Painlevé Equations — Nonlinear Special Functions

Peter A Clarkson

School of Mathematics, Statistics and Actuarial Science
University of Kent, Canterbury, CT2 7NF, UK
P.A.Clarkson@kent.ac.uk

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Outline

1. Introduction
2. Classical solutions of the **second Painlevé equation**

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

and the **second Painlevé σ-equation**

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2$$

3. Painlevé Challenges
   - Equivalence problem
   - Numerical solution of Painlevé equations

Classical Special Functions

- **Airy, Bessel, Whittaker, Kummer, hypergeometric functions**
- Special solutions in terms of rational and elementary functions (for certain values of the parameters)
- Solutions satisfy **linear** ordinary differential equations and **linear** difference equations
- Solutions related by **linear** recurrence relations

Painlevé Transcendents — Nonlinear Special Functions

- Special solutions such as rational solutions, algebraic solutions and special function solutions (for certain values of the parameters)
- Solutions satisfy **nonlinear** ordinary differential equations and **nonlinear** difference equations
- Solutions related by **nonlinear** recurrence relations

Definition 1

An ODE has the **Painlevé property** if its solutions have **no movable singularities except poles**.

Definition 2

An ODE has the **Painlevé property** if its solutions have **no movable branch points**.

- **Single-valued**

$$w(z) = \frac{1}{z - z_0} \qquad \textbf{pole}$$

$$w(z) = \exp\left(\frac{1}{z - z_0}\right) \qquad \textbf{essential singularity}$$

- **Multi-valued**

$$w(z) = \sqrt{z - z_0} \qquad \textbf{algebraic branch point}$$

$$w(z) = \ln(z - z_0) \qquad \textbf{logarithmic branch point}$$

$$w(z) = \tan[\ln(z - z_0)] \qquad \textbf{essential singularity}$$

Reference

- **Cosgrove**, "Painlevé classification problems featuring essential singularities", *Stud. Appl. Math.*, **98** (1997) 355–433. [See also **Cosgrove**, *Stud. Appl. Math.*, **104** (2000) 1–65; **104** (2000) 171–228; **116** (2006) 321–413.]

Painlevé Equations

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 6w^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \qquad \mathrm{P_{IV}}$$

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = {} & \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) \qquad \mathrm{P_V} \\ & + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = {} & \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{\mathrm{d}w}{\mathrm{d}z} \qquad \mathrm{P_{VI}} \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\}\end{aligned}$$

where α, β, γ and δ are arbitrary constants.

Painlevé σ-Equations

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma = 0 \qquad S_{\mathrm{I}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad S_{\mathrm{II}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + \left[4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - 1\right]\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) + \lambda_0\lambda_1\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \tfrac{1}{4}\left(\lambda_0^2 + \lambda_1^2\right) \qquad S_{\mathrm{III}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad S_{\mathrm{IV}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left[2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) = 0 \qquad S_{\mathrm{V}}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left[z(z-1)\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right]^2 + \left[\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left\{2\sigma - (2z-1)\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right\} + b_1b_2b_3b_4\right]^2 = \prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + b_j^2\right) \qquad S_{\mathrm{VI}}$$

Some Properties of the Painlevé Equations

- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **Bäcklund transformations** which relate solutions of a given Painlevé equation to solutions of the same Painlevé equation, though with different values of the parameters with associated **Affine Weyl groups** that act on the parameter space.
- $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have **rational**, **algebraic** and **special function solutions** expressed in terms of the classical special functions [$\mathrm{P_{II}}$: **Airy** $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$; $\mathrm{P_{III}}$: **Bessel** $J_\nu(z)$, $Y_\nu(z)$, $J_\nu(z)$, $K_\nu(z)$; $\mathrm{P_{IV}}$: **parabolic cylinder** $D_\nu(z)$; $\mathrm{P_V}$: **Whittaker** $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ [equivalently **Kummer** $M(a,b,z)$, $U(a,b,z)$ or **confluent hypergeometric** ${}_1F_1(a;c;z)$]; $\mathrm{P_{VI}}$: **hypergeometric** ${}_2F_1(a,b;c;z)$], for certain values of the parameters.
- These rational, algebraic and special function solutions of $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$, called **classical solutions**, can usually be written in **determinantal form**, frequently as **wronskians**. Often they can be written as **Hankel determinants** or **Toeplitz determinants**.
- $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be written as a (non-autonomous) **Hamiltonian system** and the Hamiltonians satisfy a second-order, second-degree differential equations (S_{I}–S_{VI}).
- $\mathrm{P_I}$–$\mathrm{P_{VI}}$ possess **Lax pairs** (**isomonodromy problems**).
- $\mathrm{P_I}$–$\mathrm{P_{VI}}$ and S_{I}–S_{VI} form a **coalescence cascade**

$$\begin{array}{ccccccc}
\mathrm{P_{VI}} & \longrightarrow & \mathrm{P_V} & \longrightarrow & \mathrm{P_{IV}} & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{P_{III}} & \longrightarrow & \mathrm{P_{II}} & \longrightarrow & \mathrm{P_I}
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
S_{\mathrm{VI}} & \longrightarrow & S_{\mathrm{V}} & \longrightarrow & S_{\mathrm{IV}} & & \\
 & & \downarrow & & \downarrow & & \\
 & & S_{\mathrm{III}} & \longrightarrow & S_{\mathrm{II}} & \longrightarrow & S_{\mathrm{I}}
\end{array}$$

Hamiltonian Representation

P_{II} can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\mathrm{II}}}{\partial p} = p - q^2 - \tfrac{1}{2}z, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\mathrm{II}}}{\partial q} = 2qp + \alpha + \tfrac{1}{2} \tag{II}$$

where $\mathcal{H}_{\mathrm{II}}(q, p, z; \alpha)$ is the Hamiltonian defined by

$$\mathcal{H}_{\mathrm{II}}(q, p, z; \alpha) = \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q$$

Eliminating p then $q = w$ satisfies P_{II} whilst eliminating q yields

$$p\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha + \tfrac{1}{2})^2 \qquad P_{34}$$

Theorem

(**Okamoto [1986]**)

The function

$$\sigma(z; \alpha) = \mathcal{H}_{\mathrm{II}} \equiv \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q$$

satisfies

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2$$

and conversely

$$q(z; \alpha) = \frac{2\sigma''(z) + \alpha + \frac{1}{2}}{4\sigma'(z)}, \qquad p(z; \alpha) = -2\frac{\mathrm{d}\sigma}{\mathrm{d}z}$$

is a solution of (II).

