INTEGRAL OPERATORS AND DELAY DIFFERENTIAL EQUATIONS

DAVID E. GILSINN, FLORIAN A. POTRA

Mathematical and Computational Sciences Division
National Institute of Standards and Technology
100 Bureau Drive, Stop 8910
Gaithersburg, MD 20899-8910 [1]

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ABSTRACT. The monodromy operator of a linear delay differential equation with periodic coefficients is formulated as an integral operator. The kernel of this operator includes a factor formed from the fundamental solution of the linear delay differential equation. Although the properties of the fundamental solutions are known, in general there is no closed form for the fundamental solution. This paper describes a collocation procedure to approximate the fundamental solution before the integral operator is discretized. Using arguments on collectively compact operators, the eigenvalues of the discretized monodromy operator are shown to converge to the eigenvalues of the monodromy operator in integral form. The eigenvalues of the monodromy operator tell the stability of the linear delay differential equation. An application to several cases of the Van der Pol oscillator with delay will be given.

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1. Introduction

Delay differential equations have occurred in many fields from biology (MacDonald [37]) to population dynamics (Kuang [34]) to machine tool dynamics (Balchandran [11] , Zhao and Balachandran [54] , Nayfeh et al. [38]). The study of machine tool dynamics has led to many problems involving delay differential equations. For example, in turning operations a cutting tool passes over a workpiece many times successively. The forces on the tool depend on chip thickness which is dependent on the tool's current position and its position one previous revolution of the workpiece, thus introducing a delay effect. Any irregularities in a previous cut produced by the tool can affect the current cut. The delay effect of the irregularities can introduce self-sustained oscillations of the tool against the workpiece, called *regenerative chatter*. This phenomenon has been studied by Tlusty and Placek [48] and Tobias [49] as early as the 1960's. Mathematically, chatter is a stable limit cycle of the delay differential equation that models the particular machining process. To a machinist it represents undesirable motions that can damage a good surface finish. Therefore, being able to determine the nature of the stability of periodic solutions to delay differential equations is crucial to determining the quality of the workpiece surface finish. It will be seen that the nature of the stability is determined by the eigenvalues of a certain integral operator, called the monodromy operator, associated with the delay differential equation.

Many of the models of machining operations fall into the class of autonomous delay differential equations of the form

$$\dot{x} = X(x(t), x(t-h)), \tag{1}$$

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where $x, X \in \mathbb{R}^n$, $h > 0$. A main concern is the question of stability of periodic solutions of (1). See, for example, Butcher et al. [20], Engelborghs et al. [22], Luzyanina and Engelborghs [35], and Luzyanina and Roose [36].

The analysis of the stability of a periodic solution for (1) usually involves the following considerations. Let $p(t)$, $p \in \mathbb{R}^n$, be a periodic function of some period $T > 0$ that may or may not be an exact periodic solution of (1). This function may, for example, have been developed by a Galerkin method or harmonic balance. In this paper we will only consider the case of $T > h$. Then, the linear variational equation about the periodic function $p(t)$ can be written

$$\dot{z}(t) = A(t)z(t) + B(t)z(t-h), \tag{2}$$

where

$$\begin{aligned} A(t) &= X_1(p(t), p(t-h)), \\ B(t) &= X_2(p(t), p(t-h)). \end{aligned} \tag{3}$$

The subscripts represent the partial derivatives with respect to the first and second variables, respectively. Since $p(t)$ is periodic with period $T > 0$, $A(t)$ and $B(t)$ are clearly periodic, with the same period. Let $\mathcal{L} = \max\left(\|A(t)\|_\infty,\ \|B(t)\|_\infty\right)$, where $\|\cdot\|$ will be used to represent a matrix norm.

Let C_h denote the space of continuous functions from $[-h, 0]$ to $\mathbb{R}^n$, with norm in C_h given by $|\phi| = \max|\phi(s)|$ for $-h \le s \le 0$. C_h is a Banach space with respect to this norm.

We define the period map $U : C_h \to C_h$ with respect to (2) by

$$(U\phi)(s) = z(s+T), \tag{4}$$

where $z(s)$ is a solution of (2) satisfying $z(s) = \phi(s)$ for $s \in [-h, 0]$. For $T > h$, U is a compact operator on C_h, whose spectrum is at most countable with zero as the only possible limit point (Halanay [27]). Halanay [27] has also shown that the period operator U, also called the *monodromy operator*, can be represented as

$$(U\phi)(s) = Z(s+T, 0)\phi(0) + \int_{-h}^{0} Z(s+T, \alpha+h)B(\alpha+h)\phi(\alpha)\, d\alpha, \tag{5}$$

where $Z(s, \alpha)$ is the fundamental solution of (2) which satisfies (2) for $s > \alpha$, $Z(\alpha, \alpha) = I_n$, the $n \times n$ identity matrix, and $Z(s, \alpha) = 0$ for $s < \alpha$. A finite monodromy matrix is obtained by discretizing (5). The specific monodromy matrix used depends on the choice of discretization method used.

The nature of the stability of the approximate periodic solution $p(t)$ to (2) depends on the eigenvalues of the period map (4). These eigenvalues are also referred to as *characteristic multipliers*, since

$$z(s+T) = (U\phi)(s) = \lambda\phi(s) = \lambda z(s) \tag{6}$$

for some λ and some $z(s) = \phi(s)$ not identically zero for $s \in [-h,\ 0]$. $\phi(s)$ in this case will be an eigenfunction of U. Along with its relation to stability the λ can also be thought about as a measure of how close $z(s)$ is to periodicity with period $T > 0$. If $\lambda = 1$, then $z(s)$ is a periodic solution with period $T > 0$. Although the term characteristic multiplier and eigenvalue of U are sometimes used interchangeably, we will maintain the usage of the term eigenvalue when referring to operators.

The computation of the eigenvalues for (5) generally involves some form of approximation. In this paper we will use two levels of approximation. The first level is the discretization of the integral and the next level is the approximation of the fundamental solution $Z(s, \alpha)$. Only in very rare cases is the fundamental solution exactly computable. The discretization of the monodromy operator (5) produces a matrix that is referred to as a discrete monodromy matrix. However, for simplicity, when the context is clear, it is referred to as a *monodromy matrix*. It, of course, depends on the quadrature method used. The analysis of the convergence of the eigenvalues of the discrete monodromy matrices is the main aim of this paper, but it will first require estimating the fundamental solution $Z(s, \alpha)$.

In Section 2 we will give a brief discussion of the Floquet theory related to the period map. It will give the theoretical background to the properties of the monodromy operator. In Section 3 we will develop a pseudo-spectral approximation to the fundamental matrix and prove an error analysis result for the approximation. In Section 4 we will discretize the monodromy operator and form an eigenvalue problem to approximate the the eigenvalues of the discretized operators. A discussion of eigenvalue convergence will then be given in Section 5. Finally, in Section 6 we compute the characteristic multipliers for several cases of the Van der Pol oscillator with delay.

2. A Floquet Theory for Delay Differential Equations

A Floquet theory for (2) has been developed by Stokes [46]. If $\sigma(U)$ represents the spectrum of U, then for each $\lambda \in \sigma(U)$, $U\phi = \lambda\phi$ for some $\phi \in C_h$, $\phi \neq 0$. That is, the spectrum consists of eigenvalues. Furthermore, the space C_h can be decomposed as the direct sum of two invariant subspaces

$$C_h = E(\lambda) \oplus K(\lambda). \tag{7}$$

$E(\lambda)$ is finite dimensional and composed of the eigenfunctions with respect to λ of U. If $\{\psi_i\}$, $i = 1, \cdots, d$, is a basis for $E(\lambda)$ and we let Ψ be the matrix with columns ψ_j for $j = 1, \cdots, d$, then there is a matrix $\mathcal{M}$ such that

$$U\Psi = \Psi\mathcal{M}. \tag{8}$$

Thus we can think of C_h as being a countable direct sum of the invariant subspaces $E(\lambda_i)$ plus a possible remainder subspace, R. That is

$$C_h = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus R, \tag{9}$$

where R is a "remainder" set in which any solution of (2) with initial condition in R decays faster than any exponential.

For each of the $E(\lambda_i)$ there is a basis set Ψ_i, and a matrix $\mathcal{M}(\lambda_i)$. If we define an at most countable basis set $\{\Psi_i\}$, $i = 1, 2, \cdots$, then we can think about U operating on $\bigoplus_{i=1}^{\infty} E(\lambda_i)$ as being represented by an infinite dimensional matrix $\mathcal{M}_\infty$. This matrix is referred to as the monodromy matrix. Its eigenvalues are the Floquet or characteristic multipliers. Again we will avoid the term characteristic multiplier for the rest of this work in favor of the term eigenvalue. The approximate periodic solution $p(t)$ of (1) is stable if all of the eigenvalues of U are within the unit circle and unstable if there is at least one with magnitude greater than unity. We note that if $p(t)$ is an exact periodic solution of (1) then one of the eigenvalues is exactly one.

3. Approximating the Fundamental Matrices by Pseudo-spectral Collocation

Numerically approximating a solution to a delay differential equation has been studied by many authors. See, for example, Paul [40], Shampine and Thompson [43], Willé and Baker [53]. For a spectral method for solving delay differential equations with constant coefficients see Ito et al. [31]. In this paper we will seek a collocation representation for the fundamental solution. There have been some studies in which representational solutions to delay differential equations have been sought. In particular, Engelborghs et al. [22] studied collocation methods for computing periodic solutions for delay differential equations. There have been studies of spline approximations by Banks and Kappel [13] and Kemper [33]. For a symbolic method for computing fundamental solutions for time-periodic ordinary differential equations see Sinha and Butcher [44].

In this paper the collocation representation will be used to construct a sequence of monodromy operators. Each of these in turn will be discretized to form a family of collectively compact operators. We will study the convergence of this family to (5) in Section 5.2.

3.1. **The Method of Steps.** Let $C_h(a)$ denote the space of continuous functions from $[a-h,\ a]$ to $\mathbb{R}^n$, with norm in $C_h(a)$ given by $|\phi| = \max|\phi(s)|$ for $a-h \le s \le a$. Note that $C_h = C_h(0)$. We wish to solve the linear delay differential equation

$$\dot{z}(t) = A(t)z(t) + B(t)z(t-h), \tag{10}$$

where $A(t)$ and $B(t)$, given by (3), are $n \times n$ matrices of continuous functions, periodic with period $T > 0$ over the interval $[a,\ b]$ with b finite. The initial condition is given by $z(t) = \phi(t)$ on $[a-h,\ a]$, $z(a) = z_0$.

The object of the *method of steps* is to reduce the problem of directly solving the delay equation (10) to solving a finite sequence of ordinary differential equations. This method has been used for many years in delay differential equations (see e. g. Bellman and Cooke [17]).

In the present context we begin by first finding the smallest positive integer q such that $a+qh \ge b$. The integer q depends on h, but h is fixed for a given problem. We now consider the finite set of intervals $[a,\ a+h],\ [a+h,\ a+2h]\ \cdots,\ [a+(q-a)h,\ a+qh]$, where the point b falls within the last interval. If b is an exact multiple, then $a+qh = b$.

