THE CIRCLE PROBLEM, THE DIVISOR PROBLEM BESSEL SERIES EXPANSIONS, AND WEIGHTED DIVISOR SUMS

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Motivated by Two Entries in Ramanujan's Lost Notebook

With Further Assistance by

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Once Again – Mystery in the Lost Notebook

I have shown you today the highest secret of my own realization. It is supreme and most mysterious indeed.

Verse 575, Vivekachudamani of Adi Shankaracharya
Sixth Century, A.D.

Ramanujan's Passport Photo

A Page From Ramanujan's Lost Notebook

$$\underline{0 < \theta < 1.}$$

$$\left[\tfrac{x}{1}\right]\sin 2\pi\theta + \left[\tfrac{x}{2}\right]\sin 4\pi\theta + \left[\tfrac{x}{3}\right]\sin 6\pi\theta + \left[\tfrac{x}{4}\right]\sin 8\pi\theta + \cdots$$

$$= \pi x\left(\tfrac{1}{2} - \theta\right) - \tfrac{1}{4}\cot\pi\theta + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{J_1(4\pi\sqrt{\lambda\theta x})}{\sqrt{\lambda\theta}} - \frac{J_1(4\pi\sqrt{\lambda(1-\theta)x})}{\sqrt{\lambda(1-\theta)}} + \frac{J_1(4\pi\sqrt{\lambda(1+\theta)x})}{\sqrt{\lambda(1+\theta)}} - \frac{J_1(4\pi\sqrt{\lambda(2-\theta)x})}{\sqrt{\lambda(2-\theta)}} + \frac{J_1(4\pi\sqrt{\lambda(2+\theta)x})}{\sqrt{\lambda(2+\theta)}} - \cdots\cdots\right\}$$

where $[x]$ denotes the greatest integer in x if x is not an integer and $x - \frac{1}{2}$ if x is an integer.

$$\left[\tfrac{x}{1}\right]\cos 2\pi\theta + \left[\tfrac{x}{2}\right]\cos 4\pi\theta + \left[\tfrac{x}{3}\right]\cos 6\pi\theta + \left[\tfrac{x}{4}\right]\cos 8\pi\theta + \cdots$$

$$= -x\log(2\sin\pi\theta) + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{I_1(4\pi\sqrt{\lambda\theta x})}{\sqrt{\lambda\theta}} + \frac{I_1(4\pi\sqrt{\lambda(1-\theta)x})}{\sqrt{\lambda(1-\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(1+\theta)x})}{\sqrt{\lambda(1+\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(2-\theta)x})}{\sqrt{\lambda(2-\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(2+\theta)x})}{\sqrt{\lambda(2+\theta)}} + \cdots\cdots\right\},$$

where $\quad I_1(x) = H_1(x) - Y_1(x).$

The First Claim

To state Ramanujan's claims, we need to first define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{1}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

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where, as customary, $[x]$ is the greatest integer less than or equal to x.

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}.$$

The First Claim

Entry

If $0 < \theta < 1$, $x > 0$, $F(x)$ *is defined by* (1), *and* $J_1(x)$ *denotes the ordinary Bessel function of order* 1, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\frac{1}{2}-\theta\right) - \frac{1}{4}\cot(\pi\theta)$$

$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{2}$$

Convergence is NOT Obvious

$$J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right).$$

Hence, as $m, n \to \infty$, the terms of the double series on the right-hand side of (2) are asymptotically equal to

$$\frac{1}{\pi\sqrt{2}x^{1/4}m^{3/4}} \left(\frac{\cos\left(4\pi\sqrt{m(n+\theta)x} - \frac{3}{4}\pi\right)}{(n+\theta)^{3/4}} - \frac{\cos\left(4\pi\sqrt{m(n+1-\theta)x} - \frac{3}{4}\pi\right)}{(n+1-\theta)^{3/4}} \right).$$

Circle Problem

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. Different signs and different orders of the summands yield distinct representations. E.g., $r_2(5) = 8$.