Classical Solutions of the Second Painlevé Equation and the Second Painlevé σ-Equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P}_{\mathrm{II}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad S_{\mathrm{II}}$$

Classical Solutions of $\mathrm{P_{II}}$ and S_{II}

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad S_{\mathrm{II}}$$

Theorem

- $\mathrm{P_{II}}$ *and* S_{II} *have* ***rational solutions*** *if and only if* $\alpha = n$*, with* $n \in \mathbb{Z}$*.*
- $\mathrm{P_{II}}$ *and* S_{II} *have solutions expressible in terms of the Riccati equation*

$$\varepsilon\frac{\mathrm{d}w}{\mathrm{d}z} = w^2 + \tfrac{1}{2}z, \qquad \varepsilon = \pm 1 \qquad (1)$$

if and only if $\alpha = n + \frac{1}{2}$*, with* $n \in \mathbb{Z}$*. The Riccati equation* (1) *has solution*

$$w(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\varphi(z)$$

where

$$\varphi(z) = C_1\,\mathrm{Ai}(\zeta) + C_2\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/2}z$$

with $\mathrm{Ai}(\zeta)$ *and* $\mathrm{Bi}(\zeta)$ *the* ***Airy functions****.*

Rational Solutions of $\mathrm{P_{II}}$ and S_{II}

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad S_{\mathrm{II}}$$

Theorem

Define the polynomial $\varphi_j(z)$ by

$$\sum_{j=0}^{\infty} \varphi_j(z)\lambda^j = \exp\left(z\lambda - \tfrac{4}{3}\lambda^3\right)$$

and the ***Yablonskii–Vorob'ev polynomials $Q_n(z)$*** *given by*

$$Q_n(z) = c_n \mathcal{W}(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$$

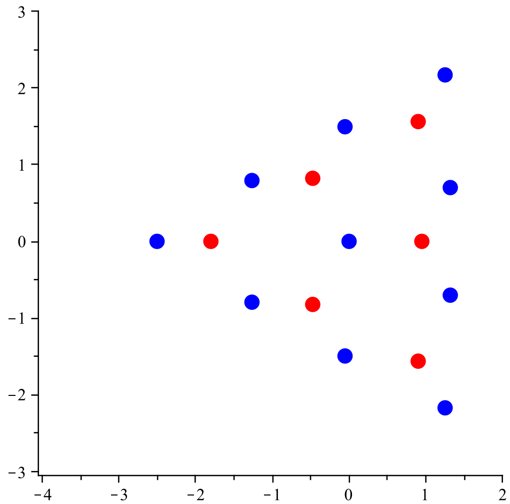
where $\mathcal{W}(\varphi_1, \varphi_3, \ldots, \varphi_{2n-1})$ is the Wronskian and c_n a constant, then

$$w(z;n) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{Q_{n-1}(z)}{Q_n(z)}, \qquad \sigma(z;n) = -\tfrac{1}{8}z^2 + \frac{\mathrm{d}}{\mathrm{d}z}\ln Q_n(z)$$

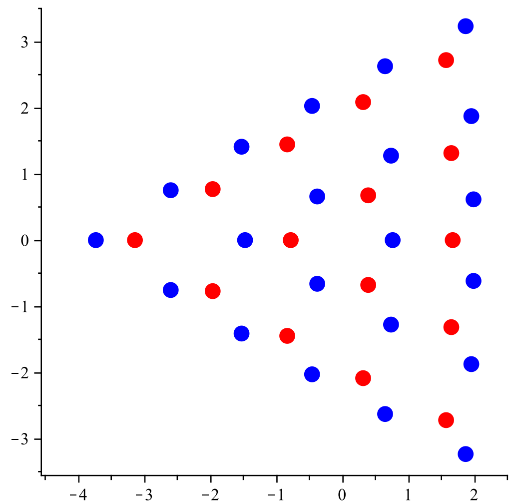
respectively satisfy $\mathrm{P_{II}}$ and S_{II} with $\alpha = n$, for $n \in \mathbb{Z}$.

Roots of some Yablonskii–Vorob’ev polynomials

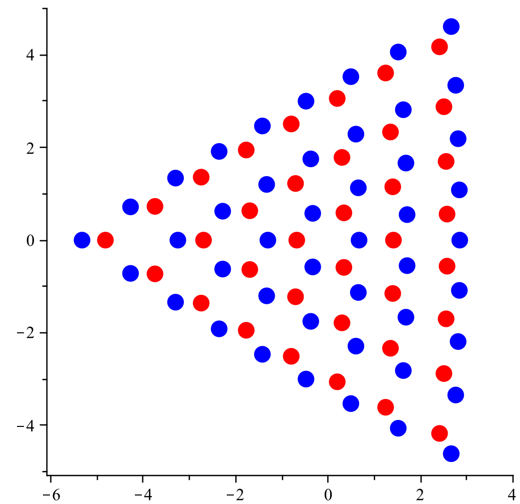
(PAC & Mansfield [2003])