At the first step,

$$\dot{z}_1(t) = A(t)z_1(t) + B(t)z_1(t-h), \tag{11}$$

where $z_1(t-h) = \phi(s)$ for some initial function $\phi \in\ C_h(a)$ and $s = t-h$. Thus the initial problem becomes an ordinary differential equation. Then, on $[a+h,\ a+2h]$ we solve

$$\dot{z}_2(t) = A(t)z_2(t) + B(t)z_2(t-h), \tag{12}$$

where $z_2(a+h) = z_1(a+h)$, $z_2(t-h) = z_1(s)$ for $s \in\ [a,\ a+h], s = t-h$. Again, we solve (12) as an ordinary differential equation. The process is continued so that on $[a+(i-1)h,\ a+ih]$, for $i = 1, 2, \cdots, q$,

$$\dot{z}_i(t) = A(t)z_i(t) + B(t)z_i(t-h), \tag{13}$$

with $z_i(a+(i-1)h) = z_{i-1}(a+(i-1)h)$. We then define $z(t)$ on $[a,b]$ as the concatenation of $z_i(t)$ for $t \in\ [a+(i-1)h,\ a+ih]$ and $i = 1, 2, \cdots, q$.

3.2. **A Sequence of Differential Equations.** Since we wish to use a Chebyshev collocation method to solve each of the differential equations (13) for $i = 1, 2, \cdots, q$, we will normalize each of the intervals $[a+(i-1)h,\ a+ih]$ to $[-1,\ 1]$ as we step through the finite sequence of differential equations (13).

The unique transformation between $[a+(i-1)h,\ a+ih]$, for $i = 1, 2, \cdots, q$, and $[-1, 1]$ is given as follows. For each $t \in\ [a+(i-1)h,\ a+ih]$ for $i = 1, 2, \cdots, q$, there is a unique $\eta \in\ [-1, 1]$ given by

$$\eta = \frac{2}{h}t - \frac{(2a+(2i-1)h)}{h}. \tag{14}$$

For $\eta \in\ [-1, 1]$ we have the unique $t \in\ [a+(i-1)h,\ a+ih]$ given by

$$t = \frac{h}{2}\eta + \frac{(2a+(2i-1)h)}{2}. \tag{15}$$

We note that the points $t \in\ [a+(i-1)h,\ a+ih]$ and $t-h \in [a+(i-2)h,\ a+(i-1)h]$ are translated to the same $\eta \in [-1, 1]$. This is clear from

$$\frac{2}{h}(t-h) - \frac{(2a+(2i-3)h}{h} = \frac{2}{h}t - \frac{(2a+(2i-1)h}{h}. \tag{16}$$

We can now shift the solving of the sequence of delay problems

$$\dot{z}_i(t) = A(t)z_i(t) + B(t)z_i(t-h), \tag{17}$$

for $t \in\ [a+(i-1)h,\ a+ih]$ and $i = 1, 2, \cdots, q$, into solving a sequence of ordinary differential equations

$$u_i'(\eta) = \frac{h}{2}\widehat{A}_i(\eta)u_i(\eta) + \frac{h}{2}\widehat{B}_i(\eta)u_{i-1}(\eta), \tag{18}$$

where, for $t \in [a+(i-1)h,\ a+ih]$,

$$\begin{aligned} u_i(-1) &= u_{i-1}(1), \\ u_i(\eta) &= z_i(t), \\ \widehat{A}_i(\eta) &= A(t), \\ \widehat{B}_i(\eta) &= B(t), \\ u_{i-1}(\eta) &= z_i(t-h). \end{aligned} \tag{19}$$

The initial function is

$$u_0(\eta) = z_1(t-h) = \phi(t-h), \quad t-h \in [a-h,\ a]. \tag{20}$$

Let the columns of the identity matrix I_n be written as $e_j = (0, \cdots, 1, \cdots, 0)^T$, where one is the j-th element and all others are zero. We can now approximate the fundamental solution for (2) on $[a, b]$ by first solving n sequences of q differential equations (18) subject to

$$\begin{aligned} u_i(-1) &= u_{i-1}(1) \\ u_0(\eta) &= 0, \quad \eta \in [-1, 1] \\ u_1(-1) &= e_j \end{aligned} \tag{21}$$

where $j = 1, \cdots, n$. We can then transform back to the t domain. In the next section we show how the Lagrange polynomials can be used to develop a collocation solution to each differential equation in the sequence (18).

3.3. **A Pseudo-spectral Collocation Algorithm.** In this section we follow the pseudo-spectral method used by Bueler [19]. Although an analysis of the stability of pseudo-spectral methods for partial differential equations has been given by Gottlieb [25], we will develop a separate error analysis result in the next section for the pseudo-spectral method described here.

We first define a sequence of projection operators. Let N be a positive integer. Let P_N be the projection operator that associates a continuous function f defined on $[-1,\ 1]$ with the unique N-th degree Lagrange polynomial interpolating through the $N+1$ Chebyshev extreme points

$$\eta_k = \cos\left(\frac{k\pi}{N}\right), \quad k = 0, 1, \cdots, N. \tag{22}$$

If the Lagrange interpolation polynomials at these points are given by

$$l_j(\eta) = \prod_{\substack{k=0 \\ k\neq j}}^{N} \left(\frac{\eta - \eta_k}{\eta_j - \eta_k}\right). \tag{23}$$

for $j = 1,\ 2, \cdots,\ N$ and $l_j(\eta_k) = \delta_{jk}$, where $\delta_{jj} = 1,\ \delta_{jk} = 0,\ j \neq k$, then

$$\left(P_N f\right)(\eta) = \sum_{k=0}^{N} f\left(\eta_k\right) l_j(\eta) \tag{24}$$

We have that $|P_N| \leq \mathcal{P}$ for some $\mathcal{P} > 0$ by the Banach-Steinhaus Theorem.

For $\eta \in [-1, 1]$ we set

$$\widehat{u}_i(\eta) = \sum_{j=0}^{N} w_j^{(i)} l_j(\eta), \tag{25}$$

where the subscript i indicates that η is associated with the unique $t \in [a+(i-1)h,\ a+ih]$, $w_j^{(i)}$ is an n-vector to be determined, and the hat is intended to indicate a solution to the problem

$$\widehat{u}_i'(\eta) = P_N\left(\frac{h}{2}\widehat{A}_i\widehat{u}_i(\eta) + \frac{h}{2}\widehat{B}_i\widehat{u}_{i-1}(\eta)\right), \tag{26}$$

subject to the collocation condition described below.

We also need to form

$$\widehat{u}_i'(\eta) = \sum_{j=0}^{N} w_j^{(i)} l_j{}'(\eta). \tag{27}$$

At the Chebyshev points we will designate

$$D_{kj} = l_j{}'(\eta_k). \tag{28}$$

The values for these derivatives are given in Gottlieb and Turkel [26] or Trefethen [50] but we state the values for D_{kj} here for completeness.

$$\begin{aligned} D_{00} &= \frac{2N^2+1}{6}, \\ D_{NN} &= -D_{00}, \\ D_{jj} &= \frac{-\eta_j}{2(1-\eta_j^2)}, \quad j = 1, 2, \cdots, N-1 \\ D_{ij} &= \frac{c_i(-1)^{i+j}}{c_j(\eta_i - \eta_j)}, \quad i \neq j, i, j = 0, \cdots, N, \end{aligned} \tag{29}$$

where

$$c_i = \begin{cases} 2, & i = 0 \text{ or } N; \\ 1, & \text{otherwise.} \end{cases} \tag{30}$$

One of the principal reasons for selecting the Chebyshev points (22) is that the pseudo-spectral differentiation matrix (29) is known exactly. Some further advantages are discussed in Salzer [42].

For notation, let

$$\begin{aligned} \widehat{u}_i(\eta) &= \left(u_{i1}(\eta), \cdots, u_{in}(\eta)\right)^T, \\ \widehat{A}_i(\eta) &= \left[\widehat{A}_{pq}^{(i)}(\eta)\right]_{p,q=1,\cdots,n}, \\ \widehat{B}_i(\eta) &= \left[\widehat{B}_{pq}^{(i)}(\eta)\right]_{p,q=1,\cdots,n}. \end{aligned} \tag{31}$$

We then write the collocation polynomial elements of $\widehat{u}_i(\eta)$ as $\widehat{u}_{ir}(\eta),\ r = 1, \cdots, n$, where

$$\widehat{u}_{ir}(\eta) = \sum_{k=0}^{N} w_{rk}^{(i)} l_k(\eta), \tag{32}$$

at the Chebyshev points (22) to get

$$\begin{aligned} \widehat{u}_{ir}(\eta_j) &= w_{rj}^{(i)}, \\ \widehat{u}_{ir}'(\eta_j) &= \sum_{k=0}^{N} w_{rk}^{(i)} D_{jk}, \\ \widehat{u}_{i-1,r}(\eta_j) &= w_{rj}^{(i-1)}. \end{aligned} \tag{33}$$

The initial conditions for the sequence of differential equation are

$$\widehat{u}_{ir}(\eta_N) = \widehat{u}_{i-1,r}(\eta_0), \tag{34}$$

or

$$w_{rN}^{(i)} = w_{r0}^{(i-1)}, \tag{35}$$

for $r = 1, \cdots, n$.

The discretized differential equations are then given by

$$
\begin{aligned}
\begin{pmatrix} \sum_{1k}^{(i)} w_{1k}^{(i)} D_{jk} \\ \vdots \\ \sum_{rk}^{(i)} w_{rk}^{(i)} D_{jk} \\ \vdots \\ \sum_{nk}^{(i)} w_{nk}^{(i)} D_{jk} \end{pmatrix} &= \frac{h}{2} \begin{bmatrix} \widehat{A}_{11}^{(i)}(\eta_j) & \cdots & \widehat{A}_{1n}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \widehat{A}_{r1}^{(i)}(\eta_j) & \cdots & \widehat{A}_{rn}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \widehat{A}_{n1}^{(i)}(\eta_j) & \cdots & \widehat{A}_{nn}^{(i)}(\eta_j) \end{bmatrix} \begin{pmatrix} w_{1j}^{(i)} \\ \vdots \\ w_{rj}^{(i)} \\ \vdots \\ w_{nj}^{(i)} \end{pmatrix} \\
&\quad + \frac{h}{2} \begin{bmatrix} \widehat{B}_{11}^{(i)}(\eta_j) & \cdots & \widehat{B}_{1n}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \widehat{B}_{r1}^{(i)}(\eta_j) & \cdots & \widehat{B}_{rn}^{(i)}(\eta_j) \\ \vdots & \cdots & \vdots \\ \widehat{B}_{n1}^{(i)}(\eta_j) & \cdots & \widehat{B}_{nn}^{(i)}(\eta_j) \end{bmatrix} \begin{pmatrix} w_{1j}^{(i-1)} \\ \vdots \\ w_{rj}^{(i-1)} \\ \vdots \\ w_{nj}^{(i-1)} \end{pmatrix}
\end{aligned} \tag{36}
$$

for $j = 0, 1, \cdots, N-1$. These provide nN equations in $n(N-1)$ unknowns. The other n equations come from the initial conditions. We can write this system in a more compact form by first defining the following vectors

$$
\begin{aligned}
w_i &= \left(w_{10}^{(i)} \cdots w_{1N}^{(i)} w_{20}^{(i)} \cdots w_{2N}^{(i)} \cdots w_{n0}^{(i)} \cdots w_{nN}^{(i)} \right)^T, \\
w_{i-1} &= \left(w_{10}^{(i-1)} \cdots w_{1N}^{(i-1)} w_{20}^{(i-1)} \cdots w_{2N}^{(i-1)} \cdots w_{n0}^{(i-1)} \cdots w_{nN}^{(i-1)} \right)^T.
\end{aligned} \tag{37}
$$

