On Multiple Zeros of Bernoulli Polynomials

Karl Dilcher

Dalhousie University, Halifax

“Special Functions in the 21st Century"
Washington, DC, April 6, 2011

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$; $B_{2n+1} = 0$ for $n \geq 1$.

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \ldots$; $B_{2n+1} = 0$ for $n \geq 1$.

- $B_n \in \mathbb{Q}$ for all n.

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $\ldots$; $B_{2n+1} = 0$ for $n \geq 1$.

- $B_n \in \mathbb{Q}$ for all n.
- Denominators are completely determined (see later)

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $\ldots$; $B_{2n+1} = 0$ for $n \geq 1$.

- $B_n \in \mathbb{Q}$ for all n.
- Denominators are completely determined (see later)
- Numerators are quite mysterious and deep.

Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $\ldots$; $B_{2n+1} = 0$ for $n \geq 1$.

- $B_n \in \mathbb{Q}$ for all n.
- Denominators are completely determined (see later)
- Numerators are quite mysterious and deep.

Applications in number theory: E.g.,

- Euler's formula

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad (n \geq 1).$$

• Related:

$$\zeta(1-n) = -\frac{B_n}{n} \quad (n \geq 2).$$

(Trivial zeros of $\zeta(s)$).

• Related:

$$\zeta(1-n) = -\frac{B_n}{n} \quad (n \geq 2).$$

(Trivial zeros of $\zeta(s)$).

• *Kummer's Theorem*:
Let p be an odd prime. If p does not divide the numerator of one of $B_2, B_4, \ldots, B_{p-3}$, then the equation

$$x^p + y^p = z^p$$

has no solutions in integers x, y, z satisfying $p \nmid xyz$.

• Related:

$$\zeta(1-n) = -\frac{B_n}{n} \quad (n \geq 2).$$

(Trivial zeros of $\zeta(s)$).

• *Kummer's Theorem*:
Let p be an odd prime. If p does not divide the numerator of one of $B_2, B_4, \dots, B_{p-3}$, then the equation

$$x^p + y^p = z^p$$

has no solutions in integers x, y, z satisfying $p \nmid xyz$.

In other words: The First Case of FLT is true.

Bernoulli polynomials:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi,$$

Bernoulli polynomials:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi,$$

or equivalently

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}.$$

Bernoulli polynomials:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi,$$

or equivalently

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}.$$

Obvious connection with Bernoulli numbers:

$$B_n(0) = B_n(1) = B_n, \qquad (n \geq 2)$$

Bernoulli polynomials:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi,$$

or equivalently

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}.$$

Obvious connection with Bernoulli numbers:

$$B_n(0) = B_n(1) = B_n, \qquad (n \geq 2)$$

Functional equation:

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

Bernoulli polynomials:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi,$$

or equivalently

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}.$$

Obvious connection with Bernoulli numbers:

$$B_n(0) = B_n(1) = B_n, \qquad (n \geq 2)$$

Functional equation:

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

This gives rise to numerous applications; e.g.,

$$1^n + 2^n + \ldots + x^n = \frac{1}{n+1}\left(B_{n+1}(x+1) - B_{n+1}\right).$$

Asymptotic Behaviour

Let $T_n(z)$ be the nth degree Taylor polynomial (about 0) of $\cos z$ (when n is even) and of $\sin z$ (when n is odd).

Asymptotic Behaviour

Let $T_n(z)$ be the nth degree Taylor polynomial (about 0) of $\cos z$ (when n is even) and of $\sin z$ (when n is odd).

Theorem (K.D., 1987)

For all $z \in \mathbb{C}$ and $n \geq 2$ we have

$$\left|(-1)^{\lfloor n/2 \rfloor}\frac{(2\pi)^n}{2n!}B_n(z+\tfrac{1}{2}) - T_n(2\pi z)\right| < 2^{-n}\exp(4\pi|z|).$$

Asymptotic Behaviour

Let $T_n(z)$ be the nth degree Taylor polynomial (about 0) of $\cos z$ (when n is even) and of $\sin z$ (when n is odd).