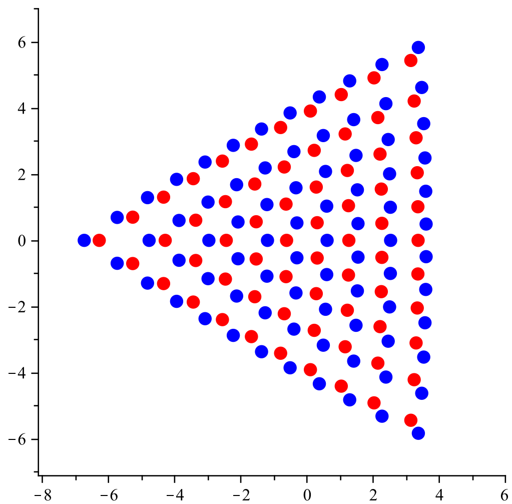
$Q_3(z), Q_4(z)$



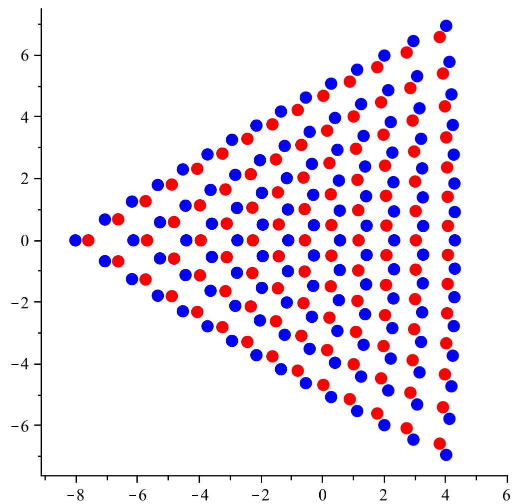
$Q_5(z), Q_6(z)$



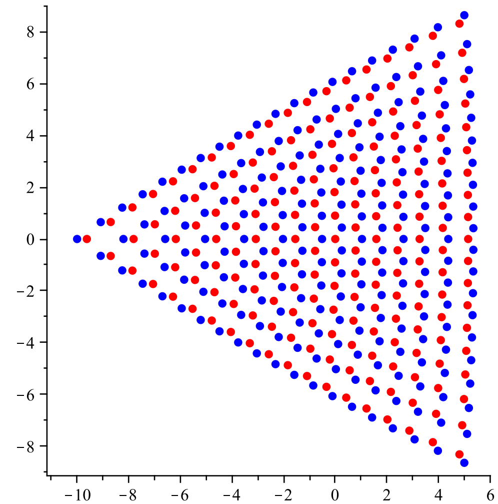
$Q_8(z), Q_9(z)$



$Q_{11}(z), Q_{12}(z)$



$Q_{14}(z), Q_{15}(z)$



$Q_{19}(z), Q_{20}(z)$

Airy Solutions of $\mathrm{P_{II}}$ and S_{II}

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha \qquad \mathrm{P_{II}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad S_{\mathrm{II}}$$

Theorem

Let

$$\varphi(z) = C_1\,\mathrm{Ai}(\zeta) + C_2\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/2}z$$

with $\mathrm{Ai}(\zeta)$ *and* $\mathrm{Bi}(\zeta)$ ***Airy functions****, and* $\tau_n(z)$ *be the Wronskian*

$$\tau_n(z) = \mathcal{W}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi}{\mathrm{d}z^{n-1}}\right)$$

then

$$w(z; n+\tfrac{1}{2}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_n(z)}{\tau_{n+1}(z)}\right), \qquad \sigma(z; n+\tfrac{1}{2}) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z)$$

respectively satisfy $\mathrm{P_{II}}$ *and* S_{II} *with* $\alpha = n + \frac{1}{2}$*, for* $n \in \mathbb{Z}$*.*

Special function solutions of Painlevé equations

	Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial	Number of parameters
$\mathrm{P_I}$	0	—			
$\mathrm{P_{II}}$	1	**Airy** $\mathrm{Ai}(z), \mathrm{Bi}(z)$	0	—	
$\mathrm{P_{III}}$	2	**Bessel** $J_\nu(z), Y_\nu(z), J_\nu(z), K_\nu(z)$	1	—	
$\mathrm{P_{IV}}$	2	**Parabolic cylinder** $D_\nu(z)$	1	**Hermite** $H_n(z)$	0
$\mathrm{P_V}$	3	**Whittaker** $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$ **Kummer** $M(a,b,z), U(a,b,z)$ **confluent hypergeometric** ${}_1F_1(a;c;z)$	2	**Associated Laguerre** $L_n^{(k)}(z)$	1
$\mathrm{P_{VI}}$	4	**hypergeometric** ${}_2F_1(a,b;c;z)$	3	**Jacobi** $P_n^{(\alpha,\beta)}(z)$	2

Application of $\mathrm{P_{III}}$ to Orthogonal Polynomials (Chen & Its [2010])

Consider the orthogonal polynomials with respect to the perturbed Laguerre weight

$$w(x;z) = x^{\alpha}\mathrm{e}^{-x-z/x}, \qquad x \in [0,\infty), \qquad \alpha > 0$$

and seek polynomials $P_n(x;z)$ which satisfy

$$\int_0^1 P_m(x;z)P_n(x;z)w(x;z)\,\mathrm{d}x = h_n(z)\delta_{m,n}$$

Consequently they satisfy the three term recurrence relation

$$xP_n(x;z) = P_{n+1}(x;z) + a_n(z)P_n(x;z) + b_n(z)P_{n-1}(x;z)$$

where $a_n(z)$ and $b_n(z)$ are expressible in terms of solutions of $\mathrm{P_{III}}$ with

$$(\alpha,\beta,\gamma,\delta) = (-2(2n+1+\nu), -2\nu, 1, -1)$$

Further if we define the Hankel determinant

$$D_n(z) = \det\left(\mu_{j+k}(z)\right)_{j,k=0}^{n-1}$$

where

$$\mu_k(z) = \int_0^\infty x^{\mu+k}\mathrm{e}^{-x-z/x}\,\mathrm{d}x = 2z^{(\nu+k+1)/2}K_{\nu+k+1}(2\sqrt{z})$$

with $K_\nu(z)$ the **modified Bessel function**, then

$$H_n(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln D_n(z)$$

satisfies a special case of S_{III}, the $\mathrm{P_{III}}$ σ-equation.

Application of $\mathrm{P_V}$ to Orthogonal Polynomials (Chen & Dai [2010])

Consider the orthogonal polynomials with respect to the Pollaczek-Jacobiweight

$$w(x;z) = x^a(1-x)^b \mathrm{e}^{-z/x}, \qquad x \in [0,1], \qquad a>0, \quad b>0$$

and seek polynomials $P_n(x;z)$ which satisfy

$$\int_0^1 P_m(x;z)P_n(x;z)w(x;z)\,\mathrm{d}x = h_n(t)\delta_{m,n}$$

Consequently they satisfy the three term recurrence relation

$$xP_n(x;z) = P_{n+1}(x;z) + a_n(z)P_n(x;z) + b_n(z)P_{n-1}(x;z)$$

where $a_n(z)$ and $b_n(z)$ are expressible in terms of solutions of $\mathrm{P_V}$ with

$$(\alpha,\beta,\gamma,\delta) = \left(\tfrac{1}{2}(2n+1+a+b)^2, -\tfrac{1}{2}b^2, a, -\tfrac{1}{2}\right)$$

Further if we define the Hankel determinant

$$D_n(z) = \det\left(\mu_{j+k}(z)\right)_{j,k=0}^{n-1}$$

where

$$\mu_k(z) = \int_0^1 x^{k+a}(1-x)^b \mathrm{e}^{-z/x}\,\mathrm{d}x = \mathrm{e}^{-z}\Gamma(1+b)U(1+b,-a-k,z)$$

with $U(\alpha,\beta,z)$ the **Kummer function** of the second kind, then

$$H_n(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln D_n(z)$$

satisfies a special case of S_{V}, the $\mathrm{P_V}$ σ-equation.