We also define matrices $\widehat{A}_i$ and $\widehat{B}_i$.

$$
\widehat{A}_i = \begin{bmatrix}
\widehat{A}_{11}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \widehat{A}_{1n}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\
0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\
0 & 0 & \widehat{A}_{11}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \widehat{A}_{1n}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\widehat{A}_{n1}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \widehat{A}_{nn}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\
0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\
0 & 0 & \widehat{A}_{n1}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \widehat{A}_{nn}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}. \tag{38}
$$

$\widehat{B}_i$ is structured in a similar manner except every $(N+1)$th row includes an element $2/h$ to take care of the initial condition by canceling the $h/2$ in (36). Thus

$$
\widehat{B}_i = \begin{bmatrix}
\widehat{B}_{11}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \widehat{B}_{1n}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\
0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\
0 & 0 & \widehat{B}_{11}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \widehat{B}_{1n}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\
\frac{2}{h} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\widehat{B}_{n1}^{(i)}(\eta_0) & 0 & \cdots & 0 & \cdots & 0 & \widehat{B}_{nn}^{(i)}(\eta_0) & 0 & 0 & \cdots & 0 \\
0 & \ddots & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & \cdots & 0 \\
0 & 0 & \widehat{B}_{n1}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 & 0 & \widehat{B}_{nn}^{(i)}(\eta_{N-1}) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \cdots & 0 & \frac{2}{h} & 0 & 0 & \cdots & 0
\end{bmatrix}. \tag{39}
$$

Then we can write the sequence of differential equations in vector form as

$$
\widehat{D} w_i = \frac{h}{2} \widehat{A}_i w_i + \frac{h}{2} \widehat{B}_i w_{i-1}, \tag{40}
$$

where $\widehat{D}$ is the Kronecker product, $\widehat{D} = D \otimes I_n$, and each D is given by

$$D = \begin{bmatrix} D_{00} & \cdots & D_{0N} \\ \vdots & \vdots & \vdots \\ D_{N-1,0} & \cdots & D_{N-1,N} \\ 0 & \cdots & 1 \end{bmatrix}. \tag{41}$$

The Kronecker product produces n blocks of D arrays down the diagonal. The unit in the lower right introduces the n initial condition, $w_{rN}^{(i)},\ r = 1, \cdots, n$.

The linear equation (40) can be solved for w_i by setting

$$M_i = \left(\widehat{D} - \frac{h}{2}\widehat{A}_i\right)^{-1} \frac{h}{2}\widehat{B}_i \tag{42}$$

and

$$w_i = M_i w_{i-1} \tag{43}$$

for $i = 2, 3, \cdots, q$. The abiliity to take the inverse here, for each i, must be determined numerically. There has been some study of the eigenvalues of the matrix D done by Trefethen and Trummer [51] as well as small perturbation of matrix eigenvalues by Kato [32], but there does not seem to be a definitive study of the large perturbations of the differentiation matrix, D.

To solve for w_1, for the fundamental solution, we need the collocation solution of

$$\widehat{u}_1'(\eta) = \frac{h}{2}\widehat{A}_1(\eta)\widehat{u}_1(\eta) \tag{44}$$

for $\eta \in\ [-1, 1]$ and

$$\widehat{u}_1(-1) = e_j. \tag{45}$$

That is, we solve n problems at each step, one for each of the initial conditions e_j. For the moment we set the initial vector as

$$w_0 = (0 \cdots u_{01} 0 \cdots u_{02} 0 \cdots u_{0n})^T, \tag{46}$$

where $u_{0r},\ r = 1, \cdots, n$, is placed in each of the $(N+1)$th elements and zero elsewhere. Then from the previous construction of $\widehat{D}$ and $\widehat{A}_1$ we have

$$w_1 = \left(\widehat{D} - \frac{h}{2}\widehat{A}_1\right)^{-1} w_0. \tag{47}$$

Now, given that we have computed

$$\widehat{u}_{ir}(\eta) = \sum_{k=0}^{N} w_{rk}^{(i)} l_k(\eta) \tag{48}$$

for $\eta \in [-1, 1]$, for $r = 1, \cdots, n$, we can compute the result for $t \in [a + (i-1)h,\ a + ih]$ by setting

$$\widehat{z}_{ir}(t) = \widehat{u}_{ir}(\eta) \tag{49}$$

for $r = 1, \cdots, n$, where

$$\eta = \frac{2}{h}t - \frac{(2a + (2i-1)h)}{h}, \tag{50}$$

or

$$\widehat{z}_{ir}(t) = \sum_{k=0}^{N} w_{rk}^{(i)} l_k\left(\frac{2}{h}t - \frac{(2a + (2i-1)h)}{h}\right). \tag{51}$$

The initial condition is

$$\widehat{u}_{ir}(\eta_N) = \widehat{u}_{i-1,r}(\eta_0). \tag{52}$$

But on $[a+(i-1)h,\ a+ih]$, $\eta_N=-1$ corresponding to $t=a+(i-1)h$, and on $[a+(i-2)h,\ a+(i-1)h]$, $\eta_0=1$ corresponding to $t=a+(i-1)h$, so that

$$\widehat{z}_{ir}(a+(i-1)h)=\widehat{z}_{i-1,r}(a+(i-1)h). \tag{53}$$

The fundamental solution is formed as n column vectors. Each j-th column vector, $j=1,\cdots,n$, is formed as follows. The initial condition $e_j=(0,\cdots,1,\cdots,0)^T$ is selected. The r-th element, $r=1,\cdots,n$, in the column is formed by concatenating the functions $z_{ir}(t)$ for $i=1,\cdots,q$. The final matrix is then denoted by $\widehat{Z}_N(t,a)$ where a is an initial point, not necessarily zero.

For further discussion of the numerical aspects of computing differentiation matrices see Baltensperger and Berrut [12], Bayliss et al. [14], Funaro [23], Solomonoff [45], and Welfert [52].

3.4. **Error Estimates.** In this section we develop an error estimate between the pseudospectral collocation of the fundamental solution of (10) and the exact fundamental solution of (10). In fact we are able to prove, using a method of Bellen [15], the next theorem, where $\mathcal{L}$ and $\mathcal{P}$ have been defined previously in Sections 1 and 3.3, respectively. Similar methods have been used by de Boor and Swartz [21], Russell and Shampine [41], and Hulme [30]. See also Bellen and Zennaro [16].

We suppose that $N+1$ Chebyshev extreme points (22) are given in $[-1,\ 1]$. Define $\Delta=\max|\eta_{i+1}-\eta_i|$. Then it is easy to show that $\Delta\le\pi/N$.

Theorem 3.1. *Let $\mathcal{L}_1=(h/2)\mathcal{P}\mathcal{L}$. Choose N sufficiently large so that $\Delta<1/\mathcal{L}_1$. If $Z(t,\,a)$ is the exact fundamental solution of (10) for $t\in\ [a,b]$, where the right hand side of (10) is assumed to have bounded derivatives of sufficiently high order, and $\widehat{Z}_N(t,a)$ is the approximate fundamental solution developed by the pseudo-spectral method of Section 3.3, then $\left\|Z-\widehat{Z}_N\right\|_\infty=O\left(\Delta^{N+2}\right)$.*

Proof. Since each of the q intervals in the method of steps is mapped to $[-1,\ 1]$ we will work with the sequence of differential equations (18). With this in mind we begin the error estimates at the first step, where the exact equation, in matrix form, is given by

$$V_1{}'(\eta)=\frac{h}{2}\widehat{A}_1(\eta)V_1(\eta), \tag{54}$$

with the initial condition $V_1(-1)=I_n$. In this section the V will represent the matrix solutions of the differrential equations. At this step we seek the matrix of Lagrange interpolation polynomials that collocate at the $N+1$ Chebyshev extreme points in $[-1,\ 1]$ that satisfy

$$\widehat{V}_1'(\eta)=\frac{h}{2}P_N\left(\widehat{A}_1(\eta)\widehat{V}_1(\eta)\right). \tag{55}$$

Since there exists a unique solution to (54) we can subtract (55) to form

$$V_1{}'(\eta)-\widehat{V}_1'(\eta)=\frac{h}{2}\widehat{A}_1(\eta)V_1(\eta)-\frac{h}{2}P_N\left(\widehat{A}_1(\eta)\widehat{V}_1(\eta)\right). \tag{56}$$

For simplicity we will re-index the Chebyshev points so that $\zeta_i=\eta_{N-i}=\cos((N-i)\pi/N)$ for $i=0,\ 1,\cdots,\ N$. We now add and subtract the projection of the right hand side of (54) to get

$$\begin{aligned} V_1(\eta)-\widehat{V}_1(\eta) &= V_1\left(\zeta_i\right)-\widehat{V}_1\left(\zeta_i\right)\\ &\quad+\frac{h}{2}\int_{\zeta_i}^{\eta}(I-P_N)\left(\widehat{A}_1(s)V_1(s)\right)\,ds\\ &\quad+\frac{h}{2}\int_{\zeta_i}^{\eta}P_N\left(\widehat{A}_1(s)\left(V_1(s)-\widehat{V}_1(s)\right)\right)\,ds, \end{aligned} \tag{57}$$

where I is the identity operator and we also use the fact that P_N is linear.

We need to compute some bounds at this point. In particular, from Lagrange interpolation theory we have

$$\left\|(I-P_N)\left(\widehat{A}_1(s)V_1(s)\right)\right\|_\infty \leq \mathcal{K}_0\Delta^{N+1}, \tag{58}$$

for some $\mathcal{K}_0$. Although $\mathcal{K}_0$ in general depends on N, we assume here that the derivatives of the right hand side of (54) are bounded. We also have $\left\|\widehat{A}_i(s)\right\|_\infty \leq \mathcal{L}$ and $\left\|\widehat{B}_i(s)\right\|_\infty \leq \mathcal{L}$ for $i=1,\ 2,\cdots,\ q$.