Theorem (K.D., 1987)

For all $z \in \mathbb{C}$ and $n \geq 2$ we have

$$\left|(-1)^{\lfloor n/2 \rfloor}\frac{(2\pi)^n}{2n!}B_n(z+\tfrac{1}{2}) - T_n(2\pi z)\right| < 2^{-n}\exp(4\pi|z|).$$

Corollary

We have uniformly on compact subsets of $\mathbb{C}$,

$$(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}(z) \to \cos(2\pi z),$$
$$(-1)^{k-1}\frac{(2\pi)^{2k+1}}{2(2k+1)!}B_{2k+1}(z) \to \sin(2\pi z).$$

Zeros

As a consequence, the real zeros of the Bernoulli polynomials converge to the zeros of $\cos(2\pi z)$, resp. $\sin(2\pi z)$.

Zeros

As a consequence, the real zeros of the Bernoulli polynomials converge to the zeros of $\cos(2\pi z)$, resp. $\sin(2\pi z)$.

This had been known before (Lense, 1934; Inkeri, 1959).

Zeros

As a consequence, the real zeros of the Bernoulli polynomials converge to the zeros of $\cos(2\pi z)$, resp. $\sin(2\pi z)$.

This had been known before (Lense, 1934; Inkeri, 1959).

It also gives an indication (though not a proof) that the complex zeros behave like those of the polynomials $T_n(z)$ (studied by Szegő, 1924).

Zeros

As a consequence, the real zeros of the Bernoulli polynomials converge to the zeros of $\cos(2\pi z)$, resp. $\sin(2\pi z)$.

This had been known before (Lense, 1934; Inkeri, 1959).

It also gives an indication (though not a proof) that the complex zeros behave like those of the polynomials $T_n(z)$ (studied by Szegő, 1924).

What was proven, though, is the existence of a parabolic zero-free region (K.D., 1983/88).

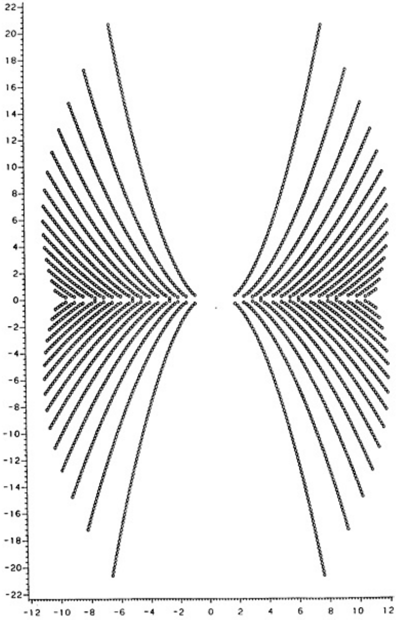
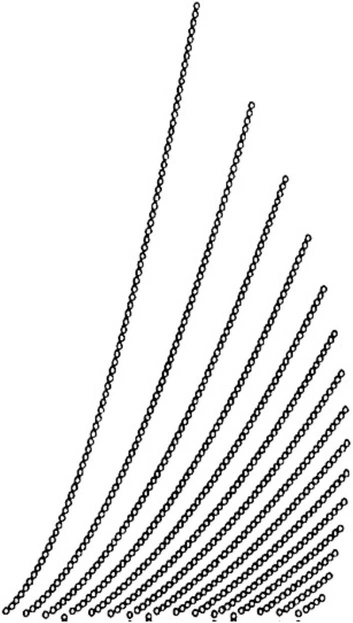


Figure 2: Complex Zeros of $E_n(x)$ $\quad 6 \le n \le 83.$



Why study zeros of Bernoulli polynomials?

Why study zeros of Bernoulli polynomials?

• Because they are there;

Why study zeros of Bernoulli polynomials?
- Because they are there;
- there are actually applications:

Why study zeros of Bernoulli polynomials?
• Because they are there;
• there are actually applications:

To show that for fixed $k \geq 2$ the diophantine equation

$$1^k + 2^k + \ldots + x^k = y^z$$

has at most finitely many solutions in x, y, z, one needs to have some knowledge of the zeros of the polynomial (in x) on the left.

Why study zeros of Bernoulli polynomials?
- Because they are there;
- there are actually applications:

To show that for fixed $k \geq 2$ the diophantine equation

$$1^k + 2^k + \ldots + x^k = y^z$$

has at most finitely many solutions in x, y, z, one needs to have some knowledge of the zeros of the polynomial (in x) on the left.

