Nonlinear Dynamics **30:** 103–154, 2002.

Estimating Critical Hopf Bifurcation Parameters for a Second-Order Delay Differential Equation with Application to Machine Tool Chatter

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(Received: 31 December 2000; accepted: 21 May 2002)

Abstract. Nonlinear time delay differential equations are well known to have arisen in models in physiology, biology and population dynamics. They have also arisen in models of metal cutting processes. Machine tool chatter, from a process called regenerative chatter, has been identified as self-sustained oscillations for nonlinear delay differential equations. The actual chatter occurs when the machine tool shifts from a stable fixed point to a limit cycle and has been identified as a realized Hopf bifurcation. This paper demonstrates first that a class of nonlinear delay differential equations used to model regenerative chatter satisfies the Hopf conditions. It then gives a precise characterization of the critical eigenvalues on the stability boundary and continues with a complete development of the Hopf parameter, the period of the bifurcating solution and associated Floquet exponents. Several cases are simulated in order to show the Hopf bifurcation occurring at the stability boundary. A discussion of a method of integrating delay differential equations is also given.

Keywords: Center manifolds, delay differential equations, exponential polynomials, Hopf bifurcation, limit cycle, machine tool chatter, normal form, semigroup of operators, subcritical bifurcation.

1. Introduction

Modeling physical systems with time delays has often arisen in physiology [1], biology [2], population dynamics [3]. Not as well known though is the fact that it has also arisen in modeling machine tool chatter. Chatter can be recognized by a characteristic noise, distinctive marks on the workpiece and by undulated or dissected chips [4]. It is a self-excited oscillation of the cutter relative to the workpiece during machining. Tlusty [4] identified two mechanisms of chatter. The first mechanism he pointed out was mode coupling in which there is simultaneous relative vibration between tool and workpiece in at least two directions in the plane of the orthogonal cut. The second mechanism is regenerative chatter that occurs when the tool removes the chip from a surface which was produced by the tool in the preceding pass. Thus, if there is relative vibration between tool and workpiece, waviness is generated on the cut surface. The tool in the next pass (next revolution in turning, next tooth in milling) encounters a wavy surface and removes a chip with periodically variable thickness. The cutting force varies periodically and produces vibration. Recent studies, though, have shown that other mechanisms can lead to cutting instabilities. Davies and Balachandran [5] have studied instabilities in milling due to impact dynamics. Davies et al. [6] have presented experimental evidence that regenerative effects may not totally explain the loss of stability of periodic motion for certain partial immersion operations. Zhao et al. [7] have simulated different instabilities that arise during partial immersion milling operations. For further analysis of the instabilities in milling, see [8–10].

In this paper, however, we will use the methods of Hassard et al. [11] to study the bifurcations exhibited by a simple model of regenerative chatter in turning operations originally presented in [12]. The cutting forces in turning operations are usually modeled as proportional to chip area, which is taken as the product of the chip width and the instantaneous chip thickness. Tlusty [4] identified the chip width as a significant parameter in the generation of chatter, since, for a sufficiently small chip width, cutting is stable without chatter, but, past a certain chip width, chatter occurs and its amplitude increases as the chip width increases. The chip thickness can be modeled as the difference between the current tool position and that at the previous tool pass, which introduces a delay term into the model.

In order to generate self-sustained oscillations the forces involve nonlinear functions of the instantaneous chip thickness. The nonlinearities arising in modeling turning machine chatter have been studied by many authors (e.g. [13–20]).

The self-sustained oscillations characteristic of machine tool chatter have been identified by a number of these authors as arising from Hopf bifurcations [16, 18]. The nature of the Hopf bifurcation for the regenerative chatter model will be studied here by identifying the direction of bifurcation, the structure of the periodic solution that bifurcates and its stability characteristic. Several methods for analyzing the nature of Hopf bifurcations have been described in the literature. Integral averaging has been used by Chow and Mallet-Paret [21], the Fredholm alternative has been used by Iooss and Joseph [23], the Implicit Function Theorem by Hale and Lunel [24], multi-scale expansion by Nayfeh et al. [16], and center manifold projection by Hassard et al. [11] and Stépán and Kalmár-Nagy [18]. Although many authors have calculated the critical coefficient that identifies whether a Hopf bifurcation at some critical value is super or subcritical by the Poincaré–Lyapunov constant, as for example given by Guckenheimer and Holmes [25], no one, to this author's knowledge, has completely demonstrated the validity of all conditions leading to Hopf bifurcation in the machine tool chatter case as well as developed the full construction of the bifurcation coefficient, the final periodic solution and the analysis of its stability. The results obtained for the model considered in this paper have also been obtained by Kalmár-Nagy and co-workers [12, 18, 26, 27].

The objectives of this paper, then, are twofold. First, to demonstrate the validity of all the conditions leading to a Hopf bifurcation for a class of nonlinear delay differential equations that includes a machine tool chatter model and, second, to compute the bifurcation coefficient, the period and the associated Floquet exponent of the resulting projected limit cycles on a center manifold. The author intends to present in as complete a fashion as possible the essence of Hassard et al.'s [11] arguments leading to the computation of these quantities, since it is these coefficients that determine the nature of the bifurcation and its stability. Other authors [18, 28] have developed the bifurcation coefficient for special cases. Liao et al. [29] have recently computed the three parameters in the case of a van der Pol oscillator with distributed delay. Although they develop the bifurcation parameter, they only apply the final formulas of Hassard et al.'s [11] algorithm to compute the period and Floquet exponent without developing them.