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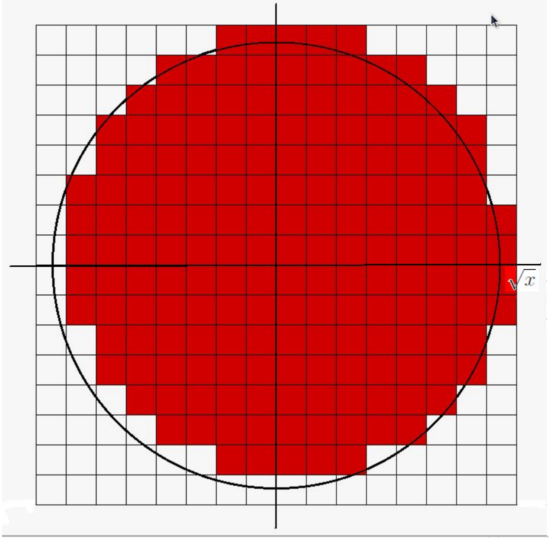
Each representation of n as a sum of two squares can be associated with a lattice point in the plane. For example, $5 = (-2)^2 + 1^2$ can be associated with the lattice point $(-2, 1)$. Then each lattice point can be associated with a unit square, say that unit square for which the lattice point is in the southwest corner.

Circle Problem



Circle Problem

$$R(x) := \sum_{0 \le n \le x}{}' r_2(n) = \pi x + P(x), \qquad (3)$$

where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2} r_2(x)$ is counted.

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$$R(x) < \pi(\sqrt{x} + \sqrt{2})^2,$$
$$R(x) > \pi(\sqrt{x} - \sqrt{2})^2,$$

$$R(x) = \pi x + O(\sqrt{x}) \qquad \text{Gauss}$$

Sierpinski (1904), Landau (1915), and Hardy (1915) independently proved that

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}). \qquad (4)$$

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0<|z|<\infty, \qquad \nu\in\mathbb{C}.$$

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$$P(x) = O(x^{1/3}). \qquad \text{Sierpinski}$$

The Error Term

$$P(x) = O(x^{a+\epsilon}), \qquad \epsilon \geq 0.$$

$a = 1/2,$	Gauss($\approx$ 1800)
$a = 1/3,$	Sierpinski(1906), Landau(1913)
$a = 33/100,$	van der Corput(1922)
$a = 37/112 = 0.3303\ldots$	Littlewood and Walfisz(1925), Hardy(1925)
$a = 27/82 = 0.3292\ldots$	van der Corput(1928)
$a = 15/46 = 0.3260\ldots$	Chih Tsung-Tao(1950), Richert(1953)
$a = 12/37 = 0.3243\ldots$	Chen(1963), Kolesnik(1969)
$a = 346/1067 = 0.3242\ldots$	Kolesnik(1973)
$a = 35/108 = 0.3240\ldots$	Kolesnik(1982)

The Error Term

$$
\begin{aligned}
a &= 7/22 = 0.3181\ldots && \text{Iwaniec and Mozzochi(1988)}\\
a &= 23/73 = 0.3150\ldots && \text{Huxley(1993)}\\
a &= 131/416 = 0.3149\ldots && \text{Huxley(2003)}
\end{aligned}
$$

The Error Term

Let $g(x)$ be a non-negative function of x.

$G(x) = \Omega(g(x)), \Omega_+(g(x)), \Omega_-(g(x))$ if there exists a positive constant K such that

$$|G(x)| > Kg(x), \qquad G(x) > Kg(x), \qquad G(x) < -Kg(x),$$

each inequality holding on some sequence $\{x_n\}$, $x_n \to \infty$.

$$P(x) = \Omega_\pm(x^{1/4}), \qquad \text{Hardy(1915)}$$

$$P(x) = \Omega_-\left((x \log x)^{1/4}\right), \qquad \text{Hardy(1915)}$$

$$P(x) = \Omega_\pm(x^{1/4}(\log\log x)^{1/4}(\log\log\log x)^{1/4}),$$

Gangadharan(1961)

$$P(x) = \Omega_+\left(x^{1/4}\exp\left(C_1(\log\log x)^{1/4}(\log\log\log x)^{-3/4}\right)\right),$$

Corrádi and Katai(1967)

The Error Term

$$P(x) = \Omega_{-}\left(x^{1/4}(\log x)^{1/4}(\log\log x)^{(\log 2)/4}\exp\left(-C_2\sqrt{\log\log\log x}\right)\right),$$

