75$, respectively $E_\alpha(-t^\alpha)$ (*full* line) with its asymptotic approximations $\exp\left[-t^\alpha/\Gamma(1+\alpha)\right]$ (*dashed* line) valid for short times, and $t^{-\alpha}/\Gamma(1-\alpha)$ (*dotted* line) valid for long times.

We have adopted log-log plots in order to better achieve such a comparison and the transition from the stretched exponential to the inverse power-law decay.

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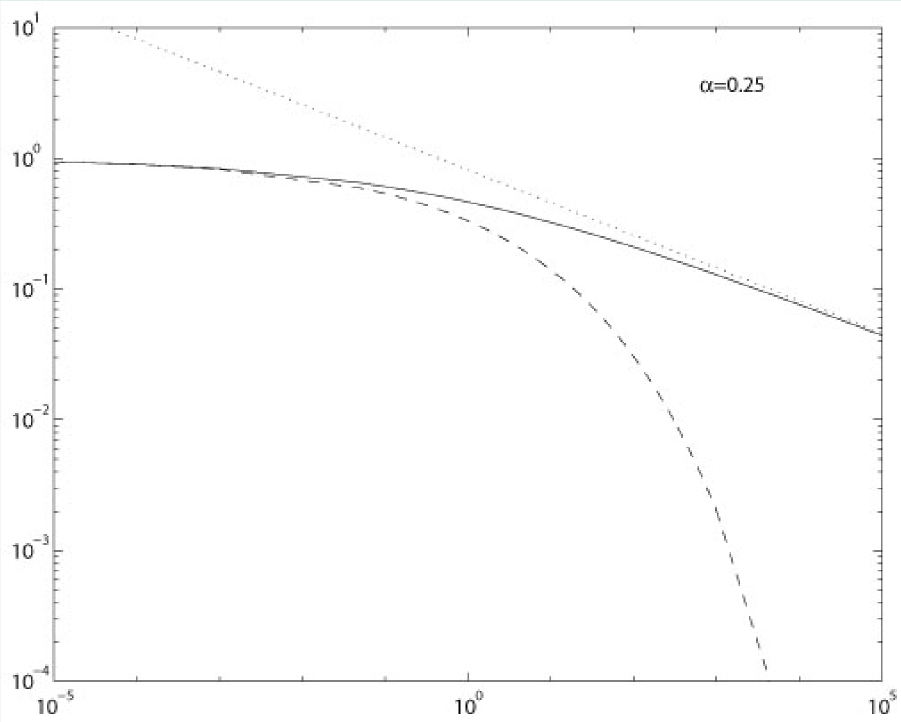


**Fig. 3a** – Log-log plot of $E_\alpha(-t^\alpha)$ for $\alpha = 0.25$ for $10^{-6} \leq t \leq 10^{6}$ .

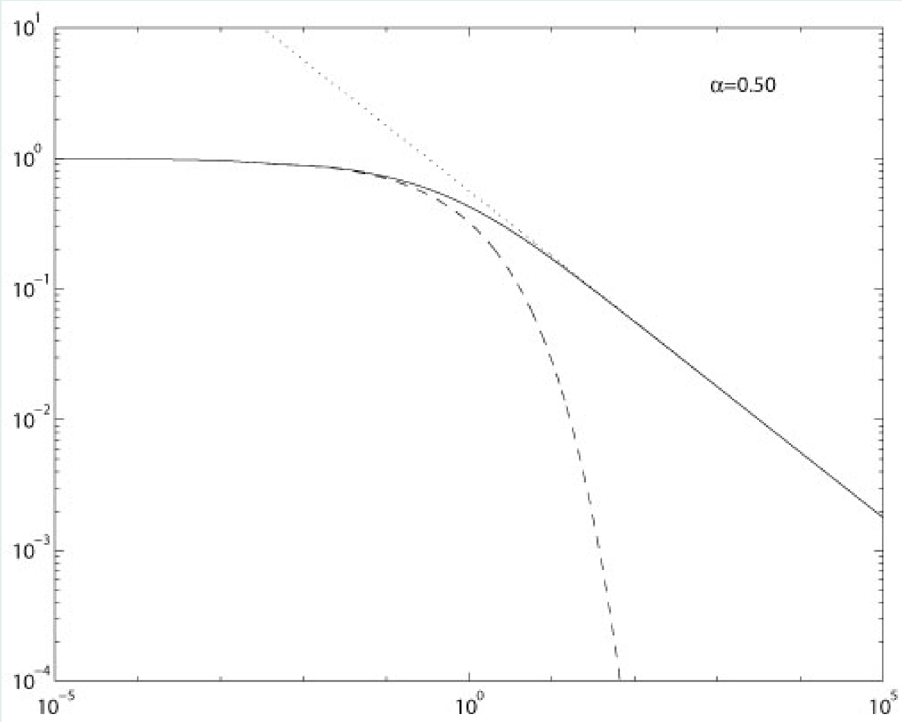


**Fig. 3b** – Log-log plot of $E_\alpha(-t^\alpha)$ for $\alpha = 0.50$ for $10^{-6} \le t \le 10^6$ .

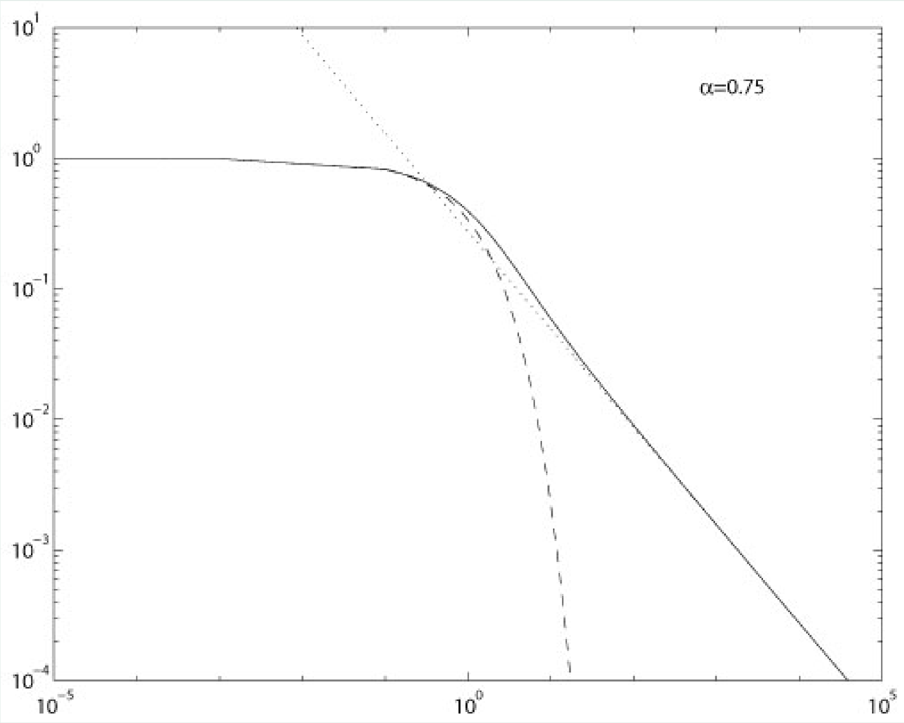


**Fig. 3c** – Log-log plot of $E_\alpha(-t^\alpha)$ for $\alpha = 0.75$ for $10^{-6} \le t \le 10^6$ .

In Figs. 4a, 4b, 4c we show some plots of the **basic fundamental solution** $u_0(t) = e_\alpha(t)$ for $\alpha = 1.25\,,\, 1.50\,,\, 1.75$, respectively.

Here the algebraic decay of the fractional oscillation can be recognized and compared with the two contributions provided by $f_\alpha$ (monotonic behaviour, dotted line) and $g_\alpha(t)$ (exponentially damped oscillation, dashed line).

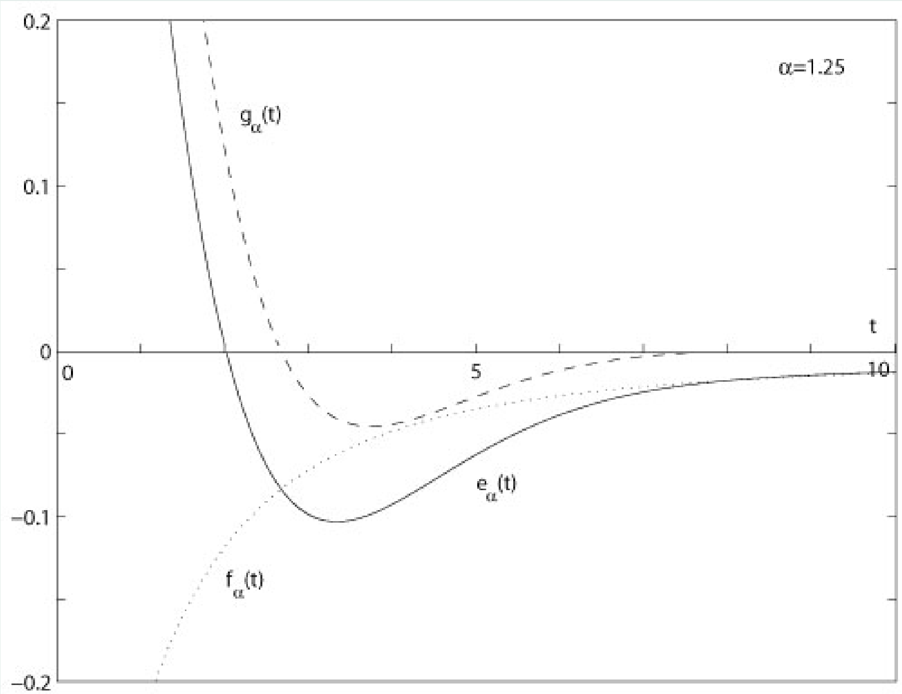


**Fig. 4a** – Decay of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.25$
*full* line $= e_\alpha(t)$ , *dashed* line $= g_\alpha(t)$ , *dotted* line $= f_\alpha(t)$ .