The model of regenerative chatter considered here falls into a class of second-order delay differential equation of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}s^2}(s) + 2\xi\frac{\mathrm{d}x}{\mathrm{d}s}(s) + x = p(\Delta x + E(\Delta x^2 + \Delta x^3)), \tag{1}$$

where $p > 0,\ 0 < \xi < 1,\ E > 0$ and

$$\begin{aligned} \Delta x &= x(s-\sigma) - x(s), \\ p &= \mu + p_c, \end{aligned} \tag{2}$$

and p_c represents a critical value of the parameter p at which bifurcation occurs. Equation (1) can also be written in vector form, with $z_1(s) = x(s)$, $z_2(s) = x'(s)$,

$$\frac{\mathrm{d}Z}{\mathrm{d}s}(s) = L(\mu)Z(s) + R(\mu)Z(s-\sigma) + f(Z(s), Z(s-\sigma), \mu), \tag{3}$$

where

$$\begin{aligned} Z(s) &= \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix}, \\ L(\mu) &= \begin{pmatrix} 0 & 1 \\ -1-(\mu+p_c) & -2\xi \end{pmatrix}, \\ R(\mu) &= \begin{pmatrix} 0 & 0 \\ \mu+p_c & 0 \end{pmatrix}, \\ f(z(s), z(s-\sigma), \mu) &= \\ &\begin{pmatrix} 0 \\ (\mu+p_c)E\,(z_1(s-\sigma)-z_1(s))^2 + (\mu+p_c)E\,(z_1(s-\sigma)-z_1(s))^3 \end{pmatrix}. \end{aligned} \tag{4}$$

Although this class of equations has been studied by Campbell et al. [28] where they computed the Hopf bifurcation parameter, they did not compute the period of the bifurcating solution or the Floquet exponent. The current paper is more comprehensive in that it covers more analytical ground, although the essentials are given in [11]. Since that book is quite extensive to read in its entirety, this paper distills the methods involved into a more compact form. It will not only show the application of the methods to Equation (3) but also introduce the background theory involved since this theory is intimately connected with the procedures for computing the bifurcation parameter, the period and Floquet exponent and the final form of the periodic solution.

The paper will be divided as follows. Section 2 contains the statement of the main Hopf bifurcation theorem that will be established in this paper. Section 3 includes the demonstration that (3) satisfies the conditions of the theorem. In Section 4, Equation (3) will be converted into an equivalent operator equation that will be projected onto a center manifold. A Floquet stability analysis will then be performed on the normal form of this projected equation. Finally the bifurcated periodic solution for (3) will be recovered. In Section 5 the main results will be applied to the analysis of machine tool chatter for the model of Stépán and Kalmár-Nagy [18]. A computational algorithm used to integrate the time delay differential equations in this paper will be given in the Appendix. In order to get an overview of the paper a reader can initially skip individual proofs of lemmas and concentrate on reading the introductory paragraphs of sections, definitions, and statements of lemmas. This should provide sufficient background in order to understand the results in the application Section 5.

2. The Fundamental Theorem

The following statement of the Hopf bifurcation result for the delay differential equation (3) is due to Hassard et al. [11] and Kazarinoff et al. [30]. Another statement and proof based on the implicit function theorem is given in [24]. A more general statement of the theorem is possible but will not be given here.

THEOREM 2.1. *Let*

$$Z'(s) = L(\mu)Z(s) + R(\mu)Z(s-\sigma) + f(Z(s), Z(s-\sigma), \mu), \tag{5}$$

where f contains nonlinear terms and L, R, f depend analytically on μ. $Z, f \in R^2$. Furthermore, suppose there exists a family of pairs of complex, simple, conjugate eigenvalues $\lambda(\mu)$, $\bar{\lambda}(\mu)$ such that

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \tag{6}$$

where α, ω are real and

$$\begin{aligned} \alpha(0) &= 0, \\ \omega(0) &> 0, \\ \alpha'(0) &\neq 0, \end{aligned} \tag{7}$$

and all other eigenvalues have negative real parts.

Then, (5) has a family of periodic solutions $Z(s, \varepsilon)$, parameterized by ε, such that

1. *There is an $\varepsilon_0 > 0$ and an analytic function*

$$\mu(\varepsilon) = \mu_2\varepsilon^2 + \mu_3\varepsilon^3 + \cdots, \tag{8}$$

for $0 < \varepsilon < \varepsilon_0$, where

$$Z(s, \varepsilon) = \mathcal{P}(s, \mu(\varepsilon)), \tag{9}$$

and $\mathcal{P}(s, \mu(\varepsilon))$ is a periodic solution of (5).

2. *The period $\mathcal{T}(\varepsilon)$ of $Z(s, \varepsilon)$ is an analytic function of ε and takes the form*

$$\mathcal{T}(\varepsilon) = \frac{2\pi}{\omega_c}(1 + \tau_2\varepsilon^2 + \cdots), \tag{10}$$

where ω_c is the frequency at the critical parameter p_c.

3. *The two Floquet exponents associated with the projection of $Z(s, \varepsilon)$ on a center manifold are 0 and*

$$\beta(\varepsilon) = \beta_2\varepsilon^2 + \beta_3\varepsilon^3 + \cdots, \tag{11}$$

for $0 < \varepsilon < \varepsilon_0$.

4. *$Z(s, \varepsilon)$ is orbitally asymptotically stable with asymptotic phase if $\beta(\varepsilon) < 0$ and unstable if $\beta(\varepsilon) > 0$.*

When $\mu_2 > 0$ the bifurcating periodic solutions are said to be *supercritical* and when $\mu_2 < 0$ they are said to be *subcritical*.