Hafner(1981)

$$P(x) = \Omega_{-}\left(x^{1/4}(\log x)^{1/4}\frac{(\log\log x)^{3(2^{1/3}-1)/4}}{(\log\log\log x)^{5/8}}\right).$$

Soundararajan(2003)

$$\frac{1}{4}\log 2 = 0.1732\ldots \qquad \frac{3(2^{1/3}-1)}{4} = 0.1949\ldots$$

Key Identities

(Hardy) For $\sigma = \text{Re } s > 0$,

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2 + 4\pi^2 n)^{3/2}}.$$

(Ramanujan) If $a, b > 0$,

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}}.$$

Differentiate w.r.t. b. Let $a \to 0$.

The First Claim – Now Proved by BCB, S. Kim, A. Zaharescu

Theorem

Let $F(x)$ be defined by (1), let $J_1(x)$ denote the ordinary Bessel function of order 1, let $0 < \theta < 1$, and let $x > 0$. Then

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\frac{1}{2}-\theta\right) - \frac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{5}$$

Version of BCB & Zaharescu

Theorem

If $0 < \theta < 1$, $x > 0$, *and* $J_1(x)$ *denotes the ordinary Bessel function of order* 1, *then*

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Weighted divisor sums and Bessel function series (with A. Zaharescu), Math. Ann. **335** (2006), 249–283.

Eliminating Bessel Functions

Theorem

For $0 < \theta < 1$ *and* $x > 0$,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) - \pi x\left(\tfrac{1}{2} - \theta\right)$$

$$= \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{n+\theta}\sin^2\left(\frac{\pi(n+\theta)x}{m}\right) - \frac{1}{n+1-\theta}\sin^2\left(\frac{\pi(n+1-\theta)x}{m}\right)\right).$$

Corollary

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + 2\sqrt{x}$$

$$\times \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}.$$

"Equivalence" of Formulations of
Sierpinski, Hardy, and Ramanujan

$$r_2(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d-1)/2}. \tag{6}$$

Therefore, by (6),

$$\sum_{k=1}^{\infty} r_2(k) \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx})$$

$$= 4 \sum_{k=1}^{\infty} \sum_{\substack{d|k \\ d \text{ odd}}} (-1)^{(d-1)/2} \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx})$$

$$= 4\sqrt{x} \lim_{N\to\infty} \sum_{m,n\le N} \left(\frac{J_1(2\pi\sqrt{m(4n+1)x})}{\sqrt{m(4n+1)}} - \frac{J_1(2\pi\sqrt{m(4n+3)x})}{\sqrt{m(4n+3)}} \right)$$

Equivalence

$$= 2\sqrt{x} \lim_{N\to\infty} \sum_{m,n\le N} \left(\frac{J_1(4\pi\sqrt{m(n+\frac{1}{4})x})}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1(4\pi\sqrt{m(n+\frac{3}{4})x})}{\sqrt{m(n+\frac{3}{4})}} \right).$$

Theorem

If $0 < \theta < 1$, $x > 0$, *and* $J_\nu(x)$ *denotes the ordinary Bessel function of order* ν, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\frac{1}{2}-\theta\right) - \frac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x}\lim_{N\to\infty}\sum_{m=1}^{N}\sum_{n=0}^{N}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

Summary

Three Orders of Summation

1. Iterated Double Sum (sum on n first, then sum on m); Ramanujan; Berndt, Sun Kim, and Zaharescu
2. Iterated Double Sum (sum on m first, then sum on n); Berndt and Zaharescu
3. Symmetric Sum (m, n tend to ∞ together); Ramanujan(?) Berndt, Sun Kim, and Zaharescu

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Can one show the inversion in order of summation directly?