Now let

$$e_i = \max_{-1\leq\eta\leq\zeta_i}\left\|V_1(\eta)-\widehat{V}_1(\eta)\right\|_\infty. \tag{59}$$

Then,

$$\begin{aligned} e_{i+1} \quad \leq \quad & e_i + \frac{h}{2}\int_{\zeta_i}^{\zeta_{i+1}}\left\|(I-P_N)\left(\widehat{A}_1(s)V_1(s)\right)\right\|_\infty\ ds \\ & +\frac{h}{2}\int_{\zeta_i}^{\zeta_{i+1}}\left\|P_N\left(\widehat{A}_1(s)\left(V_1(s)-\widehat{V}_1(s)\right)\right)\right\|_\infty\ ds, \end{aligned} \tag{60}$$

or, letting $\Delta = \max\left(\zeta_{i+1}-\zeta_i\right)$,

$$e_{i+1} \leq e_i + \frac{h}{2}\mathcal{K}_0\Delta^{N+2} + \frac{h}{2}\mathcal{P}\mathcal{L}\Delta e_{i+1}. \tag{61}$$

If we set $\mathcal{K}_1 = \frac{h}{2}\mathcal{K}_0$, then (61) becomes

$$e_{i+1} \leq e_i + \mathcal{K}_1\Delta^{N+2} + \mathcal{L}_1\Delta e_{i+1}. \tag{62}$$

For Δ sufficiently small,

$$e_{i+1} \leq \left(\frac{1}{1-\mathcal{L}_1\Delta}\right)e_i + \left(\frac{\mathcal{K}_1}{1-\mathcal{L}_1\Delta}\right)\Delta^{N+2}, \tag{63}$$

for $i=0,\ 1,\cdots,\ N$. To simplify a little, let

$$\alpha = \frac{1}{1-\mathcal{L}_1\Delta}, \quad \beta = \left(\frac{\mathcal{K}_1}{1-\mathcal{L}_1\Delta}\right)\Delta^{N+2}, \tag{64}$$

then, by a simple geometric series argument,

$$e_N \leq \alpha^N e_0 + \left(\frac{\alpha^N-1}{\alpha-1}\right)\beta. \tag{65}$$

For the first step $e_0 = 0$, so that

$$e_N \leq \left(\frac{\alpha^N-1}{\alpha-1}\right)\beta = \left[\left(\frac{1}{1-\mathcal{L}_1\Delta}\right)^N - 1\right]\frac{\mathcal{K}_1}{\mathcal{L}_1}\Delta^{N+1}. \tag{66}$$

Now, for $\Delta = \max\left(\zeta_{i+1}-\zeta_i\right) \leq \pi/N < 1/\mathcal{L}_1$, $\left(1/\left(1-\mathcal{L}_1\Delta\right)\right)^N$ is a decreasing sequence, so that $\left(1/\left(1-\mathcal{L}_1\Delta\right)\right)^N \leq \left(1/\left(1-\mathcal{L}_1\Delta\right)\right)$. Therefore,

$$e_N \leq \left[\left(\frac{1}{1-\mathcal{L}_1\Delta}\right)-1\right]\frac{\mathcal{K}_1}{\mathcal{L}_1}\Delta^{N+1} = \left[\frac{\mathcal{K}_1}{1-\mathcal{L}_1\Delta}\right]\Delta^{N+2} \leq 2\mathcal{K}_1\Delta^{N+2}, \tag{67}$$

for Δ sufficiently small. Therefore

$$\max_{-1\leq\eta\leq 1}\left\|V_1(\eta)-\widehat{V}_1(\eta)\right\|_\infty = O\left(\Delta^{N+2}\right). \tag{68}$$

We proceed inductively and assume that for the j-th step, $j=1,\ 2,\cdots,\ q-1$

$$\max_{-1\leq\eta\leq 1}\left\|V_j(\eta)-\widehat{V}_j(\eta)\right\|_\infty \leq \beta_j\Delta^{N+2}, \tag{69}$$

for some $\beta_j > 0$. We then have at the $(j+1)$-th step the exact problem

$$V'_{j+1}(\eta) = \frac{h}{2}\widehat{A}_{j+1}(\eta)V_{j+1}(\eta) + \frac{h}{2}\widehat{B}_{j+1}(\eta)V_j(\eta). \tag{70}$$

with the initial condition $V_{j+1}(-1) = V_j(1)$. Proceeding as in the first step we seek the collocated solution to

$$\widehat{V}'_{j+1}(\eta) = \frac{h}{2} P_N \left(\widehat{A}_{j+1}(\eta)\widehat{V}_{j+1}(\eta) + \widehat{B}_{j+1}(\eta)\widehat{V}_j(\eta) \right), \tag{71}$$

with the initial condition $\widehat{V}_{j+1}(-1) = \widehat{V}_j(1)$. Now, subtracting (71) from (70) we have on $[\zeta_i,\ \zeta_{i+1}]$

$$\begin{aligned} V_{j+1}(\eta) - \widehat{V}_{j+1}(\eta) \quad = \quad & V_{j+1}(\zeta_i) - \widehat{V}_{j+1}(\zeta_i) \\ & + \frac{h}{2}\int_{\zeta_i}^{\eta} (I - P_N)\left(\widehat{A}_{j+1}(s)V_{j+1}(s) + \widehat{B}_{j+1}(s)V_j(s) \right)\, ds \\ & + \frac{h}{2}\int_{\zeta_i}^{\eta} P_N\left(\widehat{A}_{j+1}(s)\left(V_{j+1}(s) - \widehat{V}_{j+1}(s) \right) \right)\, ds \\ & + \frac{h}{2}\int_{\zeta_i}^{\eta} P_N\left(\widehat{B}_{j+1}(s)\left(V_j(s) - \widehat{V}_j(s) \right) \right)\, ds. \end{aligned} \tag{72}$$

As before, let

$$e_i = \max_{-1\le\eta\le\zeta_i} \left\| V_{j+1}(s) - \widehat{V}_{j+1}(s) \right\|_\infty . \tag{73}$$

From Lagrange interpolation theory we have

$$\left\| (I - P_N)\left(\widehat{A}_{j+1}(s)V_{j+1}(s) + \widehat{B}_{j+1}(s)V_j(s) \right) \right\|_\infty \le \mathcal{K}_j \Delta^{N+1}, \tag{74}$$

for some $\mathcal{K}_j > 0$. From the definition of $\mathcal{L}$ we have $\left\| \widehat{A}_{j+1}(s) \right\|_\infty \le \mathcal{L}$ and $\left\| \widehat{B}_{j+1}(s) \right\|_\infty \le \mathcal{L}$. Then, from the definitions of $\mathcal{L},\ \mathcal{P},\ \mathcal{K}_j$,

$$e_{i+1} \le e_i + \frac{h}{2}\left(\mathcal{K}_j + \mathcal{P}\mathcal{L}\Delta\beta_j \right)\Delta^{N+2} + \frac{h}{2}\mathcal{P}\mathcal{L}\Delta e_{i+1}. \tag{75}$$

As in the initial case let $\mathcal{K}_{j+1} = \frac{h}{2}\left(\mathcal{K}_j + \mathcal{P}\mathcal{L}\Delta\beta_j \right)$ and $\mathcal{L}_{j+1} = \mathcal{L}_1 = \frac{h}{2}\mathcal{P}\mathcal{L}$, then for sufficiently small Δ we have

$$e_{i+1} \le \left(\frac{1}{1 - \mathcal{L}_{j+1}\Delta} \right) e_i + \left(\frac{\mathcal{K}_{j+1}}{1 - \mathcal{L}_{j+1}\Delta} \right)\Delta^{N+2}. \tag{76}$$

As before, let $\alpha = \frac{1}{1-\mathcal{L}_{j+1}\Delta}$ and $\beta = \frac{\mathcal{K}_{j+1}}{1-\mathcal{L}_{j+1}\Delta}$, then

$$e_N \le \alpha^N e_0 + \left(\frac{\alpha^N - 1}{\alpha - 1} \right)\beta\Delta^{N+2}. \tag{77}$$

But from from the inductive assumption (69) $e_0 \le \beta_j \Delta^{N+2}$, so that

$$e_N \le \alpha^N \beta_j \Delta^{N+2} + \left(\frac{\alpha^N - 1}{\alpha - 1} \right)\beta\Delta^{N+2}. \tag{78}$$

Again, for $\Delta < 1/\mathcal{L}_1$, α^N is decreasing so that $\alpha^N \le \alpha$ and therefore

$$e_N \le \left(\alpha\beta_j + \beta \right)\Delta^{N+2}. \tag{79}$$

Therefore, the induction shows that

$$\max_{1\le i\le q} \left\| V_i - \widehat{V}_i \right\|_\infty = O\left(\Delta^{N+2} \right). \tag{80}$$

But, since $Z_i(t) = V_i(\eta)$ and $\widehat{Z}_i(t) = \widehat{V}_i(\eta)$ we have from (80) and the fact that the fundamental solution $Z(t,a)$ is the concatenation of $Z_i(t,a) = V_i(\eta)$ and the collocated solution $\widehat{Z}_N(t,a)$ is the concatenation of $\widehat{Z}_{Ni}(t,a) = \widehat{V}_i(\eta)$ for $i = 1, 2, \cdots, q$, that

$$\left\| Z - \widehat{Z}_N \right\|_\infty = \max_{1\le i\le q} \left\| Z_i - \widehat{Z}_{Ni} \right\|_\infty = \max_{1\le i\le q} \left| V_i - \widehat{V}_i \right|_\infty = O\left(\Delta^{N+2} \right). \tag{81}$$

□

This theorem implies that $\widehat{Z}_N(t,v)$ converges uniformly to $Z(t,v)$ for $(t,\, v) \in [0,\, T] \times [0,\, T]$.

4. Forming the Monodromy Matrix Eigenvalue Problem

In Section 1 the the monodromy operator (5) involved knowledge of the fundamental matrix of (2). In Section 3 a collocation algorithm to approximate the fundamental matrix along with an error analysis of the collocation method was developed. In this section we will define an approximate monodromy operator and use it to form a matix eigenvalue problem to estimate the monodromy operator eigenvalues.

4.1. **Discretized Operators.** To approximate the monodromy operator (5) we will require a quadrature rule that satisfies

$$\sum_{k=1}^{P+1} v_k f\left(s_k\right) \to \int_{-h}^{0} f(s)\ ds \tag{82}$$

as $P \to \infty$ for each continuous function $f \in C_h$. The rule is satisfied if

$$\sum_{k=1}^{P+1} |v_k| \leq M, \tag{83}$$

for some $M > 0$ and $P = 1,\ 2, \cdots$. This condition is necessary to prove certain compactness results later.

Let $-h = s_1 < s_2 < \cdots < s_{P+1} = 0$, and define

$$\left(U_P\phi\right)(s) = Z(s+T,0)\phi(0) + \sum_{k=1}^{P+1} v_k Z\left(s+T, s_k+h\right) B\left(s_k+h\right) \phi\left(s_k\right) \tag{84}$$

for $\phi \in C_h$. From the theory of delay differential equations (Hale and Lunel [29]) it is known that $Z(t,u)$ is a continuous function on $[0,T] \times [0,T]$ and thus also uniformly continuous there. Furthermore, $B(t)$ is continuous and periodic in $[0,T]$.