The proof of this theorem will not be given in its most general form. For this the reader is referred to [11]. However it will be developed in sufficient detail needed for the particular delay differential equation being studied. It is hoped that applying the methods directly to (3) will demonstrate the essential arguments used by Hassard et al. [11]. The necessary constructions will be given in a sequence of lemmas.

3. Linear Stability Analysis

In this section we will develop the stability properties for the linear part of (1), which can be written as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2}(s) + 2\xi \frac{\mathrm{d}x}{\mathrm{d}s}(s) + x(s) = p\,(x(s-\sigma) - x(s))\,, \tag{12}$$

where again $0 < \xi < 1$, $p > 0$, $p = \mu + p_c$ and p_c represents a critical value of the parameter p at which bifurcation occurs. In vector form this equation can be written as

$$\frac{\mathrm{d}Z}{\mathrm{d}s}(s) = L(\mu)Z(s) + R(\mu)Z(s-\sigma). \tag{13}$$

We will show that: (1) the characteristic equation of (13) is given by

$$\lambda^2 + 2\xi\lambda + (1+p) - p\,\mathrm{e}^{-\lambda\sigma} = 0 \tag{14}$$

and (2) it has a family of pairs of complex, simple, conjugate eigenvalues $\lambda(\mu), \bar{\lambda}(\mu)$ that satisfy the conditions

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \tag{15}$$

where α, ω are real and

$$\begin{aligned} \alpha(0) &= 0, \\ \omega(0) &> 0, \\ \alpha'(0) &\neq 0, \end{aligned} \tag{16}$$

and finally (3) that all other eigenvalues have negative real parts.

In the next sections the hypotheses of the Hopf bifurcation theorem will be shown to be satisfied. The proof of the main result will then start in Section 4.

3.1. The Characteristic Equation

Following Hale [42], introduce the following trial solution

$$Z(s) = C\,\mathrm{e}^{\lambda s} \tag{17}$$

into the linear system

$$Z'(s) = L(\mu)Z(s) + R(\mu)Z(s-\sigma) \tag{18}$$

and set the determinant of the resulting system to zero, where $C \in R^2$, $L(\mu)$, and $R(\mu)$ are given by (4). This yields the transcendental characteristic equation

$$\chi(\lambda) = \lambda^2 + 2\xi\lambda + (1+p) - p\,\mathrm{e}^{-\lambda\sigma} = 0. \tag{19}$$

This form of equation, sometimes called an *exponential polynomial*, has been studied by Avellar and Hale [31], Bellman and Cooke [32], Hale and Lunel [24], Kuang [3], Pinney [33], Stepan [34]. The solutions are called the *eigenvalues* of Equation (18). In general there are an infinite number of eigenvalues. For a discussion of the general expansion of solutions of Equation (18) in terms of the eigenvalues, see [32] or [33]. Before developing the families of conjugate eigenvalues, however, we wish to characterize certain critical eigenvalues of Equation (19) of the form $\lambda = i\omega$.

3.2. Characterizing Critical Eigenvalues

We will find in the next result that eigenvalues for Equation (19) of the form $\lambda = i\omega$ exist only for special combinations of p and σ. We introduce the following definition.

DEFINITION 3.1. A triple (ω, σ, p), where ω, σ, p are real, will be called a *critical eigen triple* of (19) if $\lambda = i\omega, \sigma, p$ simultaneously satisfy Equation (19).

The next lemma characterizes the critical eigen triples for linear delay equations of the form (19) but can be modified as needed for linear delay systems with different coefficients.

LEMMA 3.1. *Critical eigen triples of Equation* (19) *satisfy the following properties:*

1. (ω, σ, p) *is a critical eigen triple of Equation* (19) *if and only if* $(-\omega, \sigma, p)$ *is also.*
2. *For* $\omega > 1$ *there is a uniquely defined sequence* $\sigma_r = \sigma_r(\omega)$, $r = 0, 1, 2, \ldots$, *and a uniquely defined* $p = p(\omega)$ *such that* (ω, σ_r, p), $r = 0, 1, 2, \ldots$, *are critical eigen triples.*
3. *If* (ω, σ, p) *is a critical eigen triple, with* $\omega > 1$, *then* $p \geq 2\xi(1+\xi)$. *That is, no critical eigen triple for (19) exists for* $p < 2\xi(1+\xi)$.
4. *For* $p = 2\xi(1+\xi)$ *there is a unique* $\omega > 1$ *and a unique sequence* σ_r, $r = 0, 1, 2, \ldots$, *such that* (ω, σ_r, p) *is a critical eigen triple for Equation* (19) *for* $r = 0, 1, 2, \ldots$.
5. *For* $p > 2\xi(1+\xi)$ *there exist two* ω*'s,* $\omega > 1$, *designated* ω_+, ω_- *and uniquely associated sequences* $\sigma_r^+ = \sigma_r(\omega_+)$, $\sigma_r^- = \sigma_r(\omega_-)$, $r = 0, 1, 2, \ldots$ *such that* $(\omega_+, \sigma_r^+, p)$, $(\omega_-, \sigma_r^-, p)$ *are critical eigen triples for Equation* (19) *for* $r = 0, 1, 2, \ldots$.
6. *There do not exist critical eigen triples for* $0 \leq \omega \leq 1$.