Very Brief Sketch of Proof in Symmetric Summation

$$d_\chi(n) = \sum_{d|n} \chi(d), \qquad \tau(\chi) = \sum_{n=1}^{q-1} \chi(n) e^{2\pi i n/q}$$

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \chi(n) = \sum_{1 \le n \le x}{}' d_\chi(n).$$

Lemma

If $0 < a < q$ *and* $(a, q) = 1$, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin\left(\frac{2\pi n a}{q}\right)$$

$$= -i \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a) \tau(\overline{\chi}) \sum_{1 \le n \le dx/q}{}' d_\chi(n).$$

Very Brief Sketch of Proof in Symmetric Summation

$$H(a,q,x) := \frac{\sqrt{qx}}{2}\left\{\sum_{m=1}^{\infty}\sum_{\substack{r=1\\ r\equiv a \bmod q}}^{\infty}\frac{J_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}} - \sum_{m=1}^{\infty}\sum_{\substack{r=1\\ r\equiv -a \bmod q}}^{\infty}\frac{J_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}}\right\}.$$

Theorem

If q is prime and $0 < a < q$, then

$$H(a,q,x) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right) - \pi x\left(\frac{1}{2}-\frac{a}{q}\right) + \frac{1}{4}\cot\left(\frac{a\pi}{q}\right).$$

Very Brief Sketch of Proof in Symmetric Summation

Theorem

Let q be a positive integer, and let χ be an odd primitive character modulo q. Then, for any $x > 0$,

$$\sum_{n\le x}{}' d_\chi(n) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi}) + \frac{i\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}J_1\left(4\pi\sqrt{nx/q}\right).$$

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$$+ \frac{i\sqrt{x}}{\tau(\overline{\chi})}\sum_{1\leq h<q/2}\overline{\chi}(h)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{h}{q})x}\right)}{\sqrt{m(n+\frac{h}{q})}}\right.$$
$$\left.-\frac{J_1\left(4\pi\sqrt{m(n+1-\frac{h}{q})x}\right)}{\sqrt{m(n+1-\frac{h}{q})}}\right\}.$$

Three Lemmas

Lemma

If j is any positive integer, then

$$I_j := \int_0^{1/2} \cot(\pi\theta)\sin(2\pi j\theta)d\theta = \frac{1}{2}. \tag{7}$$

Three Lemmas

Lemma

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Lemma

For $a, b > 0$,

$$\int_0^{\infty} \sin(au^2)J_1(bu)du = \frac{1}{b}\sin\left(\frac{b^2}{4a}\right).$$

Sketch of Proof of Ramanujan's Entry–Lemmas

Lemma

For any real number x,

$$-\sum_{m=1}^{\infty} \frac{\sin(2\pi mx)}{\pi m} = \begin{cases} 0, & \text{if } x \text{ is an integer}, \\ x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer}. \end{cases}$$

Sketch of Proof of Ramanujan's Entry

Define, for $0 < \theta < 1$,

$$f(\theta) := \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\} - \frac{1}{4}\cot(\pi\theta).$$

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We find the Fourier sine series of $f(\theta)$ on $(0, \frac{1}{2})$, and so write

$$f(\theta) = \sum_{j=1}^{\infty} b_j \sin(2\pi j\theta).$$

Sketch of Proof of Ramanujan's Entry, Cont.

$$b_j = 2\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int_0^{1/2}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}}\right.$$

$$\left.-\frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}\sin(2\pi j\theta)d\theta-\frac{1}{2}, \qquad (8)$$

In the first set of integrals of the series on the far right-hand side of (8), set

$$u = 4\pi\sqrt{m(n+\theta)x}, \qquad \text{so that} \qquad \frac{d\theta}{\sqrt{m(n+\theta)}} = \frac{du}{2\pi m\sqrt{x}},$$

and in the second set of integrals of the series, set

$$u = 4\pi\sqrt{m(n+1-\theta)x}, \qquad \text{so that} \qquad \frac{d\theta}{\sqrt{m(n+1-\theta)}} = -\frac{du}{2\pi m\sqrt{x}}.$$

Thus, we find that, for each $j \geq 1$,

Sketch of Proof of Ramanujan's Entry, Cont.

$$
\begin{aligned}
b_j + \frac{1}{2} &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\pi m} \int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1)x}} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du \\
&= \sum_{m=1}^{\infty} \frac{1}{\pi m} \int_0^{\infty} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du. \qquad (9)
\end{aligned}
$$

Sketch of Proof of Ramanujan's Entry, Cont.