But this is, essentially, a Bernoulli polynomial.

Why study zeros of Bernoulli polynomials?
- Because they are there;
- there are actually applications:

To show that for fixed $k \geq 2$ the diophantine equation

$$1^k + 2^k + \ldots + x^k = y^z$$

has at most finitely many solutions in x, y, z, one needs to have some knowledge of the zeros of the polynomial (in x) on the left.

But this is, essentially, a Bernoulli polynomial.

This equation, and generalizations, have been extensively studied during the past 20 years.

Multiple zeros

Main topic of this talk:
Can Bernoulli polynomials have multiple zeros?

Multiple zeros

Main topic of this talk:
Can Bernoulli polynomials have multiple zeros?

This was partly answered by Brillhart:

Theorem (Brillhart, 1969)

(1) $B_{2n+1}(x)$ *has no multiple zeros for any* $n \geq 0$.

Multiple zeros

Main topic of this talk:
Can Bernoulli polynomials have multiple zeros?

This was partly answered by Brillhart:

Theorem (Brillhart, 1969)

(1) $B_{2n+1}(x)$ *has no multiple zeros for any* $n \geq 0$.

(2) *Any multiple zero of* $B_{2n}(x)$ *must be a zero of* $x^2 - x - b$*, with* b *a positive odd integer.*

Multiple zeros

Main topic of this talk:
Can Bernoulli polynomials have multiple zeros?

This was partly answered by Brillhart:

Theorem (Brillhart, 1969)

(1) *$B_{2n+1}(x)$ has no multiple zeros for any $n \geq 0$.*

(2) *Any multiple zero of $B_{2n}(x)$ must be a zero of $x^2 - x - b$, with b a positive odd integer.*

The main result is

Theorem (K.D., 2008)

$B_{2n}(x)$ has no multiple zeros.

Sketch of Proof

Some other elementary properties of Bernoulli polymomials:

$$B_n(\tfrac{1}{2}) = (2^{1-n} - 1)B_n,$$
$$B_n'(x) = nB_{n-1}(x).$$

Sketch of Proof

Some other elementary properties of Bernoulli polymomials:

$$B_n(\tfrac{1}{2}) = (2^{1-n} - 1)B_n,$$
$$B_n'(x) = nB_{n-1}(x).$$

With these, a Taylor expansion now gives

$$B_{2m}(x) = \sum_{j=0}^{m} \binom{2m}{2j}(2^{1-2j} - 1)(x - \tfrac{1}{2})^{2(m-j)} B_{2j}. \tag{1}$$

Sketch of Proof

Some other elementary properties of Bernoulli polymomials:

$$B_n(\tfrac{1}{2}) = (2^{1-n} - 1)B_n,$$
$$B_n'(x) = nB_{n-1}(x).$$

With these, a Taylor expansion now gives

$$B_{2m}(x) = \sum_{j=0}^{m} \binom{2m}{2j} (2^{1-2j} - 1)(x - \tfrac{1}{2})^{2(m-j)} B_{2j}. \qquad (1)$$

Let x_b be a zero of $x^2 - x - b$. Then

$$4(x_b - \tfrac{1}{2})^2 = 4x_b^2 - 4x_b + 1 = 4b + 1,$$

Sketch of Proof

Some other elementary properties of Bernoulli polymomials:

$$B_n(\tfrac{1}{2}) = (2^{1-n} - 1)B_n,$$
$$B_n'(x) = nB_{n-1}(x).$$

With these, a Taylor expansion now gives

$$B_{2m}(x) = \sum_{j=0}^{m} \binom{2m}{2j} (2^{1-2j} - 1)(x - \tfrac{1}{2})^{2(m-j)} B_{2j}. \qquad (1)$$

Let x_b be a zero of $x^2 - x - b$. Then

$$4(x_b - \tfrac{1}{2})^2 = 4x_b^2 - 4x_b + 1 = 4b + 1,$$

and with (1) we get

$$2^{2m} B_{2m}(x_b) = \sum_{j=0}^{m} \binom{2m}{2j} (4b+1)^{m-j} (2 - 2^{2j}) B_{2j}. \qquad (2)$$