Before proving Lemma 3.1, examine Figure 1. This figure, called a *stability chart*, graphically shows the meanings of parts (2) through (5) of the lemma. The lobe boundaries represent the potential eigen triples of Equation (19). First of all, each lobe is parameterized by the same set of ω's, $\omega > 1$. For the sake of discussion here, call the right most lobe, lobe 0, the next on the left lobe 1, etc. Part (2) then states that for a given ω there is associated a unique value on the vertical axis, called $p(\omega)$, but an infinite number of $\sigma_r(\omega)$'s, one for each lobe, depicted graphically on the horizontal axis as $1/\omega_r$, for rotation rate. The minimum value on each lobe occurs at $p = 2\xi(1+\xi)$ with an associated unique $\omega = \sqrt{1+2\xi}$ (see Equations (60) and (61) below). Finally, for each lobe there are two ω's associated with each p, denoted by ω_-

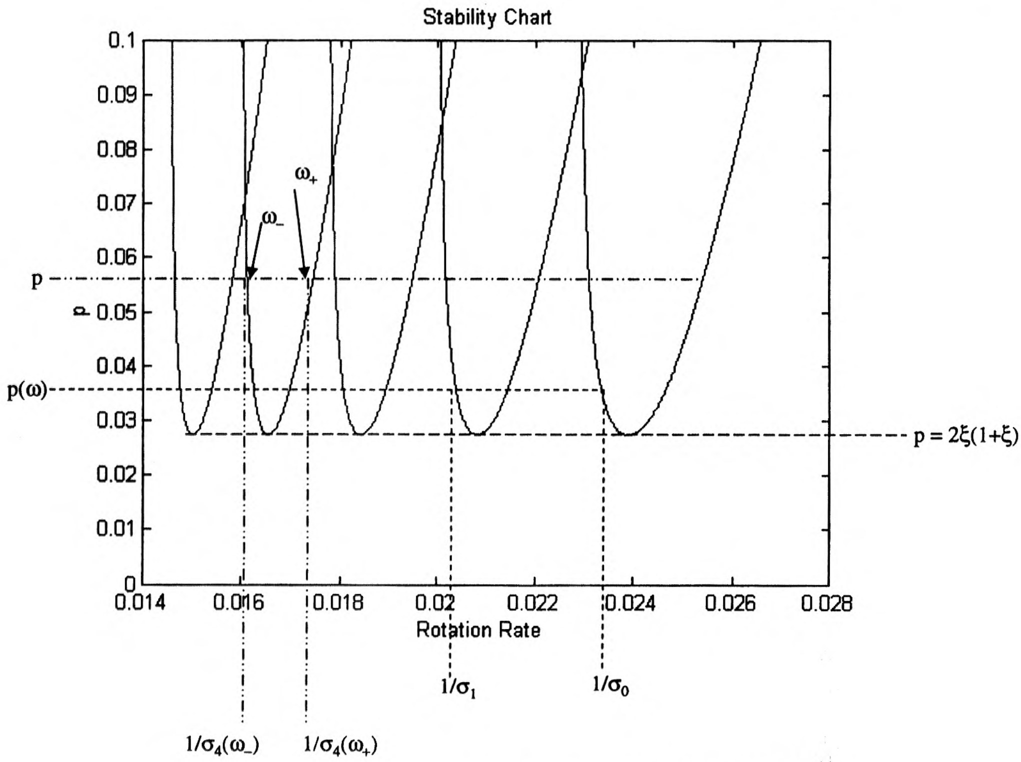

Figure 1. Stability chart with sample critical eigen triples identified.

and ω_+, where ω_- is the parameter associated with the left side of the lobe and ω_+ with the right side of the lobe. At the minimum $\omega_- = \omega_+$. Figure 1 shows that the lobes cross each other. In this paper, however, only single point Hopf crossings will be considered. The study of solutions at multiple crossings requires a separate study.

Proof of Lemma 3.1. The proof of (1) follows easily by taking the conjugate of Equation (19) and noting that the polynomial coefficients are real.

To prove (2) one can use an argument modeled after Altintas and Budak [35] (also attributed to Tlusty in 1965) to develop necessary conditions for σ_r and p. Subsequently they will be shown to be sufficient. Set

$$\Phi(\lambda) = \frac{1}{\lambda^2 + 2\xi\lambda + 1}. \tag{20}$$

Then Equation (19) becomes

$$1 + p(1 - e^{-\lambda\sigma})\Phi(\lambda) = 0. \tag{21}$$

Set $\lambda = i\omega$ and write

$$\Phi(i\omega) = G(\omega) + iH(\omega), \tag{22}$$

where

$$G(\omega) = \frac{1 - \omega^2}{(1 - \omega^2)^2 + (2\xi\omega)^2}, \tag{23}$$

$$H(\omega) = \frac{-2\xi\omega}{(1-\omega^2)^2 + (2\xi\omega)^2}. \tag{24}$$

Substitute Equation (22) into Equation (21) and separate real and imaginary parts to get

$$1 + p[(1-\cos\omega\sigma)G(\omega) - (\sin\omega\sigma)H(\omega)] = 0, \tag{25}$$

$$p[G(\omega)\sin\omega\sigma + H(\omega)(1-\cos\omega\sigma)] = 0. \tag{26}$$

From Equation (26)

$$\frac{H(\omega)}{G(\omega)} = -\frac{\sin\omega\sigma}{1-\cos\omega\sigma}. \tag{27}$$

From the definition of G, H and the fact that $\omega > 1$, Equation (27) falls into the third quadrant so that one can introduce the phase angle for Equation (22), using Equations (23) and (24), as

$$\psi = \tan^{-1}\left(\frac{H(\omega)}{G(\omega)}\right) = -\pi + \tan^{-1}\left(\frac{2\xi\omega}{\omega^2-1}\right). \tag{28}$$