Since the fundamental matrix is seldom known in practice we approximated it in Section 3. In that case we developed a sequence, $\widehat{Z}_N$, of matrices that converged uniformly to Z (Theorem 3.1). Now define a double sequence of operators on C_h by

$$\left(U_{PN}\phi\right)(s) = \widehat{Z}_N(s+T,0)\phi(0) + \sum_{k=1}^{P+1} v_k \widehat{Z}_N\left(s+T, s_k+h\right) B\left(s_k+h\right) \phi\left(s_k\right) \tag{85}$$

for $\phi \in C_h$, $P =\ 1,\ 2, \cdots,\ N =\ 1,\ 2, \cdots$.

4.2. **The Matrix Eigenvalue Problem.** From the discussion in Section 1, the stability of the approximate periodic solution $p(t)$ of (1) depends on the eigenvalues of the operator (5). In this section we will consider the discretized form of (5) given by (85) and point out some computational simplifications involved with constructing the approximate eigenvalue problem.

We discretize the interval $[-h, 0]$ into P intervals by

$$-h = s_1 < s_2 < \cdots < s_{P+1} = 0. \tag{86}$$

Then, from (5), for each $s_i \in [-h, 0]$,

$$\left(U\phi\right)\left(s_i\right) = Z(s_i+T,0)\phi(0) + \sum_{j=1}^{P+1} v_j Z(s_i+T, s_j+h) B(s_j+h)\phi(s_j) \tag{87}$$

Since $s_{P+1} = 0$, (87) can be rewritten as

$$\begin{aligned} \left(U\phi\right)\left(s_i\right) &= \sum_{j=1}^{P} v_j Z(s_i+T, s_j+h) B(s_j+h)\phi(s_j) \\ &+ \left(Z(s_i+T,0) + v_{P+1} Z(s_i+T,h) B(h)\right) \phi(s_{P+1}), \end{aligned} \tag{88}$$

where $Z(t,\alpha)$, for $0 \le \alpha \le t \le T$, is the fundamental solution of (2). Equation (88) can be put in matrix form

$$\begin{pmatrix} (U^{(P)}\phi)(s_1) \\ \vdots \\ (U^{(P)}\phi)(s_i) \\ \vdots \\ (U^{(P)}\phi)(s_{P+1}) \end{pmatrix} = \begin{bmatrix} U_{1,1}^{(P)} & \cdots & U_{1,j}^{P} & \cdots & U_{1,P+1}^{(P)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{i,1}^{(P)} & \cdots & U_{i,j}^{(P)} & \cdots & U_{i,P+1}^{(P)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{P+1,1}^{(P)} & \cdots & U_{P+1,j}^{(P)} & \cdots & U_{P+1,P+1}^{(P)} \end{bmatrix} \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix}, \tag{89}$$

where the block elements for $i = 1, \cdots, P+1, j = 1, \cdots, P$ are $U_{i,j}^{(P)} = v_j Z(s_i + T, s_j + h)B(s_j + h)$. The block elements in the last column of the matrix are given by $U_{i,P+1}^{(P)} = Z(s_i + T, 0) + v_{P+1} Z(s_i + T, h)B(h)$ for $i = 1, \cdots, P+1$.

Equation (89) is based on the assumption of an exact representation of the fundamental solution $Z(t,\alpha)$ for $0 \le \alpha \le t \le T$. However, we can in general only work with an approximate form for (89). Denote by $\widehat{Z}_N(t,\alpha)$ the collocated approximate fundamental solution for N Chebyshev points of $Z(t,\alpha)$ such that

$$\widehat{Z}_N(t,\alpha) \to Z(t,\alpha) \tag{90}$$

uniformly as $N \to \infty$ according to Theorem 3.1 for $0 \le \alpha \le t \le T$. From Section 3.3 the matrix $\widehat{Z}_N(t,\alpha)$ is the concatenation of q matrices $\widehat{Z}_{iN}(t,\alpha)$ for $i = 1, 2, \cdots, q$. Each of these i matrices $\widehat{Z}_{iN}(t,\alpha)$ can be written in the form

$$\widehat{Z}_{iN}(t,\alpha) = \sum_{k=0}^{N} W_{Nk}^{(i)} l_k \left(\frac{2}{h} t - \frac{2\alpha + (2i-1)h}{h} \right). \tag{91}$$

where $W_{Nk}^{(i)}$ is a matix of collocation coefficients. Equation (88) now becomes

$$\begin{aligned} \left(U^{(PN)}\phi\right)(s_i) &= \sum_{j=1}^{P} v_j \widehat{Z}_N(s_i + T, s_j + h)B(s_j + h)\phi(s_j) \\ &+ \left(\widehat{Z}_N(s_i + T, 0) + v_{P+1}\widehat{Z}_N(s_i + T, h)B(h)\right)\phi(s_{P+1}). \end{aligned} \tag{92}$$

The matrix in (89) should now be written as

$$\begin{pmatrix} (U^{(PN)}\phi)(s_1) \\ \vdots \\ (U^{(PN)}\phi)(s_i) \\ \vdots \\ (U^{(PN)}\phi)(s_{P+1}) \end{pmatrix} = \begin{bmatrix} U_{1,1}^{(PN)} & \cdots & U_{1,j}^{(PN)} & \cdots & U_{1,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{i,1}^{(PN)} & \cdots & U_{i,j}^{(PN)} & \cdots & U_{i,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{P+1,1}^{(PN)} & \cdots & U_{P+1,j}^{(PN)} & \cdots & U_{P+1,P+1}^{(PN)} \end{bmatrix} \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix}, \tag{93}$$

where $U_{i,j}^{(PN)} = v_j \widehat{Z}_N(s_i + T, s_j + h)B(s_j + h)$ for $i = 1, \cdots, P+1, j = 1, \cdots, P$, and $U_{i,P+1}^{(PN)} = \widehat{Z}_N(s_i + T, 0) + v_{P+1}\widehat{Z}_N(s_i + T, h)B(h)$ for $i = 1, \cdots, P+1$.

We note that it is not necessary to compute $\widehat{Z}_N(s_i + T, s_j + h)$ for every combination of (i,j). The elements $U_{i,j}^{(PN)}$ in each column j above $U_{P+1,j}^{(PN)}$ contain the matrix factor $\widehat{Z}_N(s_i + T, s_j + h)$. Since $0 \le T - h \le s_i + T \le T$ and $s_{P+1} = 0$, we need only develop the collocation polynomials for $\widehat{Z}_N(t, s_j + h)$ for $j = 1, \cdots, P+1$ using (91) for all $t \in [s_j + h,\ T]$. Then each $\widehat{Z}_N(s_i + T, s_j + h)$ in the j-th column is simply an evaluation of the collocation polynomial developed for $\widehat{Z}_N(t, s_j + h)$ at the point $s_i + T$ for $i = i, \ldots, P+1$. That is,

$$\widehat{Z}_{iN}(s_i + T, s_j + h) = \sum_{k=0}^{N} W_{Nk}^{(i)} l_k \left(\frac{2}{h}(s_i + T) - \frac{2(s_j + h) + (2i-1)h}{h} \right). \tag{94}$$

This means we only need to develop the fundamental solution collocation representation for each element in row $P+1$, which involves $P+1$ computations of the approximate fundamental matrices rather than $(P+1)^2$ computations for the full matrix. All other rows involve interpolated values of the computed collocation polynomials at specific points in $[0,\ T]$.

From (93) the relevant eigenvalue problem becomes

$$\begin{bmatrix} U_{1,1}^{(PN)} & \cdots & U_{1,j}^{(PN)} & \cdots & U_{1,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{i,1}^{(PN)} & \cdots & U_{i,j}^{(PN)} & \cdots & U_{i,P+1}^{(PN)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ U_{P+1,1}^{(PN)} & \cdots & U_{P+1,j}^{(PN)} & \cdots & U_{P+1,P+1}^{(PN)} \end{bmatrix} \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix} = \lambda \begin{pmatrix} \phi(s_1) \\ \vdots \\ \phi(s_i) \\ \vdots \\ \phi(s_{P+1}) \end{pmatrix}. \tag{95}$$

The Matlab progam *eig* can then be used to compute the eigenvalues and eigenvectors of (95). For some related discussion of the convergence of eigenvalues of finite matrix representations to operator eigenvalues see Baker [10].

5. Operator Compactness and Eigenvalue Convergence

In this section we will establish the compactness properties of $U,\ U_P,$ and U_{PN}, as well as the convergence properties of U_P and U_{PN} to U. We finally show that the eigenvalues of U_{PN} converge to the eigenvalues of U.

5.1. Operator Compactness Properties.

Let $\mathcal{U} = \{\phi : \phi \in C_h, |\phi| \leq 1\}$ be the unit ball in C_h. An operator T acting on C_h is said to be *compact* iff the set $T\mathcal{U}$ is relatively compact. A family of operators $\mathcal{T}$ operating on C_h is said to be *collectively compact* iff the set $\mathcal{T}\mathcal{U} = \{T\phi : T \in \mathcal{T}, \phi \in \mathcal{U}\}$ is relatively compact. In this section we will prove a lemma with respect to the compactness properties of the operators (5), (84), and (85). Although Halanay [27] has shown that U is compact on C_h we will include the proof for completeness.

Lemma 5.1. *The operators U, U_P, and U_{PN} from C_h to C_h are compact for $P = 1,\ 2, , \cdots$ and $N = 1,\ 2, , \cdots$. The sets $\{U_P\}$ and $\{U_{PN}\}$ are collectively compact.*

Proof. To show that U defined by (5) is compact, we first note that the fundamental solution Z is continuous on $[0,T] \times [0,T]$ and therefore uniformly continuous. B is continuous on $[0,T]$ and is therefore bounded, $\|B\| \leq \mathcal{B}$ for some $\mathcal{B} > 0$. Z is also bounded by $\|Z\| \leq \mathcal{Z}$ for some $\mathcal{Z} > 0$. Then

$$|(U\phi)(s)| \leq \mathcal{Z}(1 + h\mathcal{B})|\phi| \tag{96}$$

where $s \in [-h, 0]$. Therefore U is continuous. U will be compact if we show that it maps the unit ball to a pre-compact set. By the Arzelà-Ascoli Theorem we only need to show that $U\phi$, for $|\phi| \leq 1$, is uniformly bounded and equicontinuous. The inequality (96) shows uniform boundedness for $|\phi| \leq 1$. To show equicontinuity let $s_1,\ s_2 \in [-h,\ 0]$ and write

$$\begin{aligned} |(U\phi)(s_1) - (U\phi)(s_2)| \quad \leq \quad & \Big\{ \|Z(s_1 + T, 0) - Z(s_2 + T, 0)\| \\ & + \int_{-h}^{0} \|Z(s_1 + T, \alpha + h) - Z(s_2 + T, \alpha + h)\| \, d\alpha \, \mathcal{B} \Big\} |\phi|. \end{aligned} \tag{97}$$

Since Z is uniformly continuous on $[0,\ T] \times [0,\ T]$ then, given $\epsilon > 0$, there is a $\delta > 0$ such that, if $|s_1 - s_2| < \delta$ and $|\phi| \leq 1$, then

$$|(U\phi)(s_1) - (U\phi)(s_2)| \leq (1 + h\mathcal{B})\epsilon, \tag{98}$$

proving equicontinuity of U for $|\phi| \leq 1$.