Main ingredients:

Theorem (von Staudt, 1840; Clausen, 1840)

- *A prime p divides the denominator of B_{2n} if and only if $p-1 \mid 2n$.*

Main ingredients:

Theorem (von Staudt, 1840; Clausen, 1840)

- *A prime p divides the denominator of B_{2n} if and only if $p-1 \mid 2n$.*
- *If $p-1 \mid 2n$, then $pB_{2n} \equiv -1 \pmod{p}$.*

Main ingredients:

Theorem (von Staudt, 1840; Clausen, 1840)

- *A prime p divides the denominator of B_{2n} if and only if $p-1 \mid 2n$.*
- *If $p-1 \mid 2n$, then $pB_{2n} \equiv -1 \pmod{p}$.*

Fix an $m \geq 1$, and consider primes p with $p-1 \mid 2m$.

Main ingredients:

Theorem (von Staudt, 1840; Clausen, 1840)

- *A prime p divides the denominator of B_{2n} if and only if $p-1 \mid 2n$.*
- *If $p-1 \mid 2n$, then $pB_{2n} \equiv -1 \pmod{p}$.*

Fix an $m \geq 1$, and consider primes p with $p-1 \mid 2m$.

If $p-1 = 2m$, or if $p-1 < 2m$ and $p \mid 4b+1$,
then easy to see: $B_{2m}(x_b) \neq 0$.

Recall:

$$2^{2m}B_{2m}(x_b) = \sum_{j=0}^{m} \binom{2m}{2j} (4b+1)^{m-j}(2-2^{2j})B_{2j}.$$

Remaining case

$p-1<2m$ and $p \nmid 4b+1$:

$$\text{Set} \quad q := \frac{2m}{p-1}; \qquad \text{then} \quad q \in \mathbb{Z}, \quad 2 \le q \le m.$$

Remaining case

$p-1<2m$ and $p\nmid 4b+1$:

$$\text{Set}\quad q:=\frac{2m}{p-1};\qquad \text{then}\quad q\in\mathbb{Z},\quad 2\le q\le m.$$

Multiply both sides of (2) with p; then

- By von Staudt - Clausen:

$$pB_{2j}\equiv\begin{cases}-1 \pmod{p} & \text{for } 2j=r(p-1),\\ & \qquad r=1,2,\dots,q;\\ 0 \pmod{p} & \text{for all other } j.\end{cases}$$

Remaining case

$p-1 < 2m$ and $p \nmid 4b+1$:

$$\text{Set} \quad q := \frac{2m}{p-1}; \qquad \text{then} \quad q \in \mathbb{Z}, \quad 2 \le q \le m.$$

Multiply both sides of (2) with p; then

- By von Staudt - Clausen:

$$pB_{2j} \equiv \begin{cases} -1 \pmod{p} & \text{for } 2j = r(p-1), \\ & \qquad r = 1, 2, \ldots, q; \\ 0 \pmod{p} & \text{for all other } j. \end{cases}$$

- By Fermat's Little Theorem, for $2j = r(p-1)$,

$$2 - 2^{2j} = 2 - 2^{r(p-1)} \equiv 2 - 1 = 1 \pmod{p}.$$

- Since $p \nmid 4b+1$,

$$(4b+1)^j = \left((4b+1)^{\frac{p-1}{2}}\right)^r \equiv \varepsilon_b^r \pmod{p},$$

where

$$\varepsilon_b = \begin{cases} 1, & 4b+1 \text{ quadratic residue} \pmod{p}; \\ -1, & 4b+1 \text{ quadratic nonresidue} \pmod{p}. \end{cases}$$

• Since $p \nmid 4b+1$,

$$(4b+1)^j = \left((4b+1)^{\frac{p-1}{2}}\right)^r \equiv \varepsilon_b^r \pmod{p},$$

where

$$\varepsilon_b = \begin{cases} 1, & 4b+1 \text{ quadratic residue } \pmod{p}; \\ -1, & 4b+1 \text{ quadratic nonresidue } \pmod{p}. \end{cases}$$

So (2) becomes

$$pB_{2m}(x_b) \equiv -\varepsilon_b^q \sum_{r=1}^{q} \binom{q(p-1)}{r(p-1)} \varepsilon_b^r \pmod{p}.$$

When $\varepsilon_b = 1$, sum is well-known to be $\equiv 1 \pmod{p}$ (Hermite, 1876).