Painlevé Challenges

1. **Equivalence problem**
   - Given an equation with the Painlevé property, how do we know which Painlevé equation, or Painlevé σ-equation, it is related to?
2. **Numerical solution of Painlevé equations**
   - How do we use the special properties of the Painlevé equations, e.g. that they are solvable by the isomonodromy method through an associated Riemann-Hilbert problem, in the development of numerical software?

Painlevé Equivalence Problem

- Given an equation with the Painlevé property, how do we know which equation, in particular a Painlevé equation or Painlevé σ-equation, it is solvable in terms of?

For linear ODEs, if we can solve the equation in terms of the classical special functions then we regard that the equation is solved.

Example

The linear ODEs

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + z^2 v = 0, \qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \mathrm{e}^{2z} w = 0,$$

respectively have the solutions

$$v(z) = \sqrt{z}\left\{C_1 J_{1/4}\left(\tfrac{1}{2}z^2\right) + C_2 J_{-1/4}\left(\tfrac{1}{2}z^2\right)\right\}$$
$$w(z) = C_1 J_0(\mathrm{e}^z) + C_2 Y_0(\mathrm{e}^z),$$

with C_1 and C_2 arbitrary constants, $J_\nu(\zeta)$ and $Y_\nu(\zeta)$ **Bessel functions**.

MAPLE can easily find such solutions of linear ODEs.

However MAPLE is not as clever for nonlinear ODEs.

MAPLE's `odeadvisor` command will tell you that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6y^2 + x$$

is the first Painlevé equation, but gives "none" as the answer for

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6y^2 - x$$

which is obtained by making the simple transformation $x \to -x$.

Example

Consider the equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w^3 - 1 \tag{1}$$

This can be shown to possess the Painlevé property, but which equation is it equivalent to? It's not in the list of 50 equations given by **Ince [1956]**.

Equation (1) arises from the symmetry reduction

$$u(x,t) = \ln w(z), \qquad z = 2\sqrt{xt}$$

of the **Tzitzeica equation** (**Tzitzeica [1910]**)

$$u_{xt} = \exp(2u) - \exp(-u)$$

which is also known as the **Bullough-Dodd-Mikhailov-Shabat-Zhiber equation**.

Example

Consider the equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w^3 - 1 \tag{1}$$

This can be shown to possess the Painlevé property, but which equation is it equivalent to? It's not in the list of 50 equations given by **Ince [1956]**

Answer

Making the transformation

$$w(z) = x^{1/3}y(x), \qquad z = \tfrac{3}{2}x^{2/3} \tag{2}$$

yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + y^3 - \frac{1}{x} \tag{3}$$

which is the special case of $\mathrm{P_{III}}$ with $\alpha = 0$, $\beta = -1$, $\gamma = 1$ and $\delta = 0$.

Remark

The transformation (2) is suggested by the asymptotic expansions of (1) and (3)

$$\begin{aligned} w(z) &\sim 1 + \lambda z^{-1/2}\exp\left(-\sqrt{3}\,z\right), && \text{as} \quad z \to \infty \\ y(x) &\sim x^{-1/3}\left\{1 + \kappa x^{-1/3}\exp\left(-\tfrac{3}{2}\sqrt{3}\,x^{2/3}\right)\right\}, && \text{as} \quad x \to \infty \end{aligned}$$

with λ and κ constants.

Example

The equation

$$\frac{\mathrm{d}^3W}{\mathrm{d}z^3}+6W\frac{\mathrm{d}W}{\mathrm{d}z}-2W-z\frac{\mathrm{d}W}{\mathrm{d}z}=0 \tag{1}$$

arises as the scaling reduction

$$u(x,t)=\frac{W(z)}{(3t)^{2/3}},\qquad z=\frac{x}{(3t)^{1/3}}$$

of the **Korteweg-de Vries equation**

$$u_t+6uu_x+u_{xxx}=0$$

- In the literature it is frequently stated that (1) is solvable in terms of $\mathrm{P_{II}}$, though this is not obvious. Specifically the one-to-one relationship between solutions of (1) and solutions of $\mathrm{P_{II}}$

  $$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}=2\omega^3+zw+\alpha$$

  is given by

  $$W=-\frac{\mathrm{d}w}{\mathrm{d}z}-w^2,\qquad w=\frac{1}{2W-z}\left(\frac{\mathrm{d}W}{\mathrm{d}z}+\alpha\right)$$

$$\frac{\mathrm{d}^3W}{\mathrm{d}z^3} + 6W\frac{\mathrm{d}W}{\mathrm{d}z} - 2W - z\frac{\mathrm{d}W}{\mathrm{d}z} = 0 \tag{1}$$

- Multiplying (1) by $W - \frac{1}{2}z$ and integrating yields

$$(W - \tfrac{1}{2}z)\left(\frac{\mathrm{d}^2W}{\mathrm{d}z^2} + 2W^2 - zW\right) + \tfrac{1}{2}\frac{\mathrm{d}W}{\mathrm{d}z} - \tfrac{1}{2}\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^2 = C_1 + \tfrac{1}{8}$$

  with C_1 an arbitrary constant. Letting $W = \frac{1}{2}z - v$ yields

$$v\frac{\mathrm{d}^2v}{\mathrm{d}z^2} = \tfrac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 + 2v^3 - zv^2 + C_1$$

  which is P_{34} and this explains why (1) is solvable in terms of P_{II}.