To show that $\{U_P\}$ is collectively compact, we first show that U_P is compact for each P. From (84) it is clear that the operators are uniformly bounded for $|\phi| \leq 1$, since

$$|(U_P\phi)(s)| \leq \mathcal{Z}\left(1 + \mathcal{B}\sum_{j=1}^{P+1}|v_j|\right) \leq \mathcal{Z}(1+\mathcal{B}M). \tag{99}$$

For equicontinuity, let $s_1,\ s_2 \in [-h,\ 0]$ and write

$$\begin{aligned} |(U_P\phi)(s_1) - (U_P\phi)(s_2)| \quad \leq \quad & \Bigg\{ \|Z(s_1+T,0) - Z(s_2+T,0)\| \\ & + \sum_{j=1}^{P+1}|v_j|\,\|Z(s_1+T,s_j+h) - Z(s_2+T,s_j+h)\|\ d\alpha\,\mathcal{B}\Bigg\}|\phi|. \end{aligned} \tag{100}$$

By uniform continuity of Z on $[0,T]\times[0,T]$, let $\epsilon > 0$, then there is a $\delta > 0$ such that , if $|s_1 - s_2| < \delta$ and $|\phi| \leq 1$,

$$|(U_P\phi)(s_1) - (U_P\phi)(s_2)| \leq (1+\mathcal{B}M)\epsilon. \tag{101}$$

Therefore U_P is compact for each P. The inequalities (99) and (101) show that $\{U_P\}$ is a uniformly bounded set of equicontinuous operators for $|\phi| \leq 1$, which proves that $\{U_P\}$ is pre-compact and therefore collectively compact.

Since $\widehat{Z}_N$ is uniformly continuous on $[0,T]\times[0,T]$, the proof that $\{U_{PN}\}$ is collectively compact for each P is similar to the argument above, where the operator U_{PN} is defined by (85). □

5.2. **Operator Convergence.** In this section we will prove some lemmas relating the convergence of the operators U_P and U_{PN}.

Lemma 5.2. *The operators U_{PN} and U_P satisfy*

$$\|U_{PN} - U_P\| \to 0 \tag{102}$$

uniformly in P for $N \to \infty$.

Proof. By definition

$$\begin{aligned} \|U_{PN} - U_P\| &= \sup_{|\phi|\leq 1}|U_{PN}\phi - U_P\phi|, \\ |U_{PN}\phi - U_P\phi| &= \sup_{-h\leq s\leq 0}|(U_{PN}\phi)(s) - (U_P\phi)(s)|. \end{aligned} \tag{103}$$

But

$$\begin{aligned} \|(U_{PN}\phi)(s) - (U_P\phi)(s)\| \quad \leq \quad & \Bigg\{ \left\|\widehat{Z}_N(s+T,0) - Z(s+T,0)\right\| \\ & + \sum_{j=1}^{P+1}|v_j|\left\|\widehat{Z}_N(s+T,s_j+h) - Z(s+T,s_j+h)\right\|\mathcal{B}\Bigg\}|\phi|. \end{aligned} \tag{104}$$

By the uniform convergence of $\widehat{Z}_N$ to Z, given $\epsilon > 0$, there exists $N_1 > 0$, such that, for $N \geq N_1$, $\left\|\widehat{Z}_N(t,v) - Z(t,v)\right\| < \epsilon$ for all $(t,v) \in [0,\ T]\times[0,\ T]$. Then

$$\|(U_{PN}\phi)(s) - (U_P\phi)(s)\| \leq \left(1 + \mathcal{B}\sum_{j=1}^{P+1}|v_j|\right)\epsilon|\phi| \leq (1+\mathcal{B}M)\epsilon|\phi|, \tag{105}$$

which shows pointwise convergence, uniformly in P. But from (105) the conclusion follows for $|\phi| \leq 1$. □

We will say that the double sequence of operators U_{PN} converges pointwise to U if and only if, given $\epsilon > 0$, there is a positive P_1 such that $|U_{PN}\phi - U\phi| < \epsilon$ for all $(P,\ N)$ such that $P > P_1$ and $N > P_1$.We will write $U_{PN} \to U$ as $(P,\ N) \to \infty$.

We need two final lemmas.

Lemma 5.3. *U_P converges pointwise to U.*

Proof. This follows from the quadrature rule. □

Lemma 5.4. *U_{PN} converges pointwise to U as $(P,\ N) \to \infty$.*

Proof. Write for $\phi \in C_h$

$$|U_{PN}\phi - U\phi| \leq |U_{PN}\phi - U_P\phi| + |U_P\phi - U\phi| \tag{106}$$

By Lemma 5.2, given $\epsilon > 0$, there exists $N_1 > 0$ so that $|U_{PN}\phi - U_P\phi| \leq \epsilon$, for $N \geq N_1$, uniformly in P. Also, by Lemma 5.3, there exists P_1 so that $|U_P\phi - U\phi| \leq \epsilon$ for $P \geq P_1$. Select $P_2 = \max(N_1,\ P_1)$. Then for $P > P_2$ and $N > P_2$, $|U_{PN}\phi - U\phi| \leq 2\epsilon$. □

5.3. **Eigenvalue Convergence.** We will begin this section by summarizing some known results related to the compact operator U. In particular, we denote by $\sigma(U)$ and $\rho(U)$ the spectrum and resolvent sets, respectively, of U. For any $z \in \rho(U)$, $R_z(U) = (z - U)^{-1}$ is the resolvent operator. $\sigma(U)$ is countable with zero the only possible accumulation point. The nonzero numbers in $\sigma(U)$ are eigenvalues.

If $\lambda \in \sigma(U)$ is nonzero, there is a smallest integer, ν, such that $\mathcal{N}\left((\lambda - U)^\nu\right) = \mathcal{N}\left((\lambda - U)^{\nu+1}\right)$, where $\mathcal{N}$ denotes the null space. ν is called the ascent of $\lambda - U$. $\mathcal{N}\left((\lambda - U)^\nu\right)$ is finite dimensional and if $m = \dim \mathcal{N}\left((\lambda - U)^\nu\right)$, then m is called the algebraic multiplicity of λ. The vectors in $\mathcal{N}\left((\lambda - U)^\nu\right)$ are the generalized eigenfunctions of U with respect to λ. The geometric multiplicity of λ is equal to $\dim \mathcal{N}(\lambda - U)$ and is less than or equal to m.

If $\lambda \neq 0$ is an eigenvector of U with algebraic multiplicity m and Γ is a circle centered at λ, lying in $\rho(U)$, that encloses no other points of $\sigma(U)$, then the spectral projection associated with λ and U is defined by

$$E = E(\lambda) = \frac{1}{2\pi i}\int_\Gamma R_z(U)\ dz. \tag{107}$$

E is a projection of C_h onto the space of generalized eigenfunctions associated with λ and U.

We will need to generalize slightly some known theorems. The first is

Theorem 5.5. *U, U_{PN} are bounded linear operators from C_h to C_h. U is compact, $U_{PN} \to U$ pointwise, and $\{U_{PN}\}$is collectively compact. Then for each open set Ω, with $\sigma(U) \subset \Omega$, there exists a P_1 such that $\sigma\left(U_{PN}\right) \subset \Omega$ for $P > P_1$ and $N > P_1$.*

Proof. See the proof of Theorem 5.3 of Anselone and Palmer [6]. □

The next theorem says that there are eigenvalues of U_{PN} approaching any eigenvalue of U.

Theorem 5.6. *If $\lambda \in \sigma(U)$, $\lambda \neq 0$ then, for sufficiently large P and N, there exists $\lambda_{PN} \in \sigma\left(U_{PN}\right)$ such that $\lambda_{PN} \to \lambda$ as $(P,\ N) \to \infty$.*

Proof. See the proof of Theorem 4.16 of Anselone [4]. □

We next show that all eigenvalues of U_{PN}, for P and N sufficiently large, approach some eigenvalue of U.

Theorem 5.7. *Consider the linear operators U, $U_{PN}:\ C_h \to C_h$, given by (5) and (85). Let $\lambda_0 \neq 0$ be an eigenvalue of U and let $\epsilon > 0$ be less than the distance from λ_0 to the remaining part of the spectrum of U. Denote by σ_{PN} the set of eigenvalues of U_{PN} that are within distance ϵ of λ_0. Then for all sufficiently large P and N, the sum of the multiplicities*

of the eigenvalues in σ_{PN} equals the multiplicity of λ_0 and all the elements of σ_{PN} converge to λ_0. That is,

$$\max_{\lambda\in\sigma_{PN}} |\lambda-\lambda_0| \to 0 \quad as \quad (P,\ N)\to\infty. \tag{108}$$

Proof. Let Γ_0 be a circle centered at λ_0 of radius ϵ. Then, for $P,\ N$ sufficiently large $\Gamma_0 \subset \rho\left(U_{PN}\right)$ and the spectral projection

$$E_{PN} = E_{PN}\left(\lambda_0\right) = \frac{1}{2\pi i}\int_{\Gamma_0} R_z\left(U_{PN}\right)\ dz \tag{109}$$

exists and $\dim R\left(E_{PN}\left(\lambda_0\right)\right) = \dim R\left(E\left(\lambda_0\right)\right) = m$ for some m, the algebraic multiplicity of λ_0. E_{PN} is the spectral projection associated with U_{PN} and the eigenvalues of U_{PN} that are enclosed by Γ_0. Denote these eigenvalues by $\lambda_1(P,\ N),\cdots,\ \lambda_m(P,\ N)$. If Γ_1 is another circle centered at λ_0 with radius $\epsilon/2$ then again, for $P,\ N$ sufficiently large $\lambda_1(P,\ N),\cdots,\ \lambda_m(P,\ N)$ are all inside Γ_1. The argument can be continued. Therefore $\max_{\lambda\in\sigma_{PN}} |\lambda-\lambda_0| \to 0$ as $(P,\ N)\to\infty$. □

6. Examples

In these examples we will consider the class of autonomous delay differential equations of the form

$$\ddot{x} + x = X(x(t-h), \dot{x}(t-h)), \tag{110}$$

where $x, X \in R, X(0,0) = 0$. We assume that X is sufficiently differentiable. It is known that the solutions exist and are unique, if continuous initial condition functions are specified on the delay interval $[-h, 0]$ (Hale [28]). In order to simplify the notation, we will normalize the delay to unity. This can be done by substituting th for t. Furthermore, since the period is unknown in (110) we can introduce a normalized period of $T = 2\pi$ by replacing t by t/ω where ω is an unknown frequency. Then we can put (110) in the form

$$\omega^2\ddot{x} + x = X(x(t-\omega), \dot{x}(t-\omega)). \tag{111}$$

For these examples we will look at various cases of the Van der Pol equation with

$$X(x(t-\omega), \dot{x}(t-\omega)) = \omega\lambda(1-x(t-\omega)^2)\dot{x}(t-\omega). \tag{112}$$

A fast algorithm for constructing an approximate solution of (111) with (112) of the form

$$\tilde{x}(t) = a_0 + \sum_{n=1}^{m}\left[a_{2n}\cos nt + a_{2n-1}\sin nt\right] \tag{113}$$

along with an approximate frequency $a_{2m+1} = \tilde{\omega}$ has been given by Gilsinn [24].