- Since $p \nmid 4b+1$,

$$(4b+1)^j = \left((4b+1)^{\frac{p-1}{2}}\right)^r \equiv \varepsilon_b^r \pmod{p},$$

where

$$\varepsilon_b = \begin{cases} 1, & 4b+1 \text{ quadratic residue} \pmod{p}; \\ -1, & 4b+1 \text{ quadratic nonresidue} \pmod{p}. \end{cases}$$

So (2) becomes

$$pB_{2m}(x_b) \equiv -\varepsilon_b^q \sum_{r=1}^{q} \binom{q(p-1)}{r(p-1)} \varepsilon_b^r \pmod{p}.$$

When $\varepsilon_b = 1$, sum is well-known to be $\equiv 1 \pmod{p}$ (Hermite, 1876). So

$$pB_{2m}(x_b) \equiv -1 \pmod{p},$$

and there can be no multiple zero.

Remaining case, $\varepsilon_b = -1$: Set

$$S_p(q) := \sum_{r=1}^{q} \binom{q(p-1)}{r(p-1)} (-1)^r.$$

Remaining case, $\varepsilon_b = -1$: Set

$$S_p(q) := \sum_{r=1}^{q} \binom{q(p-1)}{r(p-1)} (-1)^r.$$

Lemma

$$S_p(q) \equiv \begin{cases} -1 \pmod{p}, & q \ \mathit{odd}; \\ 0 \pmod{p}, & q = k(p+1); \\ 1 \pmod{p}, & q \ \mathit{even}, q \neq k(p+1). \end{cases}$$

Remaining case, $\varepsilon_b = -1$: Set

$$S_p(q) := \sum_{r=1}^{q} \binom{q(p-1)}{r(p-1)} (-1)^r.$$

Lemma

$$S_p(q) \equiv \begin{cases} -1 \pmod{p}, & q \ \mathit{odd}; \\ 0 \pmod{p}, & q = k(p+1); \\ 1 \pmod{p}, & q \ \mathit{even}, q \neq k(p+1). \end{cases}$$

Proof: Case q odd is obvious, by symmetry.

Remaining case, $\varepsilon_b = -1$: Set

$$S_p(q) := \sum_{r=1}^{q} \binom{q(p-1)}{r(p-1)} (-1)^r.$$

Lemma

$$S_p(q) \equiv \begin{cases} -1 \pmod{p}, & q \ \textit{odd}; \\ 0 \pmod{p}, & q = k(p+1); \\ 1 \pmod{p}, & q \ \textit{even}, q \neq k(p+1). \end{cases}$$

Proof: Case q odd is obvious, by symmetry.
The other cases are more difficult; $(2p-2)$th roots of units are used; $S_p(q)$ is considered a linear recurrence sequence.

Lemma means:

The only case that remains open is the case $p+1 \mid q$ and $\varepsilon_b = -1$.

Lemma means:

The only case that remains open is the case $p+1 \mid q$ and $\varepsilon_b = -1$.

To deal with this case, we use the fact that if x_b is a multiple zero of $B_{2m}(x)$, it must be a zero of $B_{2m-1}(x)$.

Lemma means:

The only case that remains open is the case $p+1 \mid q$ and $\varepsilon_b = -1$.

To deal with this case, we use the fact that if x_b is a multiple zero of $B_{2m}(x)$, it must be a zero of $B_{2m-1}(x)$.

This is easy to exclude, using again the Lemma.