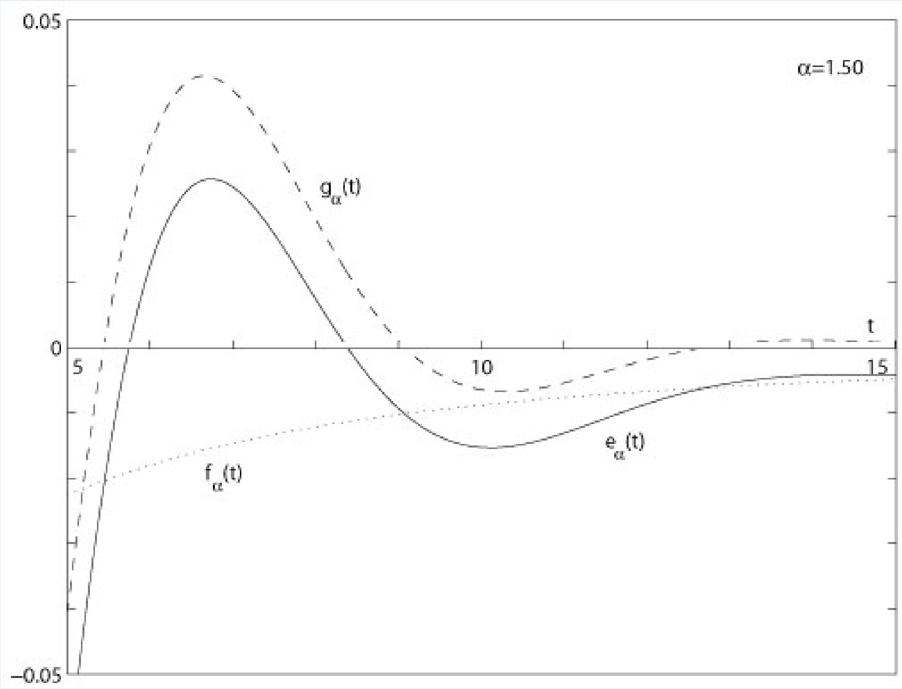


**Fig. 4b** – Decay of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.50$
*full* line $= e_\alpha(t)$, *dashed* line $= g_\alpha(t)$, *dotted* line $= f_\alpha(t)$.

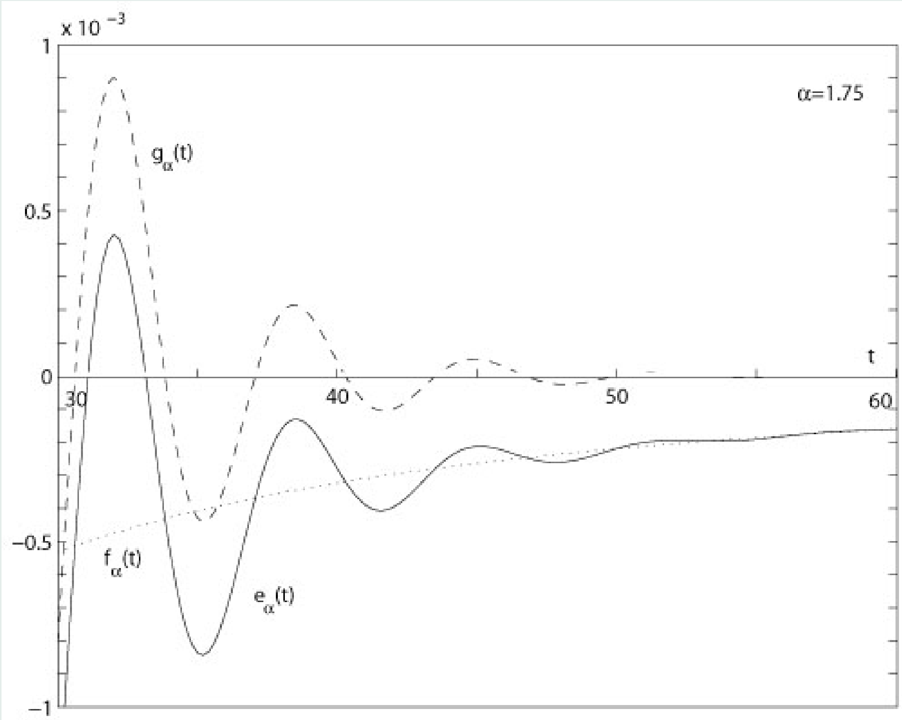


**Fig. 4c** – Decay of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.75$
*full* line $= e_\alpha(t)$, *dashed* line $= g_\alpha(t)$, *dotted* line $= f_\alpha(t)$.

## The zeros of the solutions of the fractional oscillation

Now we find it interesting to carry out some investigations about the zeros of the basic fundamental solution $u_0(t) = e_\alpha(t)$ in the case (b) of fractional oscillations. For the second fundamental solution and the impulse-response solution the analysis of the zeros can be easily carried out analogously.

Recalling the first equation in (4.13b), the required zeros of $e_\alpha(t)$ are the solutions of the equation

$$e_\alpha(t) = f_\alpha(t) + \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos\,(\pi/\alpha)}\,\cos\left[t\,\sin\,\left(\frac{\pi}{\alpha}\right)\right] = 0\,. \quad (4.15)$$

We first note that the function $e_\alpha(t)$ exhibits an *odd* number of zeros, in that $e_\alpha(0) = 1$, and, for sufficiently large $t$, $e_\alpha(t)$ turns out to be permanently negative, as shown in (4.14b) by the sign of $\Gamma(1-\alpha)$.

The smallest zero lies in the first positivity interval of $\cos\left[t\sin\left(\pi/\alpha\right)\right]$, hence in the interval $0 < t < \pi/[2\,\sin\left(\pi/\alpha\right)]$; all other zeros can only lie in the succeeding positivity intervals of $\cos\left[t\,\sin\left(\pi/\alpha\right)\right]$, in each of these two zeros are present as long as

$$\frac{2}{\alpha}\,\mathrm{e}^{t\,\cos\left(\pi/\alpha\right)} \geq |f_\alpha(t)|\,. \tag{4.16}$$

When $t$ is sufficiently large the zeros are expected to be found approximately from the equation

$$\frac{2}{\alpha}\,\mathrm{e}^{t\,\cos\left(\pi/\alpha\right)} \approx \frac{t^{-\alpha}}{|\Gamma(1-\alpha)|}\,, \tag{4.17}$$

obtained from (4.15) by ignoring the oscillation factor of $g_\alpha(t)$ and taking the first term in the asymptotic expansion of $f_\alpha(t)$. As we have shown in a report of ours (1995), such approximate equation turns out to be useful when $\alpha \to 1^+$ and $\alpha \to 2^-$.

For $\alpha \longrightarrow 1^+$, only one zero is present, which is expected to be very far from the origin in view of the large period of the function $\cos\left[t\,\sin\left(\pi/\alpha\right)\right]$. In fact, since there is no zero for $\alpha = 1$, and by increasing $\alpha$ more and more zeros arise, we are sure that only one zero exists for $\alpha$ sufficiently close to 1. Putting $\alpha = 1 + \epsilon$ the asymptotic position $T_*$ of this zero can be found from the relation (4.17) in the limit $\epsilon \longrightarrow 0^+$. Assuming in this limit the first-order approximation, we get

$$T_* \sim \log\left(\frac{2}{\epsilon}\right), \tag{4.18}$$

which shows that $T_*$ tends to infinity slower than $1/\epsilon$, as $\epsilon \longrightarrow 0$. For details see our 1995 report.

For $\alpha \to 2^-$, there is an increasing number of zeros up to infinity since $e_2(t) = \cos t$ has infinitely many zeros [in $t_n^* = (n + 1/2)\pi$ , $n = 0, 1, \ldots$]. Putting now $\alpha = 2 - \delta$ the asymptotic position $T_*$ for the largest zero can be found again from (4.17) in the limit $\delta \to 0^+$. Assuming in this limit the first-order approximation, we get

$$T_* \sim \frac{12}{\pi\,\delta} \log\left(\frac{1}{\delta}\right) . \qquad (4.19)$$

For details see our 1995 report.

Now, for $\delta \to 0^+$ the length of the positivity intervals of $g_\alpha(t)$ tends to $\pi$ and, as long as $t \le T_*$, there are two zeros in each positivity interval. Hence, in the limit $\delta \to 0^+$, there is in average one zero per interval of length $\pi$, so we expect that $N_* \sim T_*/\pi$ .

Remark : For the above considerations we got inspiration from an interesting paper by Wiman (1905), who at the beginning of the XX-th century, after having treated the Mittag-Leffler function in the complex plane, considered the position of the zeros of the function on the negative real axis (without providing any detail). Our expressions of $T_*$ are in disagreement with those by Wiman for numerical factors; however, the results of our numerical studies carried out in our 1995 report confirm and illustrate the validity of our analysis.

Here, we find it interesting to analyse the phenomenon of the transition of the (odd) number of zeros as $1.4 \le \alpha \le 1.8$. For this purpose, in Table I we report the intervals of amplitude $\Delta\alpha = 0.01$ where these transitions occur, and the location $T_*$ (evaluated within a relative error of $0.1\%$ ) of the largest zeros found at the two extreme values of the above intervals.

We recognize that the transition from 1 to 3 zeros occurs as $1.40 \le \alpha \le 1.41$, that one from 3 to 5 zeros occurs as $1.56 \le \alpha \le 1.57$, and so on. The last transition is from 15 to 17 zeros, and it just occurs as $1.79 \le \alpha \le 1.80$.

| $N_*$ | $\alpha$ | $T_*$ |
|---|---|---|
| $1 \div 3$ | $1.40 \div 1.41$ | $1.730 \div 5.726$ |
| $3 \div 5$ | $1.56 \div 1.57$ | $8.366 \div 13.48$ |
| $5 \div 7$ | $1.64 \div 1.65$ | $14.61 \div 20.00$ |
| $7 \div 9$ | $1.69 \div 1.70$ | $20.80 \div 26.33$ |
| $9 \div 11$ | $1.72 \div 1.73$ | $27.03 \div 32.83$ |
| $11 \div 13$ | $1.75 \div 1.76$ | $33.11 \div 38.81$ |
| $13 \div 15$ | $1.78 \div 1.79$ | $39.49 \div 45.51$ |
| $15 \div 17$ | $1.79 \div 1.80$ | $45.51 \div 51.46$ |

**Table I**

$N_*$ = number of zeros, $\alpha$ = fractional order, $T_*$ location of the largest zero.

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## 5. Introduction to the Wright functions

Here we provide a survey of the high transcendental functions related to the Wright special functions.

Like the functions of the Mittag-Leffler type, the functions of the Wright type are known to play fundamental roles in various applications of the fractional calculus. This is mainly due to the fact that they are interrelated with the Mittag-Leffler functions through Laplace and Fourier transformations.