Clearly, $-\pi \leq \psi \leq \pi$. Using half-angle formulas

$$\begin{aligned} \tan\psi &= -\frac{\sin\omega\sigma}{1-\cos\omega\sigma}, \\ &= -\frac{\cos\left(\frac{\omega\sigma}{2}\right)}{\sin\left(\frac{\omega\sigma}{2}\right)}, \\ &= -\cot\left(\frac{\omega\sigma}{2}\right), \\ &= \tan\left(\frac{\pi}{2} + \frac{\omega\sigma}{2} \pm n\pi\right), \end{aligned} \tag{29}$$

for $n = 0, 1, 2, \ldots$. Therefore

$$\psi = \frac{\pi}{2} + \frac{\omega\sigma}{2} \pm n\pi, \tag{30}$$

where $\omega\sigma > 0$ must be satisfied for all n. In order to satisfy this and the condition that $-\pi \leq \psi \leq \pi$, select the negative sign and

$$n = 2 + r \tag{31}$$

for $r = 0, 1, 2, \ldots$. Therefore, from Equation (30), the necessary sequence, σ_r, is given by

$$\sigma_r = \frac{\omega}{2(\psi + r\pi) + 3\pi}, \tag{32}$$

where ψ is given by Equation (28). Finally, substituting Equation (27) into Equation (25) one has the necessary condition for p as

$$p = -\frac{1}{2G(\omega)}, \tag{33}$$

where $p > 0$ since $\omega > 1$. Therefore Equations (32) and (33) are the necessary conditions for $(\omega,\ \sigma_r,\ p)$, $r = 0,\ 1,\ 2, \ldots$, to be critical eigen triples for Equation (19). Note that this also implies uniqueness. Equations (32) and (33) show how $p = p(\omega)$ and $1/\sigma_r$ are uniquely relate in Figure 1.

In order to show that Equations (32) and (33) are sufficient conditions take $\omega > 1$ and define $G(\omega)$, $H(\omega)$ by Equations (23) and (24). Define

$$\psi = \tan^{-1}\left(\frac{H(\omega)}{G(\omega)}\right) = -\pi + \tan^{-1}\left(\frac{2\xi\omega}{\omega^2 - 1}\right), \tag{34}$$

and choose

$$\sigma_r = \frac{\omega}{2(\psi + r\pi) + 3\pi}. \tag{35}$$

By substituting Equations (34) into (35) solve for

$$\theta = \tan^{-1}\left(\frac{2\xi\omega}{\omega^2 - 1}\right) = \frac{\omega\sigma_r - (2r+1)\pi}{2}. \tag{36}$$

Then, since $\omega > 1$ and $0 < \xi < 1$, one has $0 < \theta < \pi/2$. Then set

$$\sin\theta = \frac{2\xi\omega}{\sqrt{(\omega^2 - 1)^2 + (2\xi\omega)^2}}, \tag{37}$$

$$\cos\theta = \frac{\omega^2 - 1}{\sqrt{(\omega^2 - 1)^2 + (2\xi\omega)^2}}. \tag{38}$$

From Equation (36) compute

$$\sin\theta = \sin\left[\frac{\omega\sigma_r - (2r+1)\pi}{2}\right] = (-1)^{r+1}\cos\left(\frac{\omega\sigma_r}{2}\right), \tag{39}$$

$$\cos\theta = \cos\left[\frac{\omega\sigma_r - (2r+1)\pi}{2}\right] = (-1)^{r}\sin\left(\frac{\omega\sigma_r}{2}\right). \tag{40}$$

Then, by half angle formulas,

$$\sin^2\theta = \cos^2\left(\frac{\omega\sigma_r}{2}\right) = \frac{1 + \cos\omega\sigma_r}{2}, \tag{41}$$

$$\cos^2\theta = \sin^2\left(\frac{\omega\sigma_r}{2}\right) = \frac{1 - \cos\omega\sigma_r}{2}. \tag{42}$$

Combining Equations (41), (42) with Equations (37) and (38) yields

$$1 + \cos\omega\sigma_r = \frac{2(2\xi\omega)^2}{(\omega^2 - 1)^2 + (2\xi\omega)^2}, \tag{43}$$

$$1 - \cos\omega\sigma_r = \frac{2(\omega^2 - 1)^2}{(\omega^2 - 1)^2 + (2\xi\omega)^2}. \tag{44}$$

By taking the product of Equation (43) and Equation (44) one gets

$$\sin^2\omega\sigma_r = 1 - \cos^2\omega\sigma_r = \frac{4(2\xi\omega)^2(\omega^2 - 1)^2}{[(\omega^2 - 1)^2 + (2\xi\omega)^2]^2}, \tag{45}$$

which implies that

$$\sin \omega\sigma_r = \pm\frac{2(2\xi\omega)(\omega^2 - 1)}{(\omega^2 - 1)^2 + (2\xi\omega)^2}. \tag{46}$$

In order to select the sign, start with $0 < \theta < \pi/2$. Then, from Equation (36), $(2r + 1)\pi < \omega\sigma_r < 2(r + 1)\pi$ which implies that $\sin \omega\sigma_r < 0$. Therefore

$$\sin \omega\sigma_r = -\frac{2(2\xi\omega)(\omega^2 - 1)}{(\omega^2 - 1)^2 + (2\xi\omega)^2}. \tag{47}$$

Define $G(\omega)$, $H(\omega)$ and p by Equations (23), (24), and (33) respectively. Then, using Equations (44) and (47), one gets

$$1 + p[(1 - \cos \omega\sigma_r)G(\omega) - (\sin \omega\sigma_r)H(\omega)] = 0, \tag{48}$$

$$p[G(\omega) \sin \omega\sigma_r + H(\omega)(1 - \cos \omega\sigma_r)] = 0. \tag{49}$$

This is equivalent to $(\omega,\ \sigma_r,\ p)$ being a critical eigen triple for Equation (19).