We now apply the second lemma with $a = j/(8\pi mx)$ and $b = 1$, after which we apply the third lemma. Hence, we deduce from (9) that

$$\begin{aligned} b_j &= \sum_{m=1}^{\infty} \frac{\sin(2\pi mx/j)}{\pi m} - \frac{1}{2} \\ &= \begin{cases} -\dfrac{1}{2}, & \text{if } x/j \text{ is an integer,} \\ -\dfrac{x}{j} + \left[\dfrac{x}{j}\right], & \text{if } x/j \text{ is not an integer,} \end{cases} \\ &= F\left(\frac{x}{j}\right) - \frac{x}{j}, \end{aligned}$$

upon using the definition (1) of $F(x)$.

Sketch of Proof of Ramanujan's Entry, Cont.

$$f(\theta) = \sum_{j=1}^{\infty} F\left(\frac{x}{j}\right)\sin(2\pi j\theta) - x\sum_{j=1}^{\infty}\frac{\sin(2\pi\theta j)}{j}$$
$$= \sum_{j=1}^{\infty} F\left(\frac{x}{j}\right)\sin(2\pi j\theta) - x\pi\left(\frac{1}{2} - \theta\right),$$

where we employed the third lemma once again. Remembering the definition of $f(\theta)$, we see that we have completed the proof, except for establishing the convergence and uniform convergence with respect to θ.

Sketch of Proof of Ramanujan's Entry, Cont.

Let $a = \sqrt{4\pi x}$

$$S_1(a,\theta) := \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m^{3/4}}\left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3}{4}\pi)}{(n+1-\theta)^{3/4}}\right).$$

Sketch of Proof of Ramanujan's Entry, Cont.

Let $a = \sqrt{4\pi x}$

$$S_1(a,\theta) := \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m^{3/4}}\left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3}{4}\pi)}{(n+1-\theta)^{3/4}}\right).$$

$$S_2(a,\theta) := \sum_{m=1}^{\infty}\sum_{0\le n<m^3\log^5 m}\frac{1}{m^{3/4}}\left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right)$$

Sketch of Proof of Ramanujan's Entry, Cont.

We remove those terms where n is much smaller than m.

$$S_4(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{m^{1-\delta} \leq n < m^3 \log^5 m} \frac{1}{m^{3/4}} \left(\frac{\cos(a\sqrt{m(n+\theta)} - \frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4})}{(n+1-\theta)^{3/4}} \right).$$

Sketch of Proof of Ramanujan's Entry, Cont.

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$$S_4(a,\theta,\delta) = S_5(a,\theta,\delta) + S_6(a,\theta,\delta)$$

$$S_5 : n > m^{1+\delta}; \quad S_6 : n \le m^{1+\delta}$$

Handle S_5 in the same manner as S_3.

Sketch of Proof of Ramanujan's Entry, Cont.

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Handle S_5 in the same manner as S_3.

$$b = \frac{a(1-2\theta)}{4} = \pi\sqrt{x}(1-2\theta).$$

Sketch of Proof of Ramanujan's Entry, Cont.

$$S_7(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{m^{1-\delta} \leq n \leq m^{1+\delta}} \times \frac{\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right) \sin\left(\frac{a(1-2\theta)}{4}\sqrt{\frac{m}{n}}\right)}{m^{3/4} n^{3/4}}.$$

Sketch of Proof of Ramanujan's Entry, Cont.

Use Cauchy's criterion.

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Break up sum into small squares of size $L \times L$, where $L = [T^{\lambda}]$.

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Break up sum into small squares of size $L \times L$, where $L = [T^{\lambda}]$.
Separate two cases: x is not an integer; x is an integer.

Sketch of Proof of Ramanujan's Entry, Cont.

Use Cauchy's criterion.

$$\sum_{T\le m<2T}\ \sum_{T^{1-\delta}\le n\le(2T)^{1+\delta}} \frac{\sin\left(b\sqrt{\dfrac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\dfrac{1}{2}\right)}-\dfrac{3\pi}{4}\right)}{m^{3/4}n^{3/4}}.$$

Break up sum into small squares of size $L\times L$, where $L=[T^{\lambda}]$. Separate two cases: x is not an integer; x is an integer. Consider those points where either xm/n or xn/m or both are close to an integer.

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Approximate double sum by a double integral. Evaluate inner integral.

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$\frac{3\pi}{4}$ **is crucial.**

The Dirichlet Divisor Problem

Let $d(n)$ denote the number of positive divisors of the positive integer n.