Proof of the Lemma (sketch)

With Hermite's congruence

$$\sum_{j=0}^{q} \binom{q(p-1)}{j(p-1)} \equiv 2 \pmod{p}$$

it is easy to see (by just adding congruences) that the Lemma is equivalent to

$$\sum_{j=0}^{\lfloor q/2 \rfloor} \binom{q(p-1)}{2j(p-1)} \equiv \begin{cases} 1 \pmod{p} & \text{for } q \text{ odd}, \\ 2 \pmod{p} & \text{for } q \text{ even}, \ p+1 \nmid q, \\ \frac{3}{2} \pmod{p} & \text{for } p+1 \mid q. \end{cases}$$

The key step is the following

Lemma

Let p be an odd prime and ζ a primitive $(2p-2)$th root of unity. Define, for $q = 1, 2, \ldots,$

$$T_p(q) := \sum_{k=1}^{2p-2} \left(1+\zeta^k\right)^{(p-1)q}.$$

The key step is the following

Lemma

Let p be an odd prime and ζ a primitive $(2p-2)$th root of unity. Define, for $q = 1, 2, \ldots,$

$$T_p(q) := \sum_{k=1}^{2p-2} \left(1+\zeta^k\right)^{(p-1)q}.$$

Then

$$T_p(q) = (2p-2) \sum_{j=0}^{\lfloor q/2 \rfloor} \binom{q(p-1)}{2j(p-1)}.$$

The key step is the following

Lemma

Let p be an odd prime and ζ a primitive $(2p-2)$th root of unity. Define, for $q = 1, 2, \ldots,$

$$T_p(q) := \sum_{k=1}^{2p-2} \left(1+\zeta^k\right)^{(p-1)q}.$$

Then

$$T_p(q) = (2p-2) \sum_{j=0}^{\lfloor q/2 \rfloor} \binom{q(p-1)}{2j(p-1)}.$$

The proof is easy: Use a binomial expansion and change the order of summation.

By the theory of linear recurrence relations with constant coefficients:

By the theory of linear recurrence relations with constant coefficients:

- $\{T_p(q)\}$, $q = 1, 2, \ldots$, is such a sequence;

By the theory of linear recurrence relations with constant coefficients:

- $\{T_p(q)\}$, $q = 1, 2, \ldots$, is such a sequence;
- order is at most $2p - 2$;

By the theory of linear recurrence relations with constant coefficients:

- $\{T_p(q)\}$, $q = 1, 2, \ldots$, is such a sequence;
- order is at most $2p - 2$;
- characteristic polynomial has

$$(1 + \zeta^k)^{p-1}, \quad k = 1, 2, \ldots, 2p - 2,$$

as its roots.

By the theory of linear recurrence relations with constant coefficients:

- $\{T_p(q)\}$, $q = 1, 2, \ldots$, is such a sequence;
- order is at most $2p - 2$;
- characteristic polynomial has

$$(1 + \zeta^k)^{p-1}, \quad k = 1, 2, \ldots, 2p - 2,$$

as its roots.

This motivates the following lemma.

Lemma

Let p be an odd prime and $f_p(x)$ the unique monic polynomial that has $(1+\zeta^k)^{p-1}$, $k = 1, 2, \ldots, 2p-2$, as its roots.

Lemma

Let p be an odd prime and $f_p(x)$ the unique monic polynomial that has $(1+\zeta^k)^{p-1}$, $k = 1, 2, \ldots, 2p-2$, as its roots. Then

$$f_p(x) \equiv x \sum_{n=0}^{2p-3} a_n x^{2p-3-n} \pmod{p},$$

Lemma

Let p be an odd prime and $f_p(x)$ the unique monic polynomial that has $(1+\zeta^k)^{p-1}$, $k = 1, 2, \ldots, 2p-2$, as its roots. Then

$$f_p(x) \equiv x \sum_{n=0}^{2p-3} a_n x^{2p-3-n} \pmod{p},$$

where for $0 \le n \le p-2$ we have

$$a_n \equiv \begin{cases} (m+1)^2 \pmod{p} & \text{for} \quad n = 2m, \\ (m+1)(m+2) \pmod{p} & \text{for} \quad n = 2m+1, \end{cases}$$

and for $p-1 \le n \le 2p-3$,

$$a_n \equiv -a_{2p-3-n} \pmod{p}.$$

Lemma

Let p be an odd prime and $f_p(x)$ the unique monic polynomial that has $(1+\zeta^k)^{p-1}$, $k = 1, 2, \ldots, 2p-2$, as its roots. Then

$$f_p(x) \equiv x \sum_{n=0}^{2p-3} a_n x^{2p-3-n} \pmod{p},$$

where for $0 \le n \le p-2$ we have

$$a_n \equiv \begin{cases} (m+1)^2 \pmod{p} & \text{for} \quad n = 2m, \\ (m+1)(m+2) \pmod{p} & \text{for} \quad n = 2m+1, \end{cases}$$

and for $p-1 \le n \le 2p-3$,

$$a_n \equiv -a_{2p-3-n} \pmod{p}.$$

Proof uses various congruences and identities for binomial coefficients and finite sums.