- Equation (1) is equivalent to the equation

$$\frac{\mathrm{d}^4\sigma}{\mathrm{d}z^4} + 12\frac{\mathrm{d}\sigma}{\mathrm{d}z}\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2} + 2z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2} + \frac{\mathrm{d}\sigma}{\mathrm{d}z} = 0$$

  which is the second derivative of S_{II}, the P_{II} σ-equation, through a scaling and translation of variables.

Asymptotics for $\mathrm{P_I}$

(**Bender & Orszag [1969]; Holmes & Spence [1984]; Joshi & Kruskal [1992]**)

There are four families of solutions of the initial value problem for $\mathrm{P_I}$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 6w^2 + x, \qquad w(0) = \kappa, \quad \frac{\mathrm{d}w}{\mathrm{d}x}(0) = \mu$$

where κ and μ are arbitrary constants.

- Solutions which oscillate infinitely often, remain bounded for all finite $x < 0$, with

  $$w(x) = -\left(-\tfrac{1}{6}x\right)^{1/2} + d|x|^{-1/8}\sin\{\varphi(x)\} + o(|x|^{-1/8}), \qquad \text{as} \quad x \to -\infty$$

  where

  $$\varphi(x) = \sqrt[4]{24}\left(\tfrac{4}{5}|x|^{5/4} - \tfrac{5}{8}d^2 \ln|x| - \theta_0\right)$$

  with d and θ_0 parameters (**Qin & Lu [2008]**).
- A unique, monotone increasing, solution, which is bounded for all finite $x < 0$ (known as the **tri-tronquée solution**).
- Solutions with $w(x) \sim +\left(-\tfrac{1}{6}x\right)^{1/2}$, as $x \to -\infty$ (a **tronquée solution**).
- Solutions, each of which has a pole at a finite, real x_0, with $-\infty < x_0 < 0$.

Open Question:

- **How are these solutions related to κ and μ, e.g. how do d and θ_0 depend on κ and μ?**

Numerical Studies of $\mathrm{P_I}$

Consider the initial value problem

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = 6w^2 + x, \qquad w(0) = 0, \quad \frac{\mathrm{d}w}{\mathrm{d}x}(0) = \mu$$

where μ is an arbitrary constant. Numerical studies show that:

- $w(x)$ has at least one pole on the real axis;
- there are two special values of μ, namely μ_1 and μ_2, with the properties

  $$-0.451428 < \mu_1 < -0.451427, \qquad 1.851853 < \mu_2 < 1.851855$$

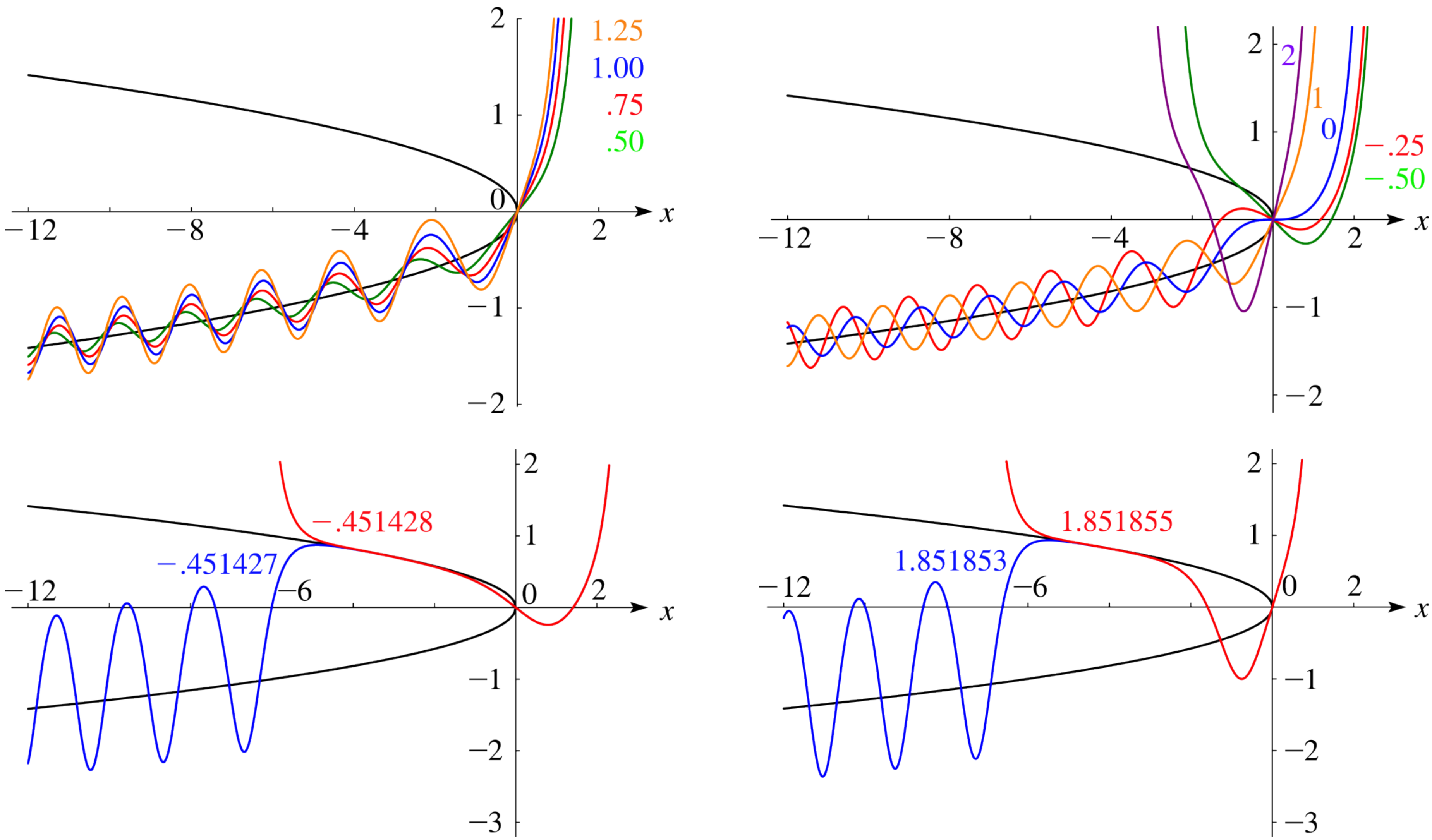
  such that:
  - if $\mu < \mu_1$, then $w(x) > 0$ for $x_0 < x < 0$, where x_0 is the first pole on negative real axis;
  - if $\mu_1 < \mu < \mu_2$, then $w(x)$ oscillates about and is asymptotic to $-\sqrt{\frac{1}{6}|x|}$;
  - if $\mu_2 < \mu$, then $w(x)$ changes sign once, from positive to negative as x passes from x_0 to 0.
- **Fornberg & Weiderman [2011]** have recently shown that

  $$\mu_1 \approx -0.4514274047417740, \qquad \mu_2 \approx 1.851854033760367$$

- The solutions with these special values **both** satisfy the boundary value probem

  $$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = 6w^2 + x, \qquad w(0) = 0, \quad w(x) \sim \sqrt{-\tfrac{1}{6}x} \quad \text{as} \quad x \to -\infty$$