The variational equation about the approximate periodic solution is

$$\dot{z} = \tilde{A}(t)z(t) + \tilde{B}(t)z(t-\tilde{\omega}), \tag{114}$$

where

$$\begin{aligned} \tilde{A}(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{\tilde{\omega}^2} & 0 \end{bmatrix}, \\ \tilde{B}(t) &= \begin{bmatrix} 0 & 0 \\ -\frac{2\lambda}{\tilde{\omega}}\tilde{x}(t-\tilde{\omega})\dot{\tilde{\omega}}(t-\tilde{\omega}) & \frac{\lambda}{\tilde{\omega}}\left(1-\tilde{x}(t-\tilde{\omega})^2\right) \end{bmatrix}, \end{aligned} \tag{115}$$

where $\tilde{\omega}$ is the computed approximate frequency.

We can now follow the development of the fundamental solution for (111) with the right hand side given by (112), as described in Section 3. At the first step we select the smallest positive integer q such that $2\pi \le q\tilde{\omega}$. We then map each interval $[(i-1)\tilde{\omega},\ i\tilde{\omega}]$ to $[-1, 1]$ for $i = 1, 2, \cdots, q$ by

$$\begin{aligned} \eta &= \frac{2}{\tilde{\omega}}t - \frac{(2i-1)\tilde{\omega}}{\tilde{\omega}}, \\ t &= \frac{\tilde{\omega}}{2}\eta + \frac{(2i-1)\tilde{\omega}}{2}. \end{aligned} \tag{116}$$

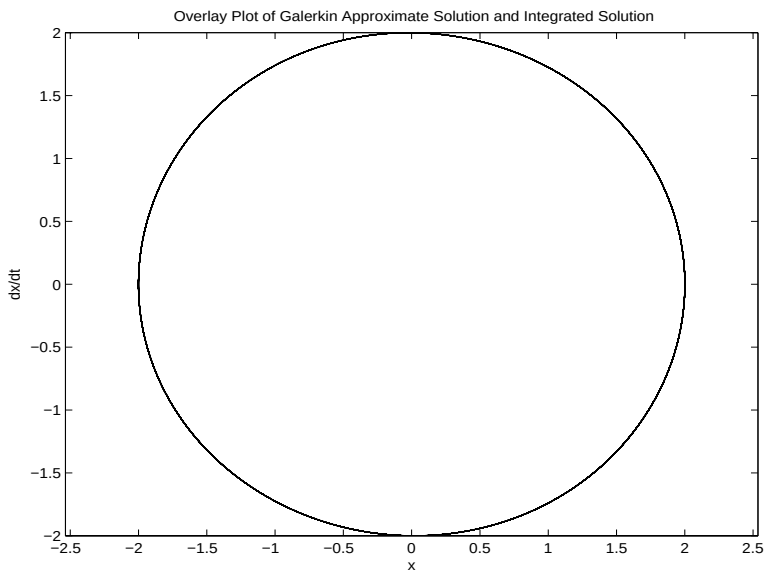

FIGURE 1. Overlay of Phase Plots of Approximate Collocated Solution and Numerically Integrated Solution for the Van der Pol Equation.

We use the Matlab script *cheb.m* from Trefethen [50] to produce $N+1$ Chebyshev points, η_j, $j = 0, \cdots, N$, as well as the spectral derivative matrix D (29). We then modify D so that the last row is all zeros except a one in position $D(N+1, N+1)$. We can then form $\widehat{D} = D \otimes I_2$. In the case of the Van der Pol equation, the matrix $\widehat{A}_i$ is constant for each $i = 1, 2, \cdots, q$, and is straightforward to construct. The matrix $\widehat{B}_i$, for $i = 1, 2, \cdots, q$, is not difficult to construct. For each $\eta_j, j = 0, \cdots, N$ compute

$$t_j = \frac{\tilde{\omega}}{2}\eta_j + \frac{(2i-1)\tilde{\omega}}{2}. \tag{117}$$

The elements of $\widehat{B}_i$, for $i = 1, 2, \cdots, q$, in (39) are then formed by using (115) where we set $t = t_j$, $j = 0, \cdots, N$. The matrix M_i in (42) can then be formed for each $i = 2, \cdots, q$. For the case of $i = 1$ the matrix M_1 is formed as

$$M_1 = \left(\widehat{D} - \frac{h}{2}\widehat{A}_1\right)^{-1}. \tag{118}$$

There are only two initial conditions to consider: $w_0(N+1) = 1, w_0(2(N+1)) = 0$ and $w_0(N+1) = 0$, $w_0(2(N+1)) = 1$. The weights at the first step are then given by (47). The final fundamental matrix is formed by following the steps from (48) to (51).

We will now examine several cases of the Van der Pol equation with different values of λ. For the first case we will take $\lambda = 0.01$. For the approximate solution we will take (115) with seven harmonics. For collocation we will use 40 Chebyshev points and interpolate over 100 steps in the interval $[0, 2\pi]$.

Figure 1 shows the overlay of the approximate solution to the Van der Pol equation developed by the Galerkin procedure with seven harmonics and the numerically integrated Van der Pol equation using dde23 in Matlab with the seven harmonic approximate solution as an initial condition on the interval $[-\tilde{\omega}, 0]$. Note that the overlay is such that one cannot distinguish the two solutions. The next figure shows the successful overlay of the fundamental solution evaluated over $[0, 2\pi]$ for both the numerically evaluated fundamental solution using dde23 and the collocated fundamental solution using the pseudospectral method.

Figure 2 shows the four components of the fundamental solution evaluated over $[0, 2\pi]$ by dde23 and collocation. The collocated fundamental solution is marked by x's on the graphs. Again, the match is extraordinarily close.

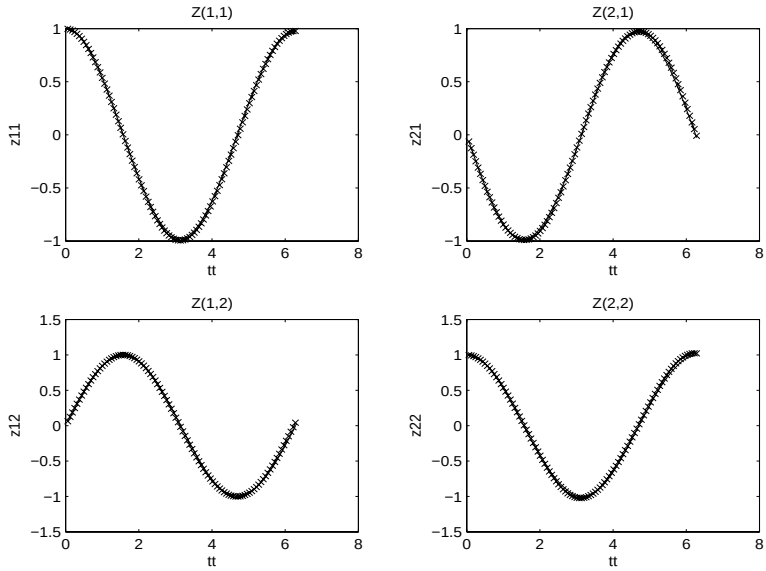

FIGURE 2. Four Elements of the Fundamental Solution of the Van der Pol Equation from 0 to 2π

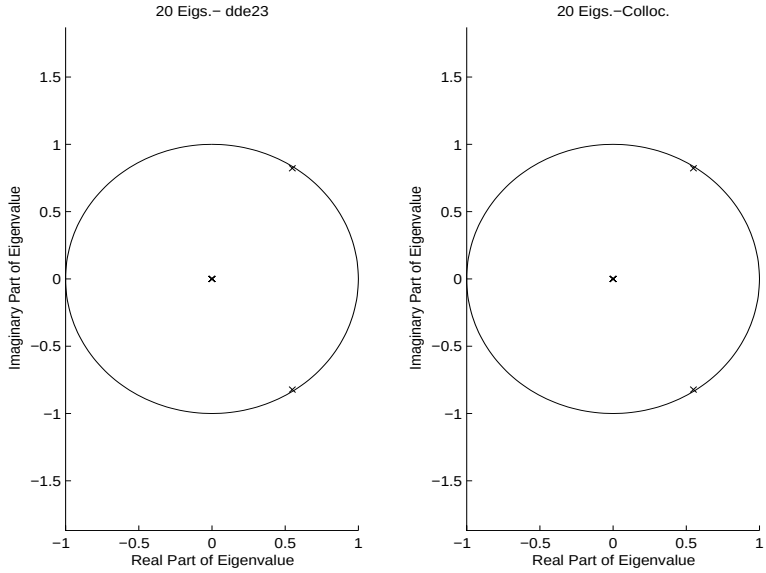

FIGURE 3. Twenty Eigenvalues of the Monodromy Operator U by dde23 and Collocation

Figure 3 shows the results of plotting the first twenty eigenvalues of the discretized monodromy operator using dde23 to evaluate the fundamental solution (left) and the collocation approach (right). Both figures show two eigenvalues symmetric about the real axis near the boundary of the unit circle. The other eighteen eigenvalues in both cases are so near the origin that they are all identified by an x at the origin. The two eigenvalues near the unit circle in the dde23 case are both of magnitude 0.9882. In the case of pseudospectral collocation the magnitudes are 0.9952 or approximately a 0.7% difference. The results are consistent with the fact that the approximate periodic solution is not exact so that none of the eigenvalues of the variational equation with respect to the approximate solution need be exactly unity. However, they are sufficiently close to a magnitude of unity which most

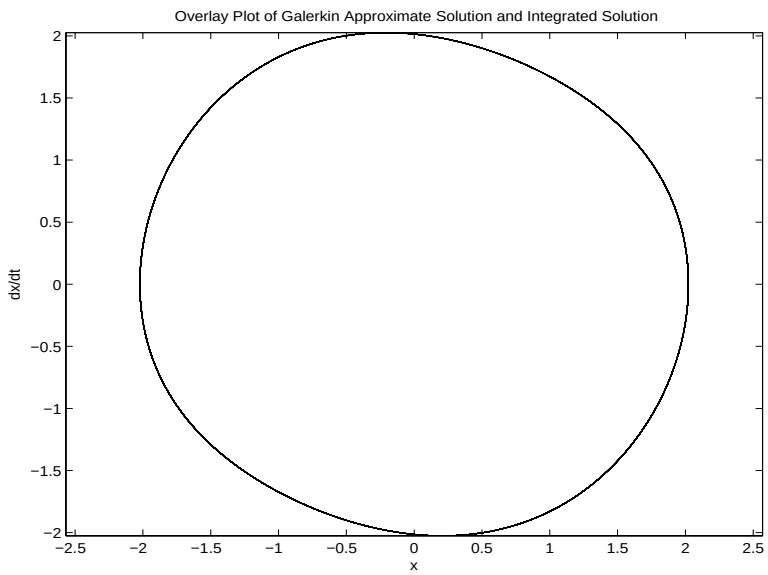

FIGURE 4. Overlay of Phase Plots of Approximate Collocated Solution and Numerically Integrated Solution for the Van der Pol Equation.

likely indicates a sufficiently good approximate solution in both cases. The quality of the approximate solution is also indicated by the absolute value of the residual, after substituting the approximate solution into the Van der Pol equation, which is 4.1020×10^{-5}. Also, the fact that all of the other eigenvalues are so near the origin is consistent with the fact that the monodromy operator and its discretizations are compact operators.