The conjecture that

$$T_p(q) \equiv \begin{cases} -2 \pmod{p} & \text{for } q \text{ odd}, \\ -4 \pmod{p} & \text{for } q \text{ even}, p+1 \nmid q, \\ -3 \pmod{p} & \text{for } p+1 \mid q, \end{cases}$$

would complete the proof. We can prove this as follows:

The conjecture that

$$T_p(q) \equiv \begin{cases} -2 \pmod{p} & \text{for } q \text{ odd}, \\ -4 \pmod{p} & \text{for } q \text{ even}, \; p+1 \nmid q, \\ -3 \pmod{p} & \text{for } p+1 \mid q, \end{cases}$$

would complete the proof. We can prove this as follows:

- Verify it for all $q \leq 2p$.

The conjecture that

$$T_p(q) \equiv \begin{cases} -2 \pmod{p} & \text{for } q \text{ odd}, \\ -4 \pmod{p} & \text{for } q \text{ even}, p+1 \nmid q, \\ -3 \pmod{p} & \text{for } p+1 \mid q, \end{cases}$$

would complete the proof. We can prove this as follows:

• Verify it for all $q \leq 2p$.
This can be done by elementary (but tricky) manipulations of congruences for binomial coefficients.

The conjecture that

$$T_p(q) \equiv \begin{cases} -2 \pmod{p} & \text{for } q \text{ odd}, \\ -4 \pmod{p} & \text{for } q \text{ even}, \ p+1 \nmid q, \\ -3 \pmod{p} & \text{for } p+1 \mid q, \end{cases}$$

would complete the proof. We can prove this as follows:

• Verify it for all $q \leq 2p$.
This can be done by elementary (but tricky) manipulations of congruences for binomial coefficients.

• Then show that the numbers given above satisfy the recurrence relation

$$a_0 T_p(n) + a_1 T_p(n-1) + \ldots + a_{2p-3} T_p(n-2p+3) \equiv 0 \pmod{p}$$

for all $n \geq 2p-2$, with the a_j as given in the previous Lemma.

The conjecture that

$$T_p(q) \equiv \begin{cases} -2 \pmod{p} & \text{for } q \text{ odd}, \\ -4 \pmod{p} & \text{for } q \text{ even}, p+1 \nmid q, \\ -3 \pmod{p} & \text{for } p+1 \mid q, \end{cases}$$

would complete the proof. We can prove this as follows:

• Verify it for all $q \le 2p$.
This can be done by elementary (but tricky) manipulations of congruences for binomial coefficients.

• Then show that the numbers given above satisfy the recurrence relation

$$a_0 T_p(n) + a_1 T_p(n-1) + \ldots + a_{2p-3} T_p(n-2p+3) \equiv 0 \pmod{p}$$

for all $n \ge 2p-2$, with the a_j as given in the previous Lemma.
This is again elementary but tricky.

The conjecture that

$$T_p(q) \equiv \begin{cases} -2 \pmod{p} & \text{for } q \text{ odd}, \\ -4 \pmod{p} & \text{for } q \text{ even}, \ p+1 \nmid q, \\ -3 \pmod{p} & \text{for } p+1 \mid q, \end{cases}$$

would complete the proof. We can prove this as follows:

• Verify it for all $q \le 2p$.
This can be done by elementary (but tricky) manipulations of congruences for binomial coefficients.

• Then show that the numbers given above satisfy the recurrence relation

$$a_0 T_p(n) + a_1 T_p(n-1) + \ldots + a_{2p-3} T_p(n-2p+3) \equiv 0 \pmod{p}$$

for all $n \ge 2p-2$, with the a_j as given in the previous Lemma.
This is again elementary but tricky.

The proof is complete.

Thank you