Painlevé I $\quad w'' = 6w^2 + x, \quad w(0) = 0, \quad w'(0) = \mu$

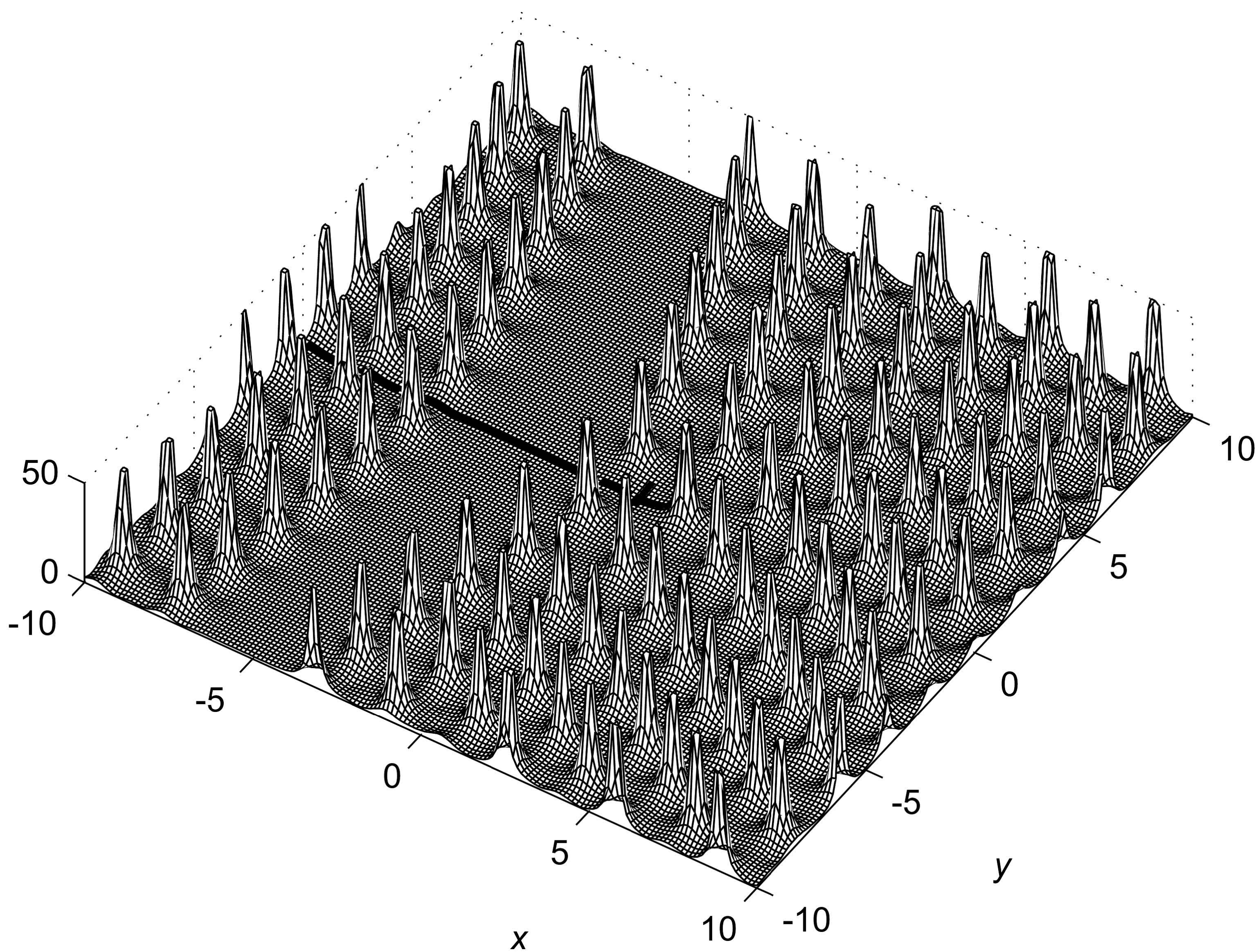


$\mu_1 \approx -0.451427404741774$ $\qquad$ $\mu_2 \approx 1.851854033760367$

Painlevé I

$$w'' = 6w^2 + x, \quad w(0) = 0, \quad w'(0) = 1.8518$$

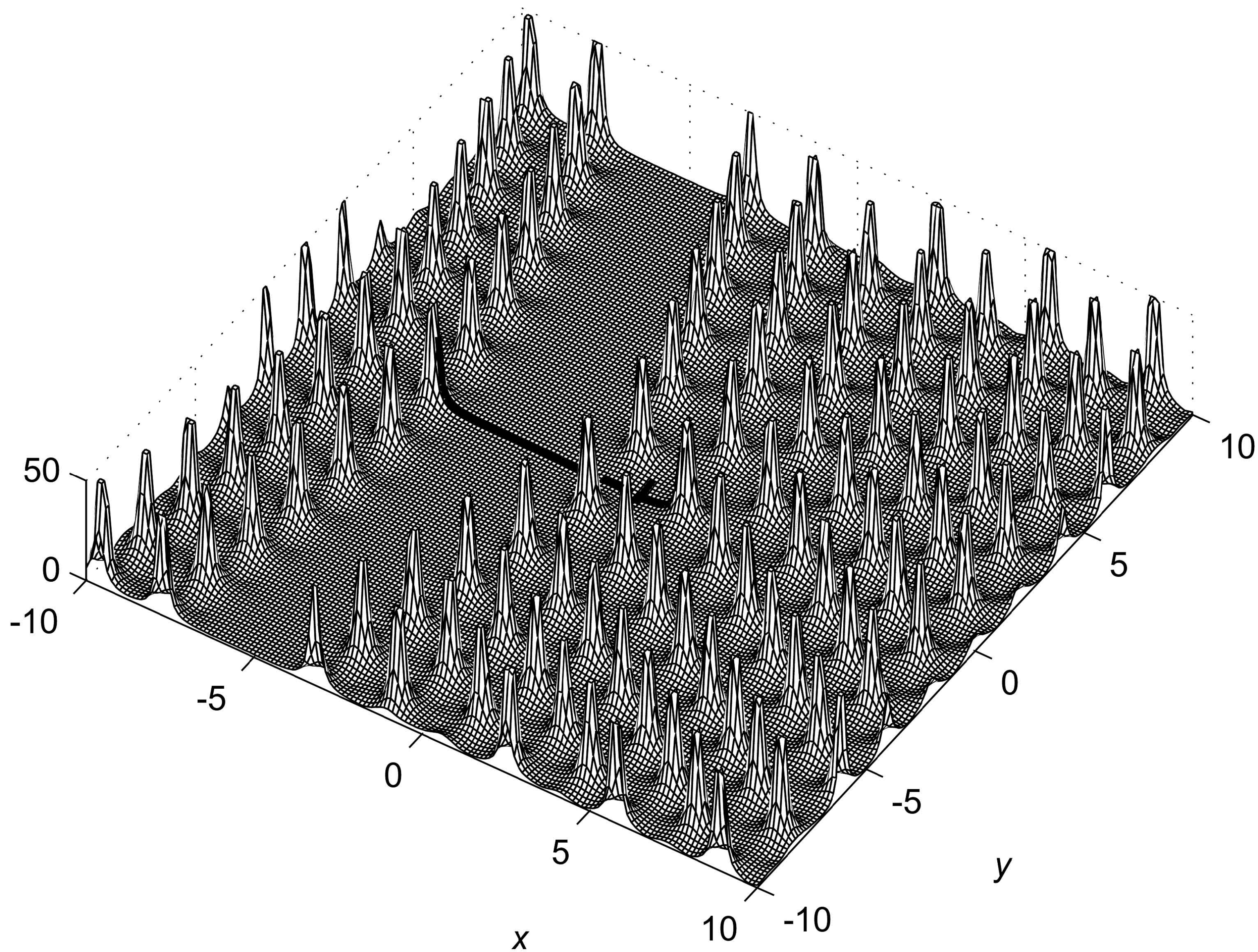
(**Fornberg & Weiderman [2011]**)



Painlevé I

$$w'' = 6w^2 + x, \quad w(0) = 0, \quad w'(0) = 1.8519$$

(**Fornberg & Weiderman [2011]**)