To prove (3), let $(\omega,\ \sigma,\ p)$ be a critical eigen triple with $\omega > 1$. Write Equation (19) as

$$(-\omega^2 + 1 + p - p \cos \omega\sigma) + i(2\xi\omega + p \sin \omega\sigma) = 0. \tag{50}$$

Define

$$\mathcal{R}(\omega,\ \sigma,\ p) = -\omega^2 + 1 + p - p \cos \omega\sigma, \tag{51}$$

$$\mathcal{I}(\omega,\ \sigma,\ p) = 2\xi\omega + p \sin \omega\sigma. \tag{52}$$

Since $(\omega,\ \sigma,\ p)$ is a critical eigen triple, $\mathcal{R}(\omega,\ \sigma,\ p) = \mathcal{I}(\omega,\ \sigma,\ p) = 0$. Then

$$\sin \omega\sigma = -\frac{2\xi\omega}{p}, \tag{53}$$

$$\cos \omega\sigma = \frac{1}{p}(-\omega^2 + 1 + p). \tag{54}$$

The squares of Equations (53) and (54) sum to one so that

$$\omega^4 + 2(2\xi^2 - 1 - p)\omega^2 + (1 + 2p) = 0. \tag{55}$$

Solving for ω^2 in Equation (55) gives

$$\omega^2 = (1 + p - 2\xi^2) \pm \sqrt{D}, \tag{56}$$

where the discriminant D is defined as

$$D = (p - 2\xi^2)^2 - 4\xi^2. \tag{57}$$

For $p > 2\xi^2$, D is increasing and $D = 0$ for

$$p = 2\xi(\xi \pm 1). \tag{58}$$

Since $p > 0$ and $0 < \xi < 1$ one must select the $+$ sign. Furthermore, since ω is real, $D \geq 0$. Therefore

$$p \geq 2\xi(\xi + 1). \tag{59}$$

In order to prove (4) note that the minimum of $D \geq 0$ occurs at

$$p_m = 2\xi(\xi + 1), \tag{60}$$

where the subscript m designates the value of p at the minimum of D. From Equation (56) and $D = 0$ the frequency at the minimum is

$$\omega_m = \sqrt{1 + 2\xi}. \tag{61}$$

Equations (60) and (61) represent the minimum value of p and the associated unique ω at each lobe in Figure 1. From the uniqueness of σ_r, in the proof of (2), we have the sequence σ_r given by

$$\sigma_r^m = \frac{2(\psi_m + r\pi) + 3\pi}{\omega_m}, \tag{62}$$

where $r = 0, 1, 2, \ldots$ and the superscript m denotes evaluation of $\sigma_r = \sigma_r(\omega)$ at $\omega = \omega_m$. Furthermore

$$\psi_m = -\pi + \tan^{-1}\left(\frac{2\xi\omega_m}{\omega_m^2 - 1}\right). \tag{63}$$

In order to show that $(\omega_m, \sigma_r^m, p_m)$ is a sequence of critical eigen triples use Equations (46) and (44) from part (2).

$$\sin\omega_m\sigma_r^m = -\frac{2(2\xi\omega_m)(\omega_m^2 - 1)}{(\omega_m^2 - 1)^2 + (2\xi\omega_m)^2}, \tag{64}$$

$$1 - \cos\omega_m\sigma_r^m = \frac{2(\omega_m^2 - 1)^2}{(\omega_m^2 - 1)^2 + (2\xi\omega_m)^2}. \tag{65}$$

Then, using Equations (60) and (61), it is fairly direct to show that $\mathcal{R}(\omega_m, \sigma_r^m, p_m) = \mathcal{I}(\omega_m, \sigma_r^m, p_m) = 0$, which proves (4).

To prove (5) use Equation (56) to define ω_+, ω_- by

$$\omega_+^2 = (1 + p - 2\xi^2) + \sqrt{p^2 - 4\xi^2 p + (4\xi^4 - 4\xi^2)}, \tag{66}$$

$$\omega_-^2 = (1 + p - 2\xi^2) - \sqrt{p^2 - 4\xi^2 p + (4\xi^4 - 4\xi^2)}. \tag{67}$$

Equations (66) and (67) give the values of ω_+ and ω_- for a given p in Figure 1. They are the same values for that p for each lobe. As an aside, note that it is not hard to show that $\mathrm{d}(\omega_+^2)/\mathrm{d}p > 0$ and $\mathrm{d}(\omega_-^2)/\mathrm{d}p < 0$. Therefore, ω_+^2 and thus ω_+ is increasing with p and ω_-^2 is decreasing and thus ω_- is also. Similar results have been obtained by Campbell et al. [28]. Again, as in the proof of part (4), define σ_r^+, σ_r^- by

$$\sigma_r^+ = \frac{2(\psi_+ + r\pi) + 3\pi}{\omega_+}, \tag{68}$$

$$\sigma_r^- = \frac{2(\psi_- + r\pi) + 3\pi}{\omega_-}, \tag{69}$$

where

$$\psi_+ = -\pi + \tan^{-1}\left(\frac{2\xi\omega_+}{\omega_+^2 - 1}\right), \tag{70}$$

$$\psi_- = -\pi + \tan^{-1}\left(\frac{2\xi\omega_-}{\omega_-^2 - 1}\right). \tag{71}$$