We start providing the definitions in the complex plane for the general Wright function and for two special cases that we call auxiliary functions. Then we devote particular attention to the auxiliary functions in the real field, because they admit a probabilistic interpretation related to the fundamental solutions of certain evolution equations of fractional order. These equations are fundamental to understand phenomena of anomalous diffusion or intermediate between diffusion and wave propagation.

At the end we add some historical and bibliographical notes.

## 6. The Wright function $W_{\lambda,\mu}(z)$

The Wright function, that we denote by $W_{\lambda,\mu,}(z)$ is so named in honour of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions, see [Wright (1933); 1935a; 1935b].The function is defined by the series representation, convergent in the whole $z$-complex plane,

$$\boxed{W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)}\,, \ \lambda > -1\,, \ \mu \in \mathbb{C}\,,} \qquad (F.1)$$

so $W_{\lambda,\mu}(z)$ is an *entire function*. Originally Wright assumed $\lambda > 0$, and, only in 1940, he considered $-1 < \lambda < 0$, see [Wright 1940]. We note that in the handbook of the Bateman Project [Erdelyi et al. Vol. 3, Ch. 18], presumably for a misprint, $\lambda$ is restricted to be non negative.

## The integral representation

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^\mu}, \; \lambda > -1, \; \mu \in \mathbb{C}, \quad (F.2)$$

where $Ha$ denotes the Hankel path. The equivalence of the series and integral representations is easily proved using the Hankel formula for the Gamma function

$$\frac{1}{\Gamma(\zeta)} = \int_{Ha} \mathrm{e}^{u} u^{-\zeta} \, du, \; \zeta \in \mathbb{C},$$

and performing a term-by-term integration. In fact,

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^\mu} = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma} \left[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{-\lambda n} \right] \frac{d\sigma}{\sigma^\mu}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[ \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma} \sigma^{-\lambda n - \mu} \, d\sigma \right] = \sum_{n=0}^{\infty} \frac{z^n}{n! \, \Gamma[\lambda n + \mu]}.$$

It is possible to prove that the Wright function is entire of order $1/(1+\lambda)$, hence of exponential type if $\lambda \geq 0$. The case $\lambda = 0$ is trivial since $W_{0,\mu}(z) = \mathrm{e}^{z}/\Gamma(\mu)$.

## Asymptotic expansions.

For the detailed asymptotic analysis in the whole complex plane for the Wright functions, the interested reader is referred to Wong and Zhao (1999a),(1999b), who have considered the two cases $\lambda \geq 0$ and $-1 < \lambda < 0$ separately, including a description of Stokes' discontinuity and its smoothing.

In the second case, that, as a matter of fact is the most interesting for us, we set $\lambda = -\nu \in (-1, 0)$, and we recall the asymptotic expansion originally obtained by Wright himself, that is valid in a suitable sector about the negative real axis as $|z| \to -\infty$,

$$W_{-\nu,\mu}(z) = Y^{1/2-\mu} \, e^{-Y} \left[ \sum_{m=0}^{M-1} A_m \, Y^{-m} + O(|Y|^{-M}) \right], \quad Y = Y(z) = (1-\nu) \, (-\nu^{\nu} \, z)^{1/(1-\nu)}, \qquad (F.3)$$

where the $A_m$ are certain real numbers.

## Generalization of the Bessel functions.

For $\lambda = 1$ and $\mu = \nu + 1 \geq 0$ the Wright functions turn out to be related to the well known Bessel functions $J_\nu$ and $I_\nu$ by the identities:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(-\frac{z^2}{4}\right), \qquad (F.4)$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(\frac{z^2}{4}\right). \qquad (F.5)$$

In view of this property some authors refer to the Wright function as the *Wright generalized Bessel function* (misnamed also as the *Bessel-Maitland function*) and introduce the notation

$$J_\nu^{(\lambda)}(z) := \left(\frac{z}{2}\right)^\nu W_{\lambda,\nu+1}\left(-\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(\lambda n + \nu + 1)}, \qquad (F.6)$$

with $\lambda > 0$ and $\nu > -1$. In particular $J_\nu^{(1)}(z) := J_\nu(z)$.

## Recurrence relations

Some of the properties, that the Wright functions share with the most popular Bessel functions, were enumerated by Wright himself.

Hereafter, we quote some relevant relations from the handbook of Bateman Project Handbook, see [Erdelyi 1955, Vol. 3, Ch. 18]:

$$\lambda z \, W_{\lambda,\lambda+\mu}(z) = W_{\lambda,\mu-1}(z) + (1-\mu)\, W_{\lambda,\mu}(z)\,, \qquad (F.7)$$

$$\frac{d}{dz} W_{\lambda,\mu}(z) = W_{\lambda,\lambda+\mu}(z)\,. \qquad (F.8)$$

We note that these relations can easily be derived from the series or integral representations, (F.1) or (F.2).

## 7. The auxiliary functions of the Wright type

In his first analysis of the time fractional diffusion equation the Author [Mainardi 1993-1994], aware of the Bateman project but not of 1940 paper by Wright, introduced the two (Wright-type) entire **auxiliary functions**,

$$\boxed{F_\nu(z) := W_{-\nu,0}(-z)\,,\ 0<\nu<1\,,} \qquad (F.9)$$

and

$$\boxed{M_\nu(z) := W_{-\nu,1-\nu}(-z)\,,\ 0<\nu<1\,,} \qquad (F.10)$$

inter-related through

$$\boxed{F_\nu(z) = \nu\, z\, M_\nu(z)\,.} \qquad (F.11)$$

As a matter of fact the functions $F_\nu(z)$ and $M_\nu$ are particular cases of $W_{\lambda,\mu}(z)$ by setting $\lambda = -\nu$ and $\mu = 0$, $\mu = 1$, respectively.

## Series representations

$$\begin{aligned} F_\nu(z) &:= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\,\Gamma(-\nu n)}\,, \\ &:= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{n!}\, \Gamma(\nu n + 1)\, \sin(\pi \nu n)\,, \end{aligned} \qquad (F.11)$$

and

$$\begin{aligned} M_\nu(z) &:= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (1-\nu)]}\,, \\ &:= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\, \Gamma(\nu n)\, \sin(\pi \nu n)\,, \end{aligned} \qquad (F.12)$$

The second series representations in (F.11)-(F.12) have been obtained by using the well-known reflection formula for the Gamma function,

$$\Gamma(\zeta)\,\Gamma(1-\zeta) = \pi / \sin \pi\zeta\,.$$

## The integral representations

$$\boxed{F_\nu(z) := \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, d\sigma\,,} \tag{F.14}$$

$$\boxed{M_\nu(z) := \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, \frac{d\sigma}{\sigma^{1-\nu}}\,,} \tag{F.15}$$

We note that the relation (F.11), $F_\nu(z) = \nu z M_\nu(z)$, can be obtained directly from (F.12)-(F.13) with an integration by parts. In fact,

$$\begin{aligned} M_\nu(z) &= \int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, \frac{d\sigma}{\sigma^{1-\nu}} = \int_{Ha} \mathrm{e}^{\,\sigma}\left(-\frac{1}{\nu z}\frac{d}{d\sigma}\mathrm{e}^{\,-z\sigma^\nu}\right) d\sigma \\ &= \frac{1}{\nu z}\int_{Ha} \mathrm{e}^{\,\sigma - z\sigma^\nu}\, d\sigma = \frac{F_\nu(z)}{\nu z}\,. \end{aligned}$$

As usual, the equivalence of the series and integral representations is easily proved using the Hankel formula for the Gamma function and performing a term-by-term integration.

## Special cases

Explicit expressions of $F_\nu(z)$ and $M_\nu(z)$ in terms of known functions are expected for some particular values of $\nu$. In [Mainardi & Tomirotti 1994] the Authors have shown that for $\nu = 1/q$, where $q \geq 2$ is a positive integer, the auxiliary functions can be expressed as a sum of $(q-1)$ simpler entire functions. In the particular cases $q = 2$ and $q = 3$ we find

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{2}\right)_m \frac{z^{2m}}{(2m)!} = \frac{1}{\sqrt{\pi}} \exp\left(-z^2/4\right), \tag{F.16}$$

and

$$M_{1/3}(z) = \frac{1}{\Gamma(2/3)} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)_m \frac{z^{3m}}{(3m)!} - \frac{1}{\Gamma(1/3)} \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)_m \frac{z^{3m+1}}{(3m+1)!} \tag{F.17}$$

$$= 3^{2/3}\, \mathrm{Ai}\left(z/3^{1/3}\right),$$

where *Ai* denotes the *Airy function*.

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Furthermore, it can be proved that $M_{1/q}(z)$ satisfies the differential equation of order $q-1$

$$\frac{d^{q-1}}{dz^{q-1}} M_{1/q}(z) + \frac{(-1)^q}{q} z\, M_{1/q}(z) = 0\,, \qquad (F.18)$$

subjected to the $q-1$ initial conditions at $z=0$, derived from (F.15),

$$M^{(h)}_{1/q}(0) = \frac{(-1)^h}{\pi} \Gamma[(h+1)/q]\, \sin[\pi\,(h+1)/q]\,, \qquad (F.19)$$

with $h=0,\, 1,\, \ldots\, q-2$. We note that, for $q \geq 4$, Eq. (F.18) is akin to the *hyper-Airy* differential equation of order $q-1$, see *e.g.* [Bender & Orszag 1987].

We point out that the most relevant applications of our auxiliary functions, are when the variable is real. More precisely, from now on, we will consider functions that are defined either on the positive real semi-axis $\mathbb{R}^+$ or on all of $\mathbb{R}$ in a symmetric way. We agree to denote the variable with $r$ or $x$ or $t$ when it is restricted to $\mathbb{R}^+$ and with $|x|$ for all of $\mathbb{R}$.

**The asymptotic representation of $M_\nu(r)$ as $r \to \infty$**

of the function $M_\nu(r)$ as $r \to \infty$. Choosing as a variable $r/\nu$ rather than $r$, the computation of the requested asymptotic representation by the saddle-point approximation yields, see [Mainardi & Tomirotti 1994],

$$M_\nu(r/\nu) \sim a(\nu)\, r^{(\nu - 1/2)/(1-\nu)} \exp\left[-b(\nu)\, r^{1/(1-\nu)}\right], \qquad (F.20)$$

where

$$a(\nu) = \frac{1}{\sqrt{2\pi\,(1-\nu)}} > 0\,, \quad b(\nu) = \frac{1-\nu}{\nu} > 0\,. \qquad (F.21)$$

The above evaluation is consistent with the first term in Wright's asymptotic expansion (F.3) after having used the definition (F.10).