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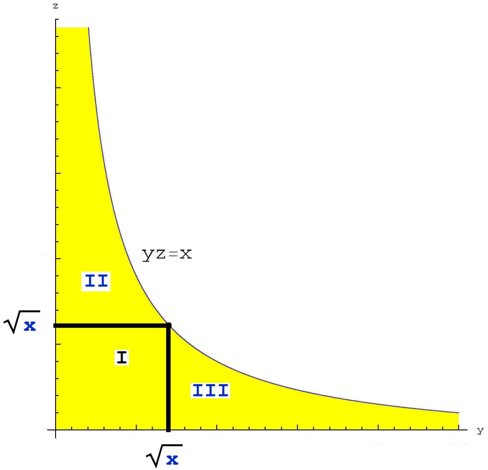
$$D(x) := \sum_{n \le x}{}' d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \Delta(x),$$

where the prime on the summation sign on the left-hand side indicates that if x is an integer then only $\frac{1}{2}d(x)$ is counted, γ is Euler's constant, and $\Delta(x)$ is the "error term."

Dirichlet's Argument

$$D(x) = \sum_{n \le x} \sum_{d | n} 1 = \sum_{d \le x} \left[\frac{x}{d} \right]$$

The Divisor Problem



Dirichlet's Argument

$$
\begin{aligned}
D(x) = \sum_{d \le x} \left[\frac{x}{d}\right] &= \sum_{d \le \sqrt{x}} \left[\frac{x}{d}\right] + \sum_{d \le \sqrt{x}} \left[\frac{x}{d}\right] - \left[\sqrt{x}\right]\left[\sqrt{x}\right] \\
&= 2 \sum_{d \le \sqrt{x}} \left(\frac{x}{d} + O(1)\right) - \left(\sqrt{x} - \{\sqrt{x}\}\right)^2 \\
&= 2x \sum_{d \le \sqrt{x}} \frac{1}{d} + O\left(\sqrt{x}\right) - x + O\left(\sqrt{x}\right) \\
&= 2x \left(\log \sqrt{x} + \gamma + O\left(1/\sqrt{x}\right)\right) - x + O\left(\sqrt{x}\right) \\
&= x \log x + (2\gamma - 1)x + O\left(\sqrt{x}\right).
\end{aligned}
$$

The Divisor Problem, Cont.

Let $Y_\nu(z)$ be the second solution of Bessel's differential equation of order ν, usually so denoted, and let $K_\nu(z)$ denote the modified Bessel function of order ν, usually so denoted.
Following Ramanujan, we set

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z). \tag{10}$$

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Voronoï (1903)

$$\Delta(x) = \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx})$$

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$$\Delta(x) = O(x^{1/3}\log x)$$

Ramanujan's Second Bessel Series Expansion

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer.} \end{cases} \tag{11}$$

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Entry

Let $F(x)$ be defined by (11), and let $I_1(x)$ be defined by (10). For $x>0$ and $0<\theta<1$,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta)=\frac{1}{4}-x\log(2\sin(\pi\theta))$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}}+\frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

Asymptotic Formulas

As $z \to \infty$,

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi\nu}{4} - \frac{\pi}{4}\right) + O\left(\frac{1}{z^{3/2}}\right),$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} + O\left(e^{-z} \frac{1}{z^{3/2}}\right),$$

Why Things Don't Work As They Did Before

1. $K_\nu(z)$ and $Y_\nu(z)$ have singularities at the origin.
2. The Fourier cosine series method does work, but additional difficulties arise.
3. There is no "cancellation" in the pairs of Bessel series terms.

Getting Rid of the Bessel Functions

It suffices to prove the following theorem.

Theorem

If $x > 0$ and $0 < \theta < 1$, then

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) - \frac{1}{4} + x\log(2\sin(\pi\theta))$$

$$= \frac{\sqrt{x}}{2\pi}\left\{\sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi^2(n+\theta)x}{m}\right)\right.\right.$$

$$\left. - \int_0^M \sin\left(\frac{2\pi^2(n+\theta)x}{t}\right)dt\right\} + \frac{1}{n+1-\theta}$$

$$\left.\left.\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi^2(n+1-\theta)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi^2(n+1-\theta)x}{t}\right)dt\right\}\right\}\right.$$

Notes

1. The individual series and integrals above each diverge by themselves.
2. The order of summation has been reversed from what Ramanujan claimed.
3. We need to assume that the double series converges for one value of θ. Then it converges for all values of θ.