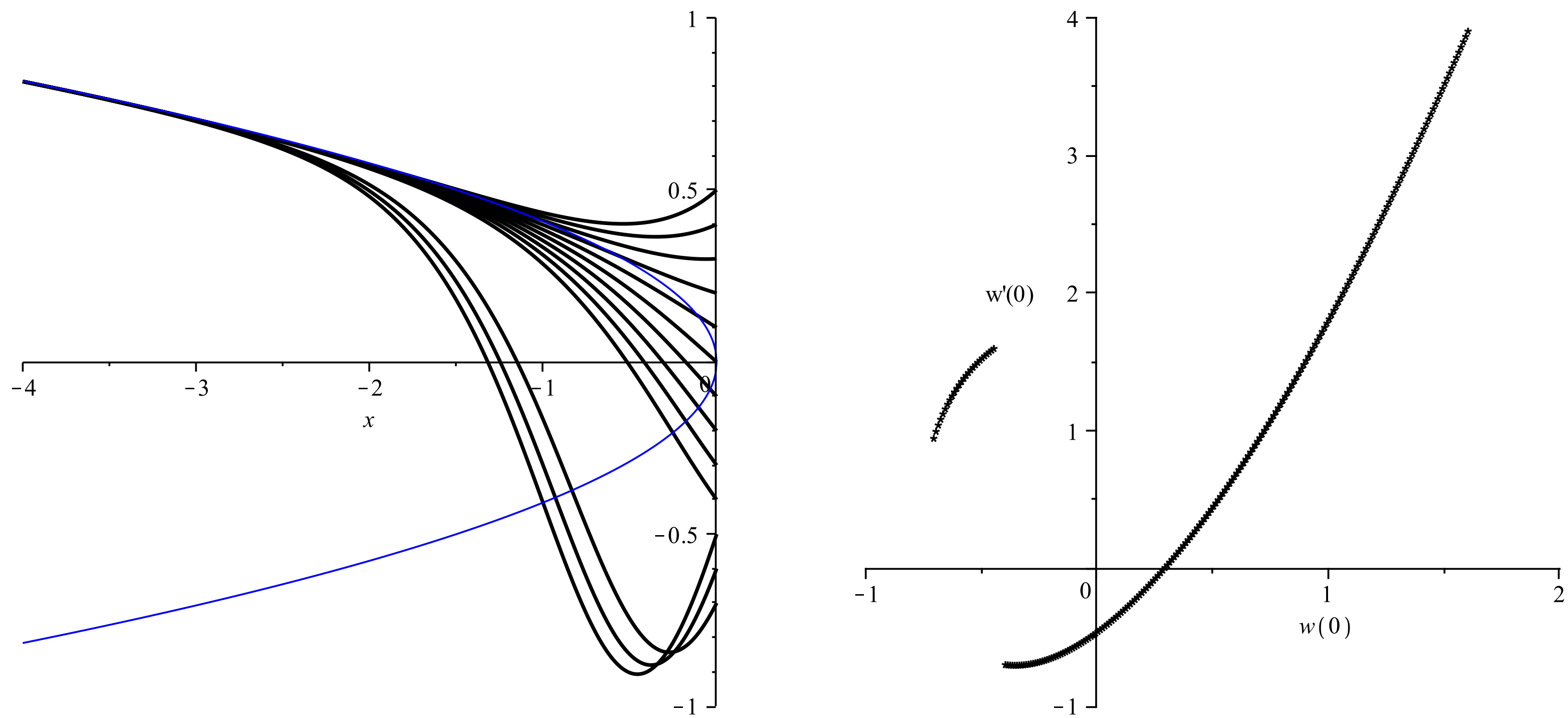


Boundary-Value Problem for P_{I}

Consider

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 6w^2 + x \qquad \begin{cases} w(0) = \kappa, \\ w(x) \sim \sqrt{-\tfrac{1}{6}x}, \qquad \text{as} \quad x \to -\infty \end{cases}$$

with κ an arbitrary parameter. There are two solutions of this BVP for several values of κ, though (naively) using MAPLE's numerical BVP solver only gives one solution.