In the next case we let $\lambda = 0.1$ and performed four subcases. First we used an approximate solution of seven harmonics and took 40 Chebyshev points for collocation. We then changed the harmonics to 11 harmonics. For the third subcase we used seven harmonics but increased the Chebyshev collocation points to 50. For the last subcase we used 11 harmonics and 50 Chebyshev points. There were no significant changes in the results. The figures below represent the subcase of seven harmonics and 50 Chebyshev points. In this case the absolute value of the residual is given as 5.0481×10^{-4}.

Figure 4 shows the tight overlay of the approximate solution with seven harmonics and the numerically integrated solution. Note that the limit cycle is a slightly skewed form of a circle as expected.

Figure 5 shows the four components of the fundamental solution evaluated over $[0, 2\pi]$ by dde23 and collocation. The collocated fundamental solution is marked by x's on the graphs. In this case the numerically integrated fundamental solution differs from the collocated fundamental solution. The collocated fundamental solution in Figure 5 is more consistent with the collocated fundamental solution in Figure 2, which suggests that dde23 is having a more difficult time constructing the fundamental solution in this case. The relative tolerances in dde23 were set to 10^{-6} and the absolute tolerances to 10^{-8}.

Figure 6 again shows the first two eigenvalues symmetric about the real axis in both cases. For the case using dde23 the magnitude of the first two eigenvalues is 0.8815 and for the pseudospectral collocation it is 0.9527. These values represent an approximate 11% change in the case of numerical integration and approximately 4% change in the case of pseudospectral collocation from the eigenvalues for $\lambda = 0.01$. In both cases the magnitudes of the eigenvalues are less than unity. Both of the preceding cases showed that the approximate limit cycle was stable.

We now examine an unstable case. In this case we take $\lambda = 1.2$ again with seven harmonics and 50 Chebyshev points. Figure 7 clearly shows an instability in the numerical integration by dde23. Figure 8, for the fundamental solutions, also reflects the instability.

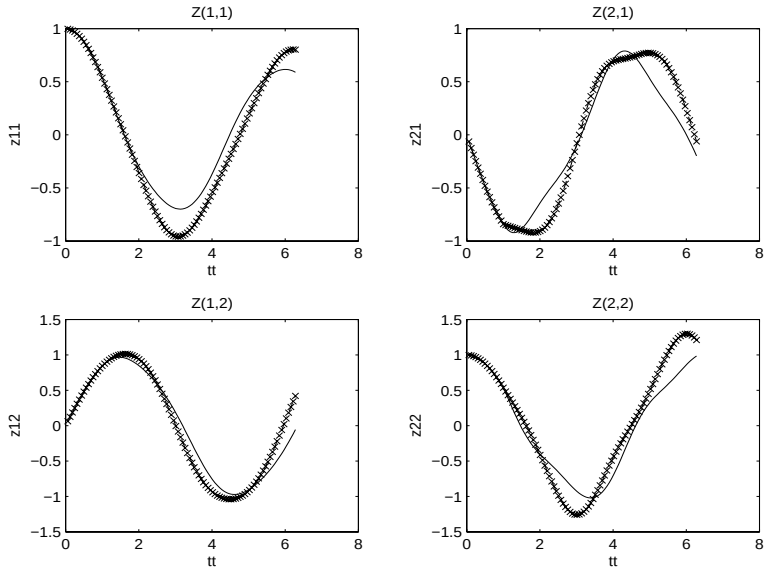

FIGURE 5. Four Elements of the Fundamental Solution of the Van der Pol Equation from 0 to 2π

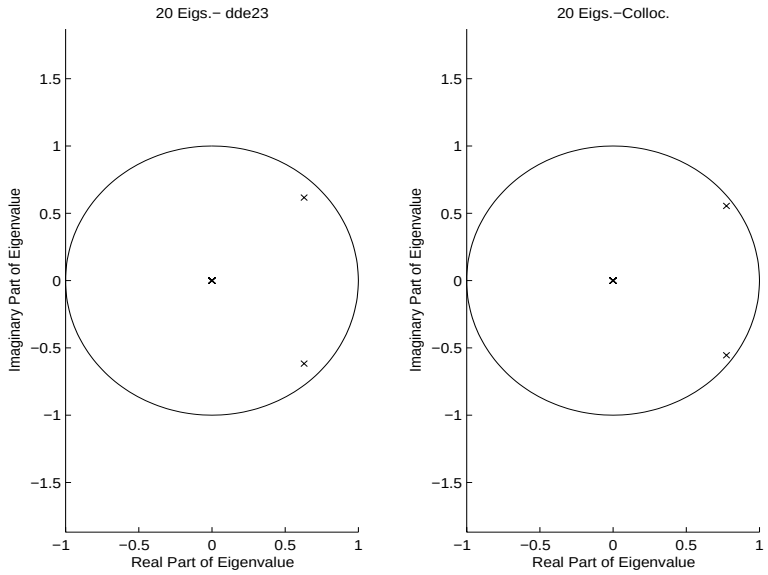

FIGURE 6. Twenty Eigenvalues of the Monodromy Operator U by dde23 and Collocation

Figure 9 shows that there are eigenvalues in both cases with magnitudes greater than unity. This is consistent with the result that the approximate solutions are stable limit cycles if all of the characteristic multipliers of the variational equation with respect to the approximate solution have magnitudes less than or equal to unity and are unstable if any one eigenvalue has magnitude greater than unity.

These results for the Van der Pol equation with delay in the non-linear terms are in contrast with the results of the Van der Pol equation with no delay. In that case the limit cycle solution is stable for large values of λ.

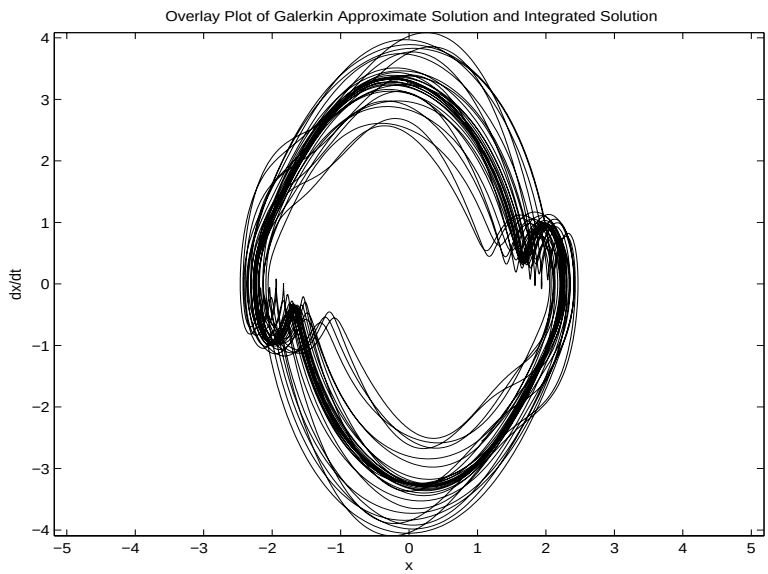

FIGURE 7. Overlay of Phase Plots of Approximate Collocated Solution and Numerically Integrated Solution for the Van der Pol Equation.

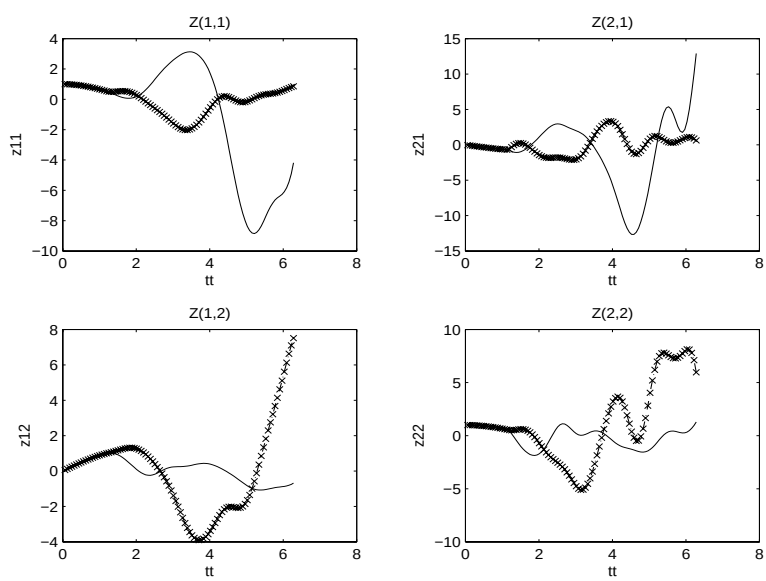

FIGURE 8. Four Elements of the Fundamental Solution of the Van der Pol Equation from 0 to 2π

7. **Disclaimer**

Certain trade names and company products are mentioned in the text or identified in an illustration in order to adequately specify the experimental procedure and equipment used. In no case does such an identification imply recommendation or endorsement by the National Institute of Standards and Technology, nor does it imply that the products are necessarily the best available for the purpose.

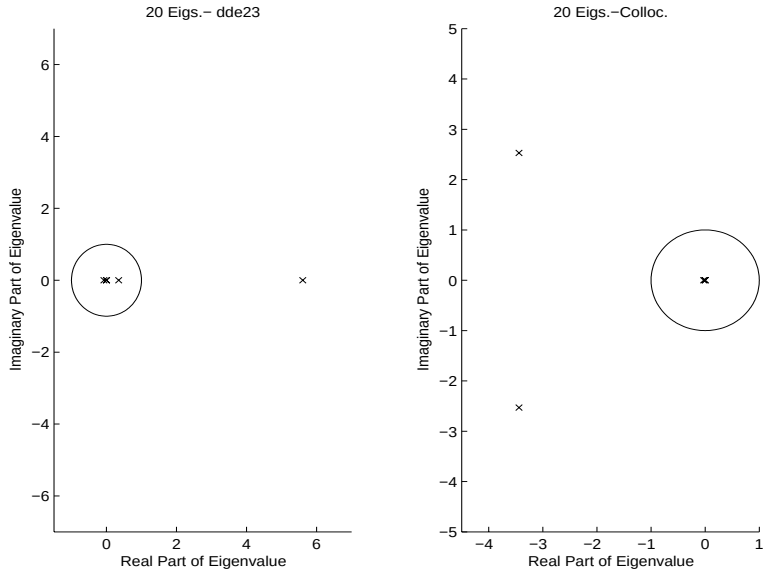

FIGURE 9. Twenty Eigenvalues of the Monodromy Operator U by dde23 and Collocation

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