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Three possible orders of summation

Notes

1. The individual series and integrals above each diverge by themselves.
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3. We need to assume that the double series converges for one value of θ. Then it converges for all values of θ.

Three possible orders of summation

(a) Iterated Double Sum (n first, m second); Ramanujan
(b) Iterated Double Sum (m first, n second); Berndt, Sun Kim, and Zaharescu
(c) Symmetric Double Sum; Berndt, Sun Kim, and Zaharescu

Numerical Calculations

$$L(x,\theta) := -\frac{1}{4} + x\log(2\sin(\pi\theta)) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2n\pi\theta)$$

$$R(x,\theta,M,N) := \frac{1}{2}\sqrt{x}\sum_{m=1}^{M}\sum_{n=0}^{N}\left(\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right)$$

Numerical Calculations

Table:

x	θ	M	N	$L(x,\theta)$	$R(x,\theta,M,N)$
0.5	0.5	20	200 500 1000	0.09657	0.10563 0.10237 0.11033
0.5	0.5	50	200 500 1000	0.09657	0.09394 0.09140 0.09834
0.5	0.5	200 500 1000	20	0.09657	0.13265 0.14249 0.11553
0.5	0.5	200 500 1000	50	0.09657	0.11492 0.12499 0.09840

Numerical Calculations

Table:

x	θ	M	N	$L(x,\theta)$	$R(x,\theta,M,N)$
1.75	0.25	20	200	0.35650	0.35218
			500		0.35655
			1000		0.35055
1.75	0.25	50	200	0.35650	0.39195
			500		0.39685
			1000		0.39056
1.75	0.25	200	20	0.35650	0.34695
		500			0.36495
		1000			0.34645
1.75	0.25	200	50	0.35650	0.32983
		500			0.34832
		1000			0.32934

Identities for Trigonometric Sums

Theorem

Let $I_1(x)$ be defined by (10)*. If* $0 < \theta, \sigma < 1$ *and* $x > 0$*, then*

$$\sum_{nm\le x}{}' \cos(2\pi n\theta)\cos(2\pi m\sigma)$$

$$= \frac{1}{4} + \frac{\sqrt{x}}{4}\sum_{n,m\ge 0}\left\{ \frac{I_1(4\pi\sqrt{(n+\theta)(m+\sigma)x})}{\sqrt{(n+\theta)(m+\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+\sigma)x})}{\sqrt{(n+1-\theta)(m+\sigma)}} + \frac{I_1(4\pi\sqrt{(n+\theta)(m+1-\sigma)x})}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+1-\sigma)x})}{\sqrt{(n+1-\theta)(m+1-\sigma)}} \right\}.$$

Identities for Trigonometric Sums

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d). \tag{12}$$

Theorem

If χ_1 and χ_2 are non-principal even primitive characters modulo p and q, respectively, then

$$\sum_{n\le x}{}' d_{\chi_1,\chi_2}(n) = \frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}} \sum_{n=1}^{\infty} d_{\overline{\chi_1},\overline{\chi_2}}(n)\left(\frac{x}{n}\right)^{\frac{1}{2}} I_1\left(4\pi\sqrt{\frac{nx}{pq}}\right).$$

Identities for Trigonometric Sums

Theorem

If p and q are primes, and $0 < a < p$ and $0 < b < q$, then

$$\sum_{nm \le x}{}' \cos(2\pi na/p)\cos(2\pi mb/q)$$

$$= \frac{1}{4} + \frac{\sqrt{pqx}}{4} \sum_{\substack{n,m=0 \\ n \equiv \pm a \bmod p \\ m \equiv \pm b \bmod q}}^{\infty} \frac{I_1(4\pi\sqrt{nmx/pq})}{\sqrt{nm}}.$$

Identities for Trigonometric Sums

$$\sum_{nm \leq x}{}' \cos(2\pi n\theta)\sin(2\pi m\sigma)$$

$$\sum_{nm \leq x}{}' nm \sin(2\pi n\theta)\sin(2\pi m\sigma)$$

Ramanujan in Better Health at Cambridge