Again

$$\sin\omega_+\sigma_r^+ = -\frac{2(2\xi\omega_+)(\omega_+^2 - 1)}{(\omega_+^2 - 1)^2 + (2\xi\omega_+)^2}, \tag{72}$$

$$1 - \cos\omega_+\sigma_r^+ = \frac{2(\omega_+^2 - 1)^2}{(\omega_+^2 - 1)^2 + (2\xi\omega_+)^2}, \tag{73}$$

$$\sin\omega_-\sigma_r^- = -\frac{2(2\xi\omega_-)(\omega_-^2 - 1)}{(\omega_-^2 - 1)^2 + (2\xi\omega_-)^2}, \tag{74}$$

$$1 - \cos\omega_-\sigma_r^- = \frac{2(\omega_-^2 - 1)^2}{(\omega_-^2 - 1)^2 + (2\xi\omega_-)^2}, \tag{75}$$

and it is not hard, but tedious, to show, as in [4], $\mathcal{R}(\omega_+,\ \sigma_r^+,\ p) = \mathcal{I}(\omega_+,\ \sigma_r^+,\ p) = \mathcal{R}(\omega_-,\ \sigma_r^-,\ p) = \mathcal{I}(\omega_-,\ \sigma_r^-,\ p) = 0$.

Finally, to prove (6), let $0 \le \omega \le 1$. Clearly $(0,\ \sigma,\ p)$ is not a critical eigen triple since $\mathcal{R}(0,\ \sigma,\ p) = 1$. To show that $(1,\ \sigma,\ p)$ is not a critical eigen triple first compute

$$\mathcal{R}(1,\ \sigma,\ p) = p(1 - \cos\sigma), \tag{76}$$

$$\mathcal{I}(1,\ \sigma,\ p) = 2\xi + p\sin\sigma. \tag{77}$$

Now $\mathcal{R}(1,\ \sigma,\ p) = 0$ if and only if $\sigma = 2k\pi,\ k = 0,\ \pm 1,\ \pm 2, \ldots$. But then $\mathcal{I}(1,\ \sigma,\ p) = 2\xi \neq 0$. Conversely, if $\mathcal{I}(1,\ \sigma,\ p) = 0$ then

$$\sin\sigma = -\frac{2\xi}{p} \tag{78}$$

and therefore

$$\cos^2\sigma = 1 - \frac{4\xi^2}{p^2}. \tag{79}$$

Since $p \ge 2\xi(1+\xi)$, $1 - (4\xi^2/p^2) > 0$, so that

$$\cos\sigma = \pm\sqrt{1 - \frac{4\xi^2}{p^2}}. \tag{80}$$

In the negative case, $1 - \cos\sigma = 1 + \sqrt{1 - 4\xi^2/p^2} > 0$. In the positive case, $1 - \cos\sigma = 1 - \sqrt{1 - 4\xi^2/p^2} = (4\xi^2/p^2)/(1 + \sqrt{1 - 4\xi^2/p^2}) \neq 0$. Therefore $\mathcal{R}(1,\ \sigma,\ p) \neq 0$. Finally, for $0 < \omega < 1$, $\mathcal{R}(\omega,\ \sigma,\ p)$ can be written as

$$\mathcal{R}(\omega,\ \sigma,\ p) = (1 - \omega^2) + p(1 - \cos\omega\sigma), \tag{81}$$

which is strictly positive since $(1 - \omega^2) > 0$ and $(1 - \cos\omega\sigma) \ge 0$. □

An immediate consequence of this lemma is the following corollary.

COROLLARY 3.1. *Given a fixed* $r = 0, 1, \ldots,$ *if* (ω_0, σ_r, p) *is a critical eigen triple,* $\omega_0 > 0$, *then there cannot be another critical eigen triple* (ω_1, σ_r, p), $\omega_1 > 0$, $\omega_1 \neq \omega_0$. *Furthermore, since* $(-\omega_0, \sigma_r, p)$ *is also a critical eigen triple, there can be no critical eigen triple* (ω_2, σ_r, p), $\omega_2 < 0$, $\omega_2 \neq -\omega_0$.

Proof. This follows from parts (1), (2) and (6) of Lemma 3.1. □

This result does not preclude two or more separate lobes from crossing. It only refers to a fixed lobe.

3.3. Families of Eigenvalues

LEMMA 3.2. *There is a family of simple, conjugate eigenvalues* $\lambda(\mu)$, $\bar{\lambda}(\mu)$, *of Equation* (19), *such that*

$$\lambda(\mu) = \alpha(\mu) + i\omega_0(\mu), \tag{82}$$

where α, ω_0 *are real and*

$$\begin{aligned} \alpha(0) &= 0, \\ \omega_0(0) &> 0, \\ \alpha'(0) &> 0. \end{aligned} \tag{83}$$

Proof. Given an eigenvalue of Equation (19) it is clear that the conjugate is also a solution from Lemma 3.1, part (1). To show the existence of the family of eigenvalues we appeal to the Implicit Function Theorem [36]. Since we are interested in eigenvalues in the neighborhood of the critical point (σ_c, p_c) let

$$\omega_0 = \omega + \omega_c \tag{84}$$

and set

$$\begin{aligned} p &= \mu + p_c, \\ \lambda &= \alpha + i(\omega + \omega_c), \end{aligned} \tag{85}$$

where ω_c is the frequency at p_c. Insert Equation (85) into Equation (19) and separate the real and imaginary parts as

$$\chi(\lambda) = \chi(\alpha, \omega, \mu) = \mathcal{R}(\alpha, \omega, \mu) + i\mathcal{I}(\alpha, \omega, \mu), \tag{86}$$

where

$$\begin{aligned} \mathcal{R}(\alpha, \omega, \mu) &= \alpha^2 - (\omega + \omega_c)^2 + 2\xi\alpha + 1 + (\mu + p_c) - (\mu + p_c)e^{-\alpha\sigma_c}\cos\sigma_c(\omega + \omega_c), \\ \mathcal{I}(\alpha, \omega, \mu) &= 2\alpha(\omega + \omega_c) + 2\xi(\omega + \omega_c) + (\mu + p_c)e^{-\alpha\sigma_c}\sin\sigma_c(\omega + \omega_c). \end{aligned} \tag{87}$$