We point out that in the limit $\nu \to 1^-$ the function $M_\nu(r)$ tends to the Dirac generalized function $\delta(r-1)$.

## Plots of $M_\nu(|x|)$

To gain more insight of the effect of the parameter $\nu$ on the behaviour close to and far from the origin, we will adopt both linear and logarithmic scale for the ordinates.

In Figs. F.1 and F.2 we compare the plots of the $M_\nu(|x|)$-Wright functions in $|x| \le 5$ for some rational values in the ranges $\nu \in [0, 1/]$ and $\nu \in [1/2, 1]$, respectively. Thus in Fig. F.1 we see the transition from $\exp(-|x|)$ for $\nu = 0$ to $1/\sqrt{\pi}\exp(-x^2)$ for $\nu = 1/2$, whereas in Fig. F.2 we see the transition from $1/\sqrt{\pi}\exp(-x^2)$ to the delta functions $\delta(x \pm 1)$ for $\nu = 1$.

In plotting $M_\nu(|x|)$ at fixed $\nu$ for sufficiently large $|x|$ the asymptotic representation (F.20)-(F.21) is useful since, as $|x|$ increases, the numerical convergence of the series in (F.15) becomes poor and poor up to being completely inefficient: henceforth, the matching between the series and the asymptotic representation is relevant. However, as $\nu \to 1^-$, the plotting remains a very difficult task because of the high peak arising around $x = \pm 1$.

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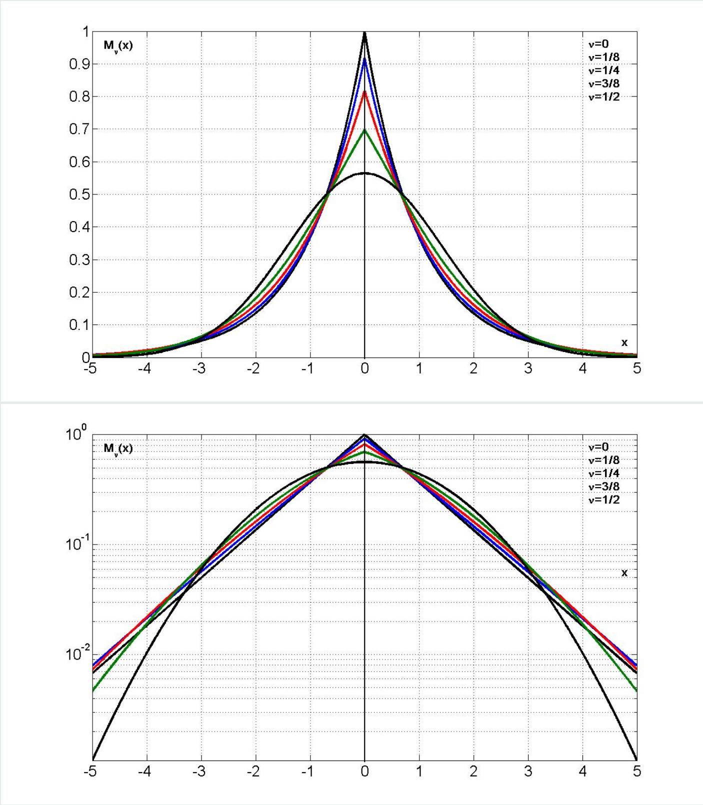


**Fig. F1** - Plots of $M_\nu(|x|)$ with $\nu = 0, 1/8, 1/4, 3/8, 1/2$ for $|x| \le 5$; top: linear scale, bottom: logarithmic scale.

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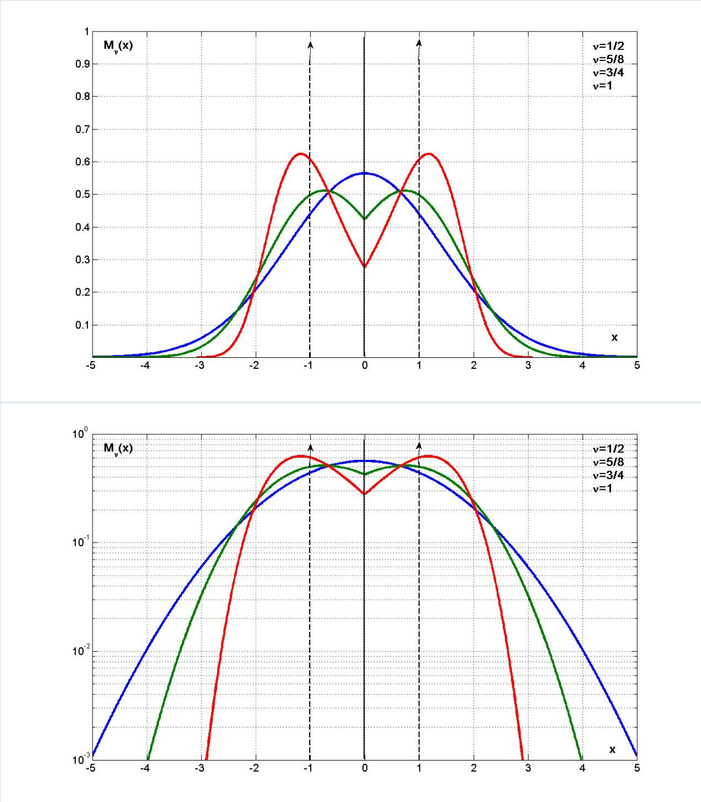


**Fig. F2** - Plots of $M_\nu(|x|)$ with $\nu = 1/2\,,\, 5/8\,,\, 3/4\,,\, 1$ for $|x| \le 5$: top: linear scale; bottom: logarithmic scale)

## The Laplace transform pairs

Let us consider the Laplace transform of the Wright function using the following notation

$$W_{\lambda,\mu}(\pm r) \div \mathcal{L}\left[W_{\lambda,\mu}(\pm r); s\right] := \int_0^\infty \mathrm{e}^{-s\,r}\, W_{\lambda,\mu}(\pm r)\, dr\,,$$

where $r$ denotes a non negative real variable, *i.e.* $0 \le r < +\infty$, and $s$ is the Laplace complex parameter.

When $\lambda > 0$ the series representation of the Wright function can be transformed term-by-term. In fact, for a known theorem of the theory of the Laplace transforms, see *e.g.* [Doetsch (197], the Laplace transform of an entire function of exponential type can be obtained by transforming term-by-term the Taylor expansion of the original function around the origin. In this case the resulting Laplace transform turns out to be analytic and vanishing at infinity.

As a consequence, we obtain the Laplace transform pair

$$\boxed{W_{\lambda,\mu}(\pm r) \,\div\, \frac{1}{s}\, E_{\lambda,\mu}\left(\pm\frac{1}{s}\right)\,,\ \lambda > 0\,,\ |s| > 0\,,} \qquad (F.22)$$

where $E_{\lambda,\mu}$ denotes the Mittag-Leffler function in two parameters. The proof is straightforward noting that

$$\sum_{n=0}^{\infty} \frac{(\pm r)^n}{n!\,\Gamma(\lambda n + \mu)} \,\div\, \frac{1}{s} \sum_{n=0}^{\infty} \frac{(\pm 1/s)^n}{\Gamma(\lambda n + \mu)}\,,$$

and recalling the series representation of the Mittag-Leffler function,

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}\,,\ \alpha > 0\,,\ \ \beta \in \mathbb{C}\,.$$

For $\lambda \to 0^+$ Eq. (F.22) provides the Laplace transform pair

$$W_{0^+,\mu}(\pm r) = \frac{e^{\pm r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s \mp 1} = \frac{1}{s} E_{0,\mu}\left(\pm \frac{1}{s}\right), \quad |s| > 1, \tag{F.23}$$

where, to remain in agreement with (F.22), we have formally put

$$E_{0,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu)} := \frac{1}{\Gamma(\mu)} E_0(z) := \frac{1}{\Gamma(\mu)} \frac{1}{1-z}, \quad |z| < 1.$$

We recognize that in this special case the Laplace transform exhibits a simple pole at $s = \pm 1$ while for $\lambda > 0$ it exhibits an essential singularity at $s = 0$.

For $-1 < \lambda < 0$ the Wright function turns out to be an entire function of order greater than 1, so that care is required in establishing the existence of its Laplace transform, which necessarily must tend to zero as $s \to \infty$ in its half-plane of convergence.

For the sake of convenience we limit ourselves to derive the Laplace transform for the special case of $M_\nu(r)$ ; the exponential decay as $r \to \infty$ of the *original* function provided by (F.20) ensures the existence of the *image* function. From the integral representation (F.13) of the $M_\nu$ function we obtain

$$M_\nu(r) \div \frac{1}{2\pi i} \int_0^\infty \mathrm{e}^{-s\,r} \left[\int_{Ha} \mathrm{e}^{\sigma - r\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}}\right] dr$$

$$= \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma}\, \sigma^{\nu-1} \left[\int_0^\infty \mathrm{e}^{-r(s+\sigma^\nu)}\, dr\right] d\sigma = \frac{1}{2\pi i} \int_{Ha} \frac{\mathrm{e}^{\sigma}\, \sigma^{\nu-1}}{\sigma^\nu + s}\, d\sigma\,.$$

Then, by recalling the integral representation of the Mittag-Leffler function,

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1}\, \mathrm{e}^{\zeta}}{\zeta^\alpha - z}\, d\zeta\,, \ \alpha > 0\,, \ z \in \mathbb{C}\,,$$

we obtain the Laplace transform pair

$$\boxed{M_\nu(r) := W_{-\nu,1-\nu}(-r) \div E_\nu(-s)\,, \ 0 < \nu < 1\,.} \qquad (F.24)$$

In this case, transforming term-by-term the Taylor series of $M_\nu(r)$ yields a series of negative powers of $s$, that represents the asymptotic expansion of $E_\nu(-s)$ as $s \to \infty$ in a sector around the positive real axis.