Numerical Studies of Painlevé Equations

- My numerical simulations were obtained using MAPLE using the `DEplot` command with option `method=dverk78`, which finds a numerical solution using a seventh-eighth order continuous Runge-Kutta method. This is easy to use, gives plots of solutions quickly with accuracy better than the human eye can detect.
- There have been several numerical studies of the **Hastings-McLeod solution** of P_{II}

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2w^3 + xw, \qquad w(x) \sim \begin{cases} \mathrm{Ai}(x), & \text{as} \quad x \to \infty \\ \left(-\tfrac{1}{2}x\right)^{1/2}, & \text{as} \quad x \to -\infty \end{cases}$$

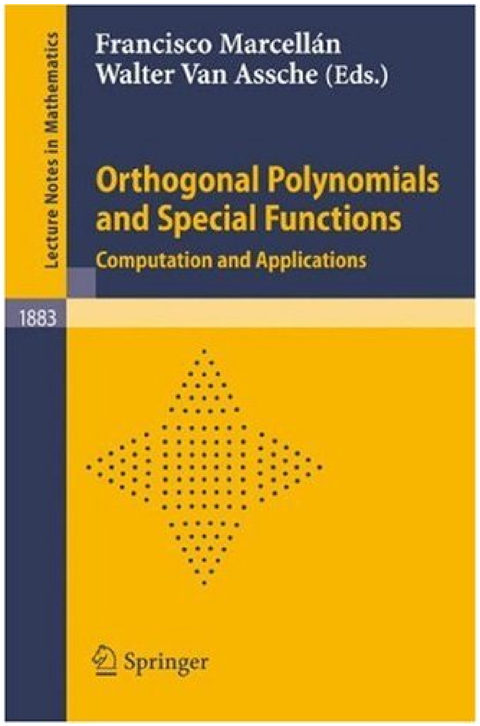
  some of which have obtained the solution to high precision [e.g. **Driscoll, Bornemann & Trefethen (2008); Edelman & Raj Rao (2005); Grava & Klein (2008); Prähofer & Spohn (2004)**].
- The Runge-Kutta method, including its variants, is a standard ODE solver. Can we do better for integrable ODEs such as the Painlevé equations?
- Painlevé equations are solvable by the isomonodromy method through an associated Riemann-Hilbert problem (inverse scattering for ODEs). How can we use this in the development of software for studying the Painlevé equations numerically?
- Should we use a "integrable discretization" of the Painlevé equations? It is well known that there **discrete Painlevé equations**, which are integrable discrete equations that tend to the associated Painlevé equations in an appropriate continuum limit.

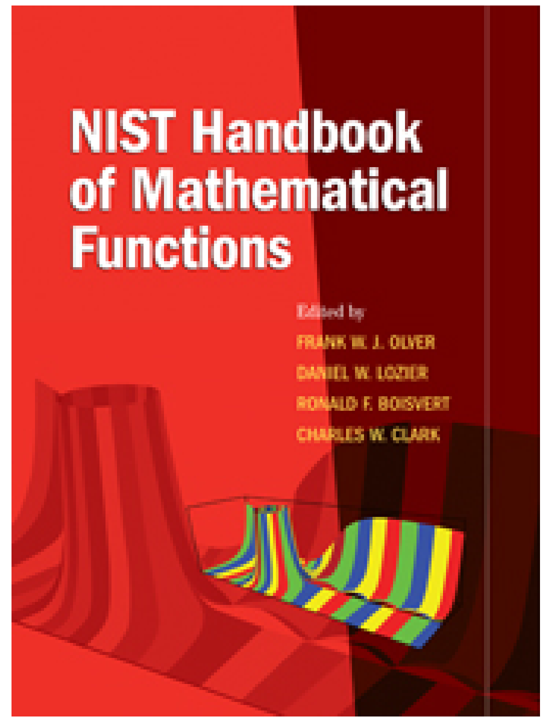
Objectives

- To provide a complete classification and unified structure of the special properties which the Painlevé equations and Painlevé σ-equations possess — the presently known results are rather fragmentary and non-systematic.
- Develop algorithmic procedures for the classification of equations with the Painlevé property.
- Develop software for numerically studying the Painlevé equations which utilizes the fact that they are integrable equations solvable using isomonodromy methods.
- To produce a general theorem on uniform asymptotics for linear systems to cover all those systems which arise as isomonodromy problems of the Painlevé equations.

Reference

P A Clarkson, Painlevé equations — nonlinear special functions, in "*Orthogonal Polynomials and Special Functions: Computation and Application*" [Editors F Marcellàn and W van Assche], *Lect. Notes Math.*, **1883**, Springer, Berlin (2006) pp 331–411





Painlevé Project

An e-site, maintained at NIST, has been established. Interested readers are asked to send to the site:

1. pointers to new work on the theory of the Painlevé equations, algebraic, analytical, asymptotic or numerical
2. pointers to new applications of the Painlevé equations
3. suggestions for possible new applications of the Painlevé equations
4. requests for specific information about the Painlevé equations.

The e-site will work as follows:

1. You must be a "subscriber to post messages to the e-site. To become a subscriber, send email to `daniel.lozier@nist.gov`
2. To post a message after becoming a subscriber, send email to PainleveProject@nist.gov. The message will be forwarded to every subscriber.
3. See `http://cio.nist.gov/esd/emaildir/lists/painleveproject/threads.html` for the complete archive of posted messages. This archive is visible to anyone, not just subscribers.
4. See `http://cio.nist.gov/esd/emaildir/lists/painleveproject/subscribers.html` for the complete list of subscribers. This list is visible to anyone, not just subscribers.