We note that (F.24) contains the well-known Laplace transform pair, see *e.g.* [Doetsch 1974],

$$M_{1/2}(r) := \frac{1}{\sqrt{\pi}} \exp\left(-r^2/4\right) \div E_{1/2}(-s) := \exp\left(s^2\right) \operatorname{erfc}(s)\ ,$$

valid $\forall s \in \mathbb{C}$

Analogously, using the more general integral representation (F.2) of the standard Wright function, we can prove that in the case $\lambda = -\nu \in (-1, 0)$ and $\mathrm{Re}(\mu) > 0$, we get

$$W_{-\nu,\mu}(-r) \div E_{\nu,\mu+\nu}(-s)\,, \; 0 < \nu < 1\,. \qquad (F.25)$$

In the limit as $\nu \to 0^+$ (thus $\lambda \to 0^-$) we formally obtain the Laplace transform pair

$$W_{0^-,\mu}(-r) := \frac{\mathrm{e}^{-r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s+1} := E_{0,\mu}(-s)\,. \qquad (F.26)$$

Therefore, as $\lambda \to 0^{\pm}$, and $\mu = 1$ we note a sort of continuity in the results (F.23) and (F.26) since

$$W_{0,1}(-r) := \mathrm{e}^{-r} \div \frac{1}{(s+1)} = \begin{cases} (1/s)\, E_0(-1/s)\,, & |s| > 1\,; \\ E_0(-s)\,, & |s| < 1\,. \end{cases} \qquad (F.27)$$

We here point out the relevant *Laplace transform pairs related to the auxiliary functions of argument* $r^{-\nu}$, see [Mainardi (1994); (1996a); (1996b)],

$$\boxed{\frac{1}{r}\, F_\nu\,(1/r^\nu) = \frac{\nu}{r^{\nu+1}}\, M_\nu\,(1/r^\nu) \;\div\; \mathrm{e}^{\,-s^\nu}\,,\;\; 0<\nu<1\,.} \qquad (F.28)$$

$$\boxed{\frac{1}{\nu}\, F_\nu\,(1/r^\nu) = \frac{1}{r^{\nu}}\, M_\nu\,(1/r^\nu) \;\div\; \frac{\mathrm{e}^{\,-s^\nu}}{s^{1-\nu}}\,,\;\; 0<\nu<1\,.} \qquad (F.29)$$

We recall that the Laplace transform pairs in (F.28) were formerly considered by [Pollard (1946)], who provided a rigorous proof based on a formal result by [Humbert (1945)]. Later [Mikusinski (1959)] got a similar result based on his theory of operational calculus, and finally, albeit unaware of the previous results, [Buchen & Mainardi (1975)] derived the result in a formal way. We note, however, that all these Authors were not informed about the Wright functions.

Hereafter we like to provide two independent proofs of (F.28) carrying out the inversion of $\exp(-s^\nu)$, either by the complex Bromwich integral formula or by the formal series method. Similarly we can act for the Laplace transform pair (F.29).

For the complex integral approach we deform the Bromwich path $Br$ into the Hankel path $Ha$, that is equivalent to the original path, and we set $\sigma = sr$. Recalling (F.13)-(F.14), we get

$$\mathcal{L}^{-1}\left[\exp\left(-s^\nu\right)\right] = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\,sr - s^\nu}\, ds = \frac{1}{2\pi i\, r}\int_{Ha} \mathrm{e}^{\,\sigma - (\sigma/r)^\nu}\, d\sigma$$
$$= \frac{1}{r}\, F_\nu\left(1/r^\nu\right) = \frac{\nu}{r^{\nu+1}}\, M_\nu\left(1/r^\nu\right)\,.$$

Expanding in power series the Laplace transform and inverting term by term we formally get, after recalling (F.12)-(F.13):

$$\mathcal{L}^{-1}\left[\exp\left(-s^\nu\right)\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\mathcal{L}^{-1}\left[s^{\nu n}\right] = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\,\frac{r^{-\nu n-1}}{\Gamma(-\nu n)}$$
$$= \frac{1}{r}\, F_\nu\left(1/r^\nu\right) = \frac{\nu}{r^{\nu+1}}\, M_\nu\left(1/r^\nu\right)\,.$$

We note the relevance of Laplace transforms (F.24) and (F.28) in pointing out the non-negativity of the Wright function $M_\nu(x)$ and the complete monotonicity of the Mittag-leffler functions $E_\nu(-x)$ for $x > 0$ and $0 < \nu < 1$. In fact, since $\exp\,(-s^\nu)$ denotes the Laplace transform of a probability density (precisely , the extremal Lévy stable density of index $\nu$, see [Feller (1971)]), the L.H.S. of (F.28) must be non-negative, and so also the L.H.S of F(24). As a matter of fact the Laplace transform pair (F.24) shows, replacing $s$ by $x$, that the spectral representation of the Mittag-Leffler function $E_\nu(-x)$ is expressed in terms of the $M$-Wright function $M_\nu(r)$, that is:

$$E_\nu(-x) = \int_0^\infty \mathrm{e}^{-rx}\, M_\nu(r)\, dr\,, \; 0 < \nu < 1\,, \; x \geq 0\,. \quad (F.30)$$

We now recognize that Eq. (F.30) is consistent with a result derived by [Pollard (1948)].

It is instructive to compare the spectral representation of $E_\nu(-x)$ with that of the function $E_\nu(-t^\nu)$. We recall

$$E_\nu(-t^\nu) = \int_0^\infty \mathrm{e}^{-rt}\, K_\nu(r)\, dr\,, \; 0 < \nu < 1\,,\; t \ge 0\,, \qquad (F.31)$$

with *spectral function*

$$K_\nu(r) = \frac{1}{\pi}\,\frac{r^{\nu-1}\,\sin(\nu\pi)}{r^{2\nu} + 2\,r^\nu\,\cos\,(\nu\pi) + 1} = \frac{1}{\pi\,r}\,\frac{\sin(\nu\pi)}{r^\nu + r^{-\nu} + 2\,\cos\,(\nu\pi)}\,. \qquad (F.32)$$

The relationship between $M_\nu(r)$ and $K_\nu(r)$ is worth to be explored. Both functions are non-negative, integrable and normalized in $\mathbb{R}^+$, so they can be adopted in probability theory as density functions.

Whereas the transition $K_\nu(r) \to \delta(r-1)$ for $\nu \to 1$ is easy to be detected numerically in view of the explicit representation (F.32), the analogous transition $M_\nu(r) \to \delta(r-1)$ is quite a difficult matter in view of its series and integral representations. In this respect see the figure hereafter.

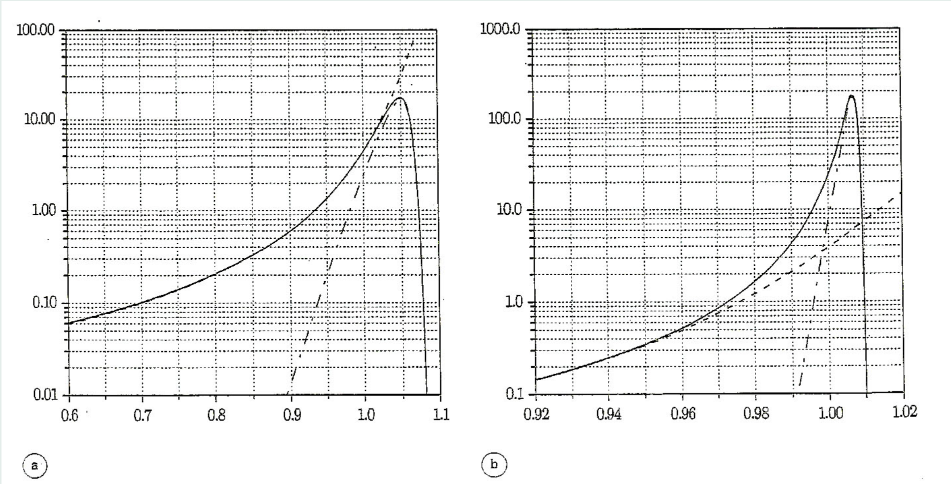


**Fig. 6** - Plots of $M_\nu(r)$ with $\nu = 1 - \epsilon$ around the maximum $r \approx 1$.

Here we compare the cases (a) $\epsilon = 0.01$ , (b) $\epsilon = 0.001$ , obtained by Pipkin's method (continuous line), 100 terms-series (dashed line) and the standard saddle-point method (dashed-dotted line).

# 8. Probability and fractional diffusion

For certain **stochastic processes of renewal type**, functions of Mittag-Leffler and Wright type can be adopted as **probability distributions**, see Mainardi-Gorenflo-Vivoli (FCAA 2005), where they are compared.

Here, we restrict our attention to the $M$-Wright functions with support both $\mathbb{R}^+$ and all of $\mathbb{R}$ (in symmetric way) that play fundamental roles in **stochastic processes of fractional diffusion**

The exponential decay for $x \to +\infty$ pointed out in Eqs (F.20)-(F.21) ensures that $M_\nu(x)$ is absolutely integrable in $\mathbb{R}^+$ and in $\mathbb{R}$. By recalling the Laplace transform pair (F.24) related to the Mittag-Leffler function, we get

$$\int_0^{+\infty} M_\nu(x)\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} M_\nu(|x|)\, dx = E_\nu(0) = 1\,. \qquad (F.33)$$

Being non-negative, $M_\nu(x)$ and $\frac{1}{2}M_\nu(|x|)$ can be interpreted as probability density functions in $\mathbb{R}^+$ and in $\mathbb{R}$, respectively. More generally, we can compute from (F.24) all the moments in $\mathbb{R}^+$, for $n = 1, 2 \ldots,$ as follows

$$\int_0^{+\infty} x^n \, M_\nu(x) \, dx = \lim_{s \to 0} \, (-1)^n \, \frac{d^n}{ds^n} \, E_\nu(-s) = \frac{\Gamma(n+1)}{\Gamma(\nu n + 1)} \, . \tag{F.34}$$

An alternative proof of Eqs. (F.33)-(F.34) can be sketched as follows:

$$\int_0^{+\infty} x^n \, M_\nu(x) \, dx = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\,\sigma} \left[ \int_0^{+\infty} \mathrm{e}^{\,-\sigma^\nu x} \, x^n \, dx \right] \frac{d\sigma}{\sigma^{1-\nu}}$$

$$= \frac{n!}{2\pi i} \int_{Ha} \frac{\mathrm{e}^{\,\sigma}}{\sigma^{\nu n+1}} \, d\sigma = \frac{\Gamma(n+1)}{\Gamma(\nu n + 1)} \, ,$$

where the exchange between the two integrals turns out to be legitimate.

## The Fourier transform of the $M$-Wright function.

$$\mathcal{F}\left[\tfrac{1}{2}M_\nu(|x|)\right] := \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, M_\nu(|x|)\, dx = \int_0^\infty \cos(\kappa x)\, M_\nu(x)\, dx = E_{2\nu}(-\kappa^2)\,.$$

$$\int_0^\infty \cos(\kappa x)\, M_\nu(x)\, dx = \sum_{n=0}^\infty (-1)^n \frac{\kappa^{2n}}{(2n)!} \int_0^\infty x^{2n}\, M_\nu(x)\, dx = \sum_{n=0}^\infty (-1)^n \frac{\kappa^{2n}}{\Gamma(2\nu n+1)} = E_{2\nu,1}(-\kappa^2)\,.$$

$$\int_0^\infty \sin(\kappa x)\, M_\nu(x)\, dx = \sum_{n=0}^\infty (-1)^n \frac{\kappa^{2n+1}}{(2n+1)!} \int_0^\infty x^{2n+1}\, M_\nu(x)\, dx = \sum_{n=0}^\infty (-1)^n \frac{\kappa^{2n+1}}{\Gamma(2\nu n+1+\nu)} = \kappa E_{2\nu,1+\nu}(-\kappa^2)\,.$$

## Relations with Lévy stable distributions

The term stable has been assigned by the French mathematician Paul Lévy, who in the 1920's years started a systematic research in order to generalize the celebrated *Central Limit Theorem* to probability distributions with infinite variance. For stable distributions we can assume the following

DEFINITION:

**If two independent real random variables with the same shape or type of distribution are combined linearly and the distribution of the resulting random variable has the same shape, the common distribution (or its type, more precisely) is said to be stable.**

The restrictive condition of stability enabled Lévy (and then other authors) to derive the **canonic form** for the Fourier transform of the densities of these distributions. Such transform in probability theory is known as **characteristic function**.

Here we follow the parameterization in [Feller (1952);(1971)] revisited in [Gorenflo & Mainardi (FCAA 1998)] and in [Mainardi-Luchko-Pagnini (FCAA 2001].

Denoting by $L_\alpha^\theta(x)$ a generic stable density in $\mathbb{R}$, where $\alpha$ is the **index of stability** and and $\theta$ the asymmetry parameter, improperly called **skewness**, its characteristic function reads:

$$L_\alpha^\theta(x) \div \widehat{L}_\alpha^\theta(\kappa) = \exp\left[-\psi_\alpha^\theta(\kappa)\right], \quad \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha \, e^{i(\text{sign}\,\kappa)\theta\pi/2}, \tag{F.35}$$

$$0 < \alpha \le 2, \; |\theta| \le \min\{\alpha, 2-\alpha\}.$$

We note that the allowed region for the parameters $\alpha$ and $\theta$ turns out to be a diamond in the plane $\{\alpha, \theta\}$ with vertices in the points $(0,0)$, $(1,1)$, $(1,-1)$, $(2,0)$, that we call the **Feller-Takayasu diamond**, see Fig. F.4. For values of $\theta$ on the border of the diamond (that is $\theta = \pm\alpha$ if $0 < \alpha < 1$, and $\theta = \pm(2-\alpha)$ if $1 < \alpha < 2$) we obtain the so-called **extremal stable densities**.

We note the **symmetry relation** $L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x)$, so that a stable density with $\theta = 0$ is symmetric

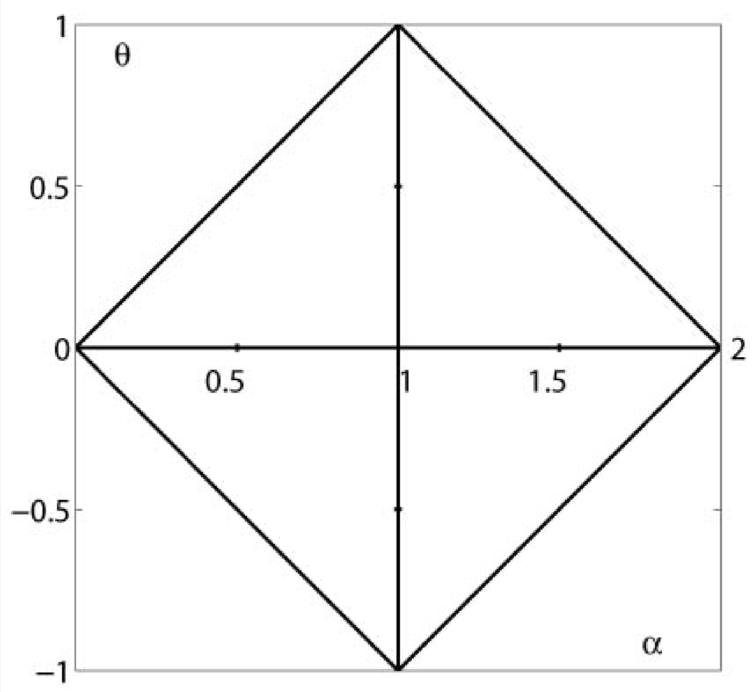


Figure 1: The Feller-Takayasu diamond for Lévy stable densities.

Stable distributions have noteworthy properties of which the interested reader can be informed from the existing literature. Here-after we recall some peculiar **Properties**:

- **The class of stable distributions possesses its own domain of attraction**, see *e.g.* [Feller (1971)].

- **Any stable density is unimodal and indeed bell-shaped**, *i.e.* its $n$-th derivative has exactly $n$ zeros in $\mathbb{R}$, see [Gawronski (1984)].

- **The stable distributions are self-similar and infinitely divisible**. These properties derive from the canonic form (F.35) through the scaling property of the Fourier transform.

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**Self-similarity** means

$$L_\alpha^\theta(x,t) \div \exp\left[-t\psi_\alpha^\theta(\kappa)\right] \Longleftrightarrow L_\alpha^\theta(x,t) = t^{-1/\alpha} L_\alpha^\theta(x/t^{1/\alpha})], \tag{F.36}$$

whre $t$ is a positive parameter. If $t$ is time, then $L_\alpha^\theta(x,t)$ is a spatial density evolving on time with self-similarity.

**Infinite divisibility** means that for every positive integer $n$, the characteristic function can be expressed as the $n$th power of some characteristic function, so that any stable distribution can be expressed as the $n$-fold convolution of a stable distribution of the same type. Indeed, taking in (F.35) $\theta = 0$, without loss of generality, we have

$$e^{-t|\kappa|^\alpha} = \left[e^{-(t/n)|\kappa|^\alpha}\right]^n \Longleftrightarrow L_\alpha^0(x,t) = \left[L_\alpha^0(x,t/n)\right]^{*n}, \tag{F.37}$$

where

$$\left[L_\alpha^0(x,t/n)\right]^{*n} := L_\alpha^0(x,t/n) * L_\alpha^0(x,t/n) * \cdots * L_\alpha^0(x,t/n)$$

is the multiple Fourier convolution in $\mathbb{R}$ with $n$ identical terms.

Only in special cases we get well-known probability distributions.

For $\alpha = 2$ (so $\theta = 0$), we recover the **Gaussian pdf**, that turns out to be the only stable density with finite variance, and more generally with finite moments of any order $\delta \geq 0$. In fact

$$L_2^0(x) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-x^2/4} .$$

All the other stable densities have finite absolute moments of order $\delta \in [-1, \alpha)$.

For $\alpha = 1$ and $|\theta| < 1$, we get

$$L_1^\theta(x) = \frac{1}{\pi} \frac{\cos(\theta\pi/2)}{[x + \sin(\theta\pi/2)]^2 + [\cos(\theta\pi/2)]^2} , \qquad (2.36)$$

which for $\theta = 0$ includes the **Cauchy-Lorentz pdf**,

$$L_1^0(x) = \frac{1}{\pi} \frac{1}{1 + x^2} .$$

In the limiting cases $\theta = \pm 1$ for $\alpha = 1$ we obtain the **singular Dirac pdf's**

$$L_1^{\pm 1}(x) = \delta(x \pm 1) .$$

In general we must recall the power series expansions provided in [Feller (1971)]. We restrict our attention to $x > 0$ since the evaluations for $x < 0$ can be obtained using the symmetry relation.

The convergent expansions of $L_\alpha^\theta(x)$ $(x > 0)$ turn out to be
for $0 < \alpha < 1\,,\quad |\theta| \le \alpha$ :

$$L_\alpha^\theta(x) = \frac{1}{\pi\, x} \sum_{n=1}^{\infty} (-x^{-\alpha})^n \, \frac{\Gamma(1+n\alpha)}{n!} \, \sin\left[\frac{n\pi}{2}(\theta-\alpha)\right] \; ; \qquad (F.38)$$

for $1 < \alpha \le 2\,,\quad |\theta| \le 2-\alpha$ :

$$L_\alpha^\theta(x) = \frac{1}{\pi\, x} \sum_{n=1}^{\infty} (-x)^n \, \frac{\Gamma(1+n/\alpha)}{n!} \, \sin\left[\frac{n\pi}{2\alpha}(\theta-\alpha)\right] \, . \qquad (F.39)$$

From the series in (F.38) and the symmetry relation we note that the extremal stable densities for $0 < \alpha < 1$ are unilateral, precisely vanishing for $x > 0$ if $\theta = \alpha$, vanishing for $x < 0$ if $\theta = -\alpha$. In particular the unilateral extremal densities $L_\alpha^{-\alpha}(x)$ with $0 < \alpha < 1$ have as Laplace transform $\exp\,(-s^\alpha)$.

From a comparison between the series expansions in (F.38)-(F.39) and in (F.14)-(F.15), we recognize that for $x > 0$ **the auxiliary functions of the Wright type are related to the extremal stable densities** as follows, see [Mainardi-Tomirotti (1997)],

$$L_\alpha^{-\alpha}(x) = \frac{1}{x}\, F_\alpha(x^{-\alpha}) = \frac{\alpha}{x^{\alpha+1}}\, M_\alpha(x^{-\alpha})\,, \quad 0 < \alpha < 1\,, \quad (F.40)$$

$$L_\alpha^{\alpha-2}(x) = \frac{1}{x}\, F_{1/\alpha}(x) = \frac{1}{\alpha}\, M_{1/\alpha}(x)\,, \quad 1 < \alpha \le 2\,. \quad (F.41)$$

In Eqs. (F.40)-(F.41), for $\alpha = 1$, the skewness parameter turns out to be $\theta = -1$, so we get the singular limit $L_1^{-1}(x) = M_1(x) = \delta(x-1)$.

More generally, all (regular) stable densities, given in Eqs. (F.38)-(F.39), were recognized to belong to the class of Fox $H$-functions, as formerly shown by [Schneider (LNP 1986)], see also [Mainardi-Pagnini-Saxena (2003)].

## Subordination formula and Fractional Diffusion

We now consider $M$-Wright functions as spatial probability densities evolving in time with self-similarity, that is

$$M_\nu(x,t) := t^{-\nu}\, M_\nu(xt^{-\nu})\,, \quad x, t \geq 0\,. \qquad (F.42)$$

These $M$-Wright functions are relevant for their composition rules proved in [Mainardi-Luchko-Pagnini (FCAA 2001)] and more generally in [Mainardi-Pagnini-Gorenflo (FCAA 2003)] by using the Mellin Transforms.

The main statement can be summarized with the THEOREM: *Let $M_\lambda(x;t)$, $M_\mu(x;t)$ and $M_\nu(x;t)$ be $M$-Wright functions of orders $\lambda, \mu, \nu \in (0,1)$ respectively, then the following composition formula holds true for any $x, t \geq 0$:*

$$M_\nu(x,t) = \int_0^\infty M_\lambda(x;\tau)\, M_\mu(\tau;t)\, d\tau\,, \ \ \textit{with} \ \ \nu = \lambda\,\mu\,. \qquad (F.43)$$

The above equation is also intended as a **subordination formula** because it can be used to define subordination among self-similar stochastic processes (with independent increments), that properly generalize the most popular Gaussian processes, to which they reduce for $\nu = 1/2$.

These more general processes are governed by time-fractional diffusion equations, as shown in recent papers of our research group, see Mura-Pagnini (JPhysA 2008], [Mura-Taqqu-Mainardi (PhysicaA 2008). Mura-Mainardi (ITSF 2009)

These general processes are referred to as **Generalized grey Brownian Motions**, that include both **Gaussian Processes** (standard Brownian motion, fractional Brownian motion) and **non-Gaussian Processes** (Schneider's grey Brownian motion), to which the interested reader is referred for details.

## The time-fractional diffusion equation

There exist three equivalent forms of the time-fractional diffusion equation of a single order, two with fractional derivative and one with fractional integral, provided we refer to the standard initial condition $u(x,0) = u_0(x)$.

Taking a real number $\beta \in (0,1)$, the time-fractional diffusion equation of order $\beta$ in the Riemann-Liouville sense reads

$$\frac{\partial u}{\partial t} = K_\beta \, D_t^{1-\beta} \, \frac{\partial^2 u}{\partial x^2} \,, \qquad (FD.1)$$

in the Caputo sense reads

$$ {}_*D_t^\beta u = K_\beta \, \frac{\partial^2 u}{\partial x^2} \,, \qquad (FD.2)$$

and in integral form

$$u(x,t) = u_0(x) + K_\beta \, \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \, \frac{\partial^2 u(x,\tau)}{\partial x^2} \, d\tau \,. \quad (FD.3)$$

where $K_\beta$ is a sort of fractional diffusion coefficient of dimensions $[K_\beta] = [L]^2[T]^{-\beta} = cm^2/sec^\beta$.

The fundamental solution (or **Green function**) $\mathcal{G}_\beta(x,t)$ for the equivalent Eqs. (FD.1) - (FD.3), that is the solution coresponding to the initial condition

$$\mathcal{G}_\beta(x,0^+) = u_0(x) = \delta(x) \qquad (FD.4)$$

can be expressed in terms of the $M$-Wright function

$$\mathcal{G}_\beta(x,t) = \frac{1}{2}\frac{1}{\sqrt{K_\beta}\,t^{\beta/2}}\,M_{\beta/2}\left(\frac{|x|}{\sqrt{K_\beta}\,t^{\beta/2}}\right)\,. \qquad (FD.5)$$

The corresponding variance can be promptly obtained

$$\sigma^2_\beta(t) := \int_{-\infty}^{+\infty} x^2\,\mathcal{G}_\beta(x,t)\,dx = \frac{2}{\Gamma(\beta+1)}\,K_\beta\,t^\beta\,. \qquad (5.16)$$

As a consequence, for $0<\beta<1$ the variance is consistent with a process of **slow diffusion** with similarity exponent $H=\beta/2$.

**Appendix A** The fundamental solution $\mathcal{G}_\beta(x,t)$ for the time-fractional diffusion equation can be obtained by applying in sequence the Fourier and Laplace transforms to any form chosen among Eqs. (FD.1)-(FD.3). Let us devote our attention to the integral form (FD.3) using non-dimensional variables by setting $K_\beta = 1$ and adopting the notation $J_t^\beta$ for the fractional integral. Then, our Cauchy problem reads

$$\mathcal{G}_\beta(x,t) = \delta(x) + J_t^\beta \frac{\partial^2 \mathcal{G}_\beta}{\partial x^2}(x,t)\,. \qquad (A.1)$$

In the Fourier-Laplace domain, after applying formula for the Laplace transform of the fractional integral and observing $\widehat{\delta}(\kappa) \equiv 1$, we get

$$\widehat{\widetilde{G}}_\beta(\kappa,s) = \frac{1}{s} - \frac{\kappa^2}{s^\beta}\,\widehat{\widetilde{G}}_\beta(\kappa,s)\,,$$

from which

$$\widehat{\widetilde{\mathcal{G}}}_\beta(\kappa,s) = \frac{s^{\beta-1}}{s^\beta+\kappa^2}\,,\; 0<\beta\le 1\,,\; \mathrm{Re}(s)>0\,,\; \kappa\in\mathbb{R}\,. \qquad (A.2)$$

**Strategy (S1):** Recalling the Fourier transform pair

$$\frac{a}{b+\kappa^2} \stackrel{\mathcal{F}}{\leftrightarrow} \frac{a}{2b^{1/2}}\, \mathrm{e}^{\,-|x|b^{1/2}}\,, \quad a, b > 0\,, \tag{A.3}$$

and setting $a = s^{\beta-1}$, $b = s^\beta$, we get

$$\widetilde{\mathcal{G}_\beta}(x,s) = \frac{1}{2} s^{\beta/2-1}\, \mathrm{e}^{\,-|x|s^{\beta/2}}\,. \tag{A.4}$$

**Strategy (S2):** Recalling the Laplace transform pair

$$\frac{s^{\beta-1}}{s^\beta + c} \stackrel{\mathcal{L}}{\leftrightarrow} E_\beta(-ct^\beta)\,, \quad c > 0\,, \tag{A.5}$$

and setting $c = \kappa^2$, we have

$$\widehat{G_\beta}(\kappa,t) = E_\beta(-\kappa^2 t^\beta)\,. \tag{A.6}$$

Both strategies lead to the result

$$\mathcal{G}_\beta(x,t) = \frac{1}{2} M_{\beta/2}(|x|,t) = \frac{1}{2}\, t^{-\beta/2}\, M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right)\,, \tag{A.7}$$

consistent with Eq. (FD.5).

## The time-fractional drift equation

Let us finally note that the $M$-Wright function does appear also in the fundamental solution of the time-fractional drift equation. Writing this equation in non-dimensional form and adopting the Caputo derivative we have

$$ {}_*D_t^{\beta} u(x,t) = -\frac{\partial}{\partial x} u(x,t)\,, \quad -\infty < x < +\infty\,,\ t \geq 0\,, \quad (B.1) $$

where $0 < \beta < 1$ and $u(x,0^+) = u_0(x)$. When $u_0(x) = \delta(x)$ we obtain the fundamental solution (Green function) that we denote by $\mathcal{G}_\beta^*(x,t)$. Following the approach of Appendix A, we show that

$$ \mathcal{G}_\beta^*(x,t) = \begin{cases} t^{-\beta}\, M_\beta\left(\dfrac{x}{t^\beta}\right), & x > 0\,, \\ 0\,, & x < 0\,, \end{cases} \quad (B.2) $$

that for $\beta = 1$ reduces to the right running pulse $\delta(x-t)$ for $x > 0$.

In the Fourier-Laplace domain, after applying the formula for the Laplace transform of the Caputo fractional derivative and observing $\widehat{\delta}(\kappa) \equiv 1$, we get

$$s^{\beta}\,\widehat{\widetilde{G^*_\beta}}(\kappa, s) - s^{\beta-1} = +i\kappa\,\widehat{\widetilde{G^*_\beta}}(\kappa, s)\,,$$

from which

$$\widehat{\widetilde{\mathcal{G}^*_\beta}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} - i\kappa}\,,\; 0 < \beta \le 1\,, \quad \Re(s) > 0\,,\; \kappa \in \mathbb{R}\,. \quad (B.3)$$

Like in Appendix A, to determine the Green function $\mathcal{G}^*_\beta(x, t)$ in the space-time domain we can follow two alternative strategies related to the order in carrying out the inversions in (B.3).
(S1) : invert the Fourier transform getting $\widetilde{\mathcal{G}_\beta}(x, s)$ and then invert the remaining Laplace transform;
(S2) : invert the Laplace transform getting $\widehat{G^*_\beta}(\kappa, t)$ and then invert the remaining Fourier transform.

**Strategy (S1):** Recalling the Fourier transform pair

$$\frac{a}{b - i\kappa} \stackrel{\mathcal{F}}{\leftrightarrow} \frac{a}{b}\,\mathrm{e}^{-xb}\,, \quad a, b > 0\,,\; x > 0\,, \qquad (B.4)$$

and setting $a = s^{\beta-1}$, $b = s^{\beta}$, we get

$$\widetilde{\mathcal{G}^*_\beta}(x,s) = s^{\beta-1}\,\mathrm{e}^{\,-xs^\beta}\,. \qquad (B.5)$$

**Strategy (S2):** Recalling the Laplace transform pair

$$\frac{s^{\beta-1}}{s^\beta + c} \stackrel{\mathcal{L}}{\leftrightarrow} E_\beta(-ct^\beta)\,, \quad c > 0\,, \qquad (B.6)$$

and setting $c = -i\kappa$, we have

$$\widehat{G^*_\beta}(\kappa,t) = E_\beta(i\kappa t^\beta)\,. \qquad (B.7)$$

Both strategies lead to the result (B.2).

In view of Eq. (F.40) we also recall that the $M$-Wright function is related to the unilateral **extremal stable density** of index $\beta$. Then, using our notation for stable densities, we write our Green function as

$$\mathcal{G}^*_\beta(x,t) = \frac{t}{\beta}\,x^{-1-1/\beta}\,L_\beta^{-\beta}\left(tx^{-1/\beta}\right)\,, \qquad (B.8)$$

# 9. Essentials of Fractional Calculus in $\mathbb{R}^+$

The **Riemann-Liouville fractional integral** of order $\mu > 0$ is defined as

$$ {}_tJ^\mu\, f(t) := \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1}\, f(\tau)\, d\tau\,, \quad \mu > 0\,, \qquad (A.1) $$

$$ \Gamma(\mu) := \int_0^\infty \mathrm{e}^{-u} u^{\mu-1}\, du\,, \quad \Gamma(n+1) = n! \quad \textbf{Gamma function}. $$

By convention ${}_tJ^0 = I$ (Identity operator). We can prove

$$ {}_tJ^\mu\, {}_tJ^\nu = {}_tJ^\nu\, {}_tJ^\mu = {}_tJ^{\mu+\nu}\,,\ \mu, \nu \geq 0\,, \quad \textbf{semigroup property} \qquad (A.2) $$

$$ {}_tJ^\mu\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\mu)}\, t^{\gamma+\mu}\,, \quad \mu \geq 0\,, \quad \gamma > -1\,, \quad t > 0\,. \qquad (A.3) $$

The **fractional derivative** of order $\mu > 0$ in the **Riemann-Liouville** sense is defined v as the operator ${}_tD^\mu$

$$ {}_tD^\mu \, {}_tJ^\mu = I \,, \quad \mu > 0 \,. \tag{A.4} $$

If $m$ denotes the positive integer such that $m-1 < \mu \le m$ , we recognize from Eqs. (A.2) and (A.4)

$$ {}_tD^\mu \, f(t) := {}_tD^m \, {}_tJ^{m-\mu} \, f(t) \,, \tag{A.5} $$

hence

$$ {}_tD^\mu f(t) = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t \dfrac{f(\tau)\,d\tau}{(t-\tau)^{\mu+1-m}}\right], & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \tag{A.5'} $$

For completion ${}_tD^0 = I$. The semigroup property is no longer valid but

$$ {}_tD^\mu \, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)} \, t^{\gamma-\mu} \,, \quad \mu \ge 0 \,, \quad \gamma > -1 \,, \quad t > 0 \,. \tag{A.6} $$

However, the property ${}_tD^\mu = {}_tJ^{-\mu}$ is not generally valid!

The **fractional derivative** of order $\mu \in (m-1, m]$ $(m \in \mathbb{N})$ in the **Caputo** sense is defined as the operator ${}_tD_*^{\mu}$ such that

$$ {}_tD_*^{\mu}\, f(t) := {}_tJ^{m-\mu}\, {}_tD^{m}\, f(t)\,, \tag{A.7} $$

hence

$$ {}_tD_*^{\mu}\, f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\mu+1-m}}\,, & m-1<\mu<m, \\ \dfrac{d^m}{dt^m} f(t)\,, & \mu = m. \end{cases} \tag{A.7'} $$

Thus, when the order is not integer the two fractional derivatives differ in that the derivative of order $m$ does not generally commute with the fractional integral.

We point out that the **Caputo fractional derivative** satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than $m$, if its order $\mu$ is such that $m-1<\mu\le m$.

Gorenflo and Mainardi (1997) have shown the essential relationships between the two fractional derivatives (when both of them exist),

$$
{}_tD_*^\mu\, f(t) = \begin{cases} {}_tD^\mu \left[ f(t) - \displaystyle\sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^k}{k!} \right], \\ {}_tD^\mu\, f(t) - \displaystyle\sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)\, t^{k-\mu}}{\Gamma(k-\mu+1)}, \end{cases} \quad m-1 < \mu < m\,. \tag{A.8}
$$

In particular, if $m = 1$ we have

$$
{}_tD_*^\mu\, f(t) = \begin{cases} {}_tD^\mu \left[ f(t) - f(0^+) \right], \\ {}_tD^\mu\, f(t) - \dfrac{f(0^+)\, t^{-\mu}}{\Gamma(1-\mu)}, \end{cases} \quad 0 < \mu < 1\,. \tag{A.9}
$$

The **Caputo fractional derivative**, represents a sort of regularization in the time origin for the **Riemann-Liouville fractional derivative**. We note that for its existence all the limiting values $f^{(k)}(0^+) := \lim_{t\to 0^+} f^{(k)}(t)$ are required to be finite for $k = 0, 1, 2 \ldots m-1$.

We observe the different behaviour of the two fractional derivatives at the end points of the interval $(m-1, m)$ namely when the order is any positive integer: whereas ${}_tD^\mu$ is, with respect to its order $\mu$, an operator continuous at any positive integer, ${}_tD_*^\mu$ is an operator left-continuous since

$$\begin{cases} \lim\limits_{\mu\to(m-1)^+} {}_tD_*^\mu f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+), \\ \lim\limits_{\mu\to m^-} {}_tD_*^\mu f(t) = f^{(m)}(t). \end{cases} \quad (A.10)$$

We also note for $m-1 < \mu \le m$,

$${}_tD^\mu f(t) = {}_tD^\mu g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\mu-j}, \quad (A.11)$$

$${}_tD_*^\mu f(t) = {}_tD_*^\mu g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}. \quad (A.12)$$

In these formulae the coefficients $c_j$ are arbitrary constants.

We point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the **Laplace transformation**.

Writing the Laplace transform of a sufficiently well-behaved function $f(t)$ $(t \geq 0)$ as

$$\mathcal{L}\{f(t); s\} = \widetilde{f}(s) := \int_0^\infty \mathrm{e}^{-st} f(t)\, dt\,,$$

the known rule for the ordinary derivative of integer order $m \in \mathbb{N}$ is

$$\mathcal{L}\{{}_tD^m f(t); s\} = s^m \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k} f^{(k)}(0^+)\,, \quad m \in \mathbb{N}\,,$$

where

$$f^{(k)}(0^+) := \lim_{t \to 0^+} {}_tD^k f(t)\,.$$

For the **Caputo derivative** of order $\mu \in (m-1, m]$ $(m \in \mathbb{N})$ we have

$$\mathcal{L}\{\, {}_tD_*^{\mu}\, f(t); s\} = s^{\mu}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k}\, f^{(k)}(0^+)\,, \qquad f^{(k)}(0^+) := \lim_{t \to 0^+} {}_tD^k f(t)\,. \tag{A.13}$$

The corresponding rule for the **Riemann-Liouvile derivative** of order $\mu$ is

$$\mathcal{L}\{\, {}_tD_t^{\mu}\, f(t); s\} = s^{\mu}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k}\, g^{(k)}(0^+)\,, \qquad g^{(k)}(0^+) := \lim_{t \to 0^+} {}_tD^k g(t)\,, \quad g(t) := {}_tJ^{m-\mu}\, f(t)\,. \tag{A.14}$$

Thus the rule (A.14) is more cumbersome to be used than (A.13) since it requires initial values concerning an extra function $g(t)$ related to the given $f(t)$ through a fractional integral.

However, when all the limiting values $f^{(k)}(0^+)$ are finite and the order is not integer, we can prove by that all $g^{(k)}(0^+)$ vanish so that the formula (A.14) simplifies into

$$\mathcal{L}\{\,{}_tD^{\mu}\,f(t);s\} = s^{\mu}\,\widetilde{f}(s)\,, \quad m-1 < \mu < m\,. \tag{A.15}$$

For this proof it is sufficient to apply the Laplace transform to Eq. (A.8), by recalling that

$$\mathcal{L}\{t^{\nu};s\} = \Gamma(\nu+1)/s^{\nu+1}\,, \quad \nu > -1\,, \tag{A.16}$$

and then to compare (A.13) with (A.14).

For more details on the theory and applications of fractional calculus we recommend to consult in addition to the well-known books by Samko, Kilbas & Marichev (1993), by Miller & Ross (1993), by Podlubny (1999), those appeared in the last few years, by Kilbas, Srivastava & Trujillo (2006), West, Bologna & Grigolini (2003), Zaslavsky (2005), Mainardi(2010), Diethelm (2010).

## Notes on the auxiliary Wright functions

In early nineties, in his former analysis of fractional equations interpolating diffusion and wave-propagation, the present Author, see *e.g.* [Mainardi (WASCOM 1993)], has introduced the functions of the Wright type

$$F_\nu(z) := W_{-\nu,0}(-z)\,, \quad M_\nu(z) := W_{-\nu,1-\nu}(-z)$$

with $0 < \nu < 1$, inter-related through $F_\nu(z) = \nu\, z\, M_\nu(z)$ to characterize the solutions for typical boundary value problems.

Being in that time only aware of the Bateman project where the parameter $\lambda$ of the Wright function $W_{\lambda,\mu}(z)$ was erroneously restricted to non-negative values, the Author thought to have extended the original Wright function, in an original way, calling $F_\nu$ and $M_\nu$ *auxiliary functions*.

Presumably for this reason the function $M_\nu$ is referred as the *Mainardi function* in the book by Podlubny (Academic Press 1999) and in some papers including *e.g.* Balescu (CSF 2007).

It was Professor Stanković, during the presentation of the paper [Mainardi-Tomirotti (TMSF1994] at the Conference *Transform Methods and Special Functions, Sofia 1994*, who informed the Author that this extension for $-1 < \mu < 0$ was already made just by Wright himself in 1940 (following his previous papers in 1930's). In his paper devoted to the 80-th birthday of Prof. Stanković, see [Mainardi-Gorenflo-Vivoli (FCAA 2005)], the Author took the occasion to renew his personal gratitude to Prof. Stanković for this earlier information that has induced him to study the original papers by Wright and work (also in collaboration) on the functions of the Wright type for further applications, see *e.g.* [Gorenflo-Luchko-Mainardi (1999); (2000)] and [Mainardi-Pagnini (2003)].

For more mathematical details on the functions of the Wright type, the reader may be referred to the article by [Kilbas-Saigo-Trujillo (FCAA 2002)] and references therein. For the numerical point of view we like to point out the recent paper by [Luchko (FCAA 2008)], where algorithms are provided for computation of the Wright function on the real axis with prescribed accuracy.

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