The Jacobian at (0,0,0) is given by

$$\begin{aligned} J(0,0,0) &= \begin{vmatrix} \mathcal{R}_\alpha(0,0,0) & \mathcal{R}_\omega(0,0,0) \\ \mathcal{I}_\alpha(0,0,0) & \mathcal{I}_\omega(0,0,0) \end{vmatrix} \\ &= 4\xi^2 + 4\omega_c^2 + p_c^2\sigma^2 + 4\sigma\xi\omega_c^2 + 4\sigma\xi(1 + p_c) > 0, \end{aligned} \tag{88}$$

where we have used Equations (53) and (54). The subscripts of α, ω of $\mathcal{R}$ and $\mathcal{I}$ indicate partial derivatives with respect to those parameters. Therefore according to the Implicit Function Theorem there are analytic functions $\alpha(\mu)$, $\omega(\mu)$ such that $\chi(\alpha(\mu), \omega(\mu), \mu) = 0$ where $\alpha(0) = 0$, $\omega(0) = 0$ and $\omega_0(0) = \omega(0) + \omega_c = \omega_c > 0$.

To test whether $\alpha'(0) > 0$ take the implicit derivative with respect to μ of

$$\begin{aligned}\chi(\alpha(\mu), \omega(\mu), \mu) &= [\alpha(\mu) + i(\omega(\mu) + \omega_c)]^2 + 2\xi[\alpha(\mu) + i(\omega(\mu) + \omega_c)] \\ &\quad + 1 + (\mu + p_c)[1 - \mathrm{e}^{-(\alpha(\mu)+i(\omega(\mu)+\omega_c))\sigma_c}] \\ &= 0 \end{aligned} \tag{89}$$

to get, with some algebra,

$$\begin{aligned}\alpha'(0) &= \frac{[2\xi - \sigma\omega_c^2 + \sigma(1+p_c)][1-\omega_c^2] + [2\omega_c(1+\sigma\xi)][2\xi\omega_c]}{p_c[2\xi - \sigma\omega_c^2 + \sigma(1+p_c)]^2 + p_c[2\omega_c(1+\sigma\xi)]^2}, \\ \omega'(0) &= \frac{[2\xi - \sigma\omega_c^2 + \sigma(1+p_c)][2\xi\omega_c] - [1-\omega_c^2][2\omega_c(1+\sigma\xi)]}{p_c[2\xi - \sigma\omega_c^2 + \sigma(1+p_c)]^2 + p_c[2\omega_c(1+\sigma\xi)]^2}.\end{aligned} \tag{90}$$

The numerator of $\alpha'(0)$, divided by p_c, can be expanded to give

$$2\xi(1+\omega_c^2) + \sigma(1-\omega_c^2)^2 + 4\sigma\omega_c^2\xi^2 + \sigma p_c(1-\omega_c^2). \tag{91}$$

The only term in Equation (91) that can potentially cause Equation (91) to become negative is the last one. However, p_c and ω_c are related by Equation (33) which implies that

$$p_c = \frac{(1-\omega_c^2)^2 + (2\xi\omega_c)^2}{2(\omega_c^2 - 1)}. \tag{92}$$

If we substitute Equation (92) into Equation (91) one gets

$$2\xi(1+\omega_c^2) + \frac{\sigma}{2}(1-\omega_c^2)^2 + 2\sigma\omega_c^2\xi^2, \tag{93}$$

which is clearly positive so that $\alpha'(0) > 0$.

To show that $\lambda = i\omega_c$ is a simple root it is sufficient to show that $\chi'(i\omega_c) \neq 0$, which means that the first-order term of the Taylor series is non-zero. But, this is clear from Equation (19). Using Equations (53) and (54) one has

$$\chi'(i\omega_c) = [2\xi - \sigma\omega_c^2 + \sigma(1+p_c)] + i[2(1+\xi)\omega_c]. \tag{94}$$

Therefore $|\chi'(i\omega_c)| > 0$, since $\omega_c > 1$. Then, by continuity, $|\chi'(\lambda(\mu))| > 0$ for μ near 0, so that $\chi'(\lambda(\mu)) \neq 0$ for μ near 0. □

3.4. The Distribution of Non-Critical Eigenvalues

Finally, one can show

LEMMA 3.3. *Given a single critical eigenvalue of* (19) *on the imaginary axis, along with its conjugate, then all other eigenvalues have negative real parts.*

Although the general location of zeros for exponential polynomials is a non-trivial problem (see, e.g., [3, 24, 32, 34]), in the current case it will be possible to use the argument principle to demonstrate the result for Equation (19). Since Equation $\chi(\lambda)$ is an analytic function the argument principle may be stated in the following form [37]. Following Stepan [34], let Γ be a simple closed contour with $\chi(\lambda) \neq 0$ on Γ. Then

$$\frac{1}{2\pi i}\int\limits_{\Gamma^+} \frac{\chi'(\lambda)}{\chi(\lambda)}\,\mathrm{d}\lambda = \frac{1}{2\pi}\Delta_{\Gamma^+} \arg \mathrm{D}(\lambda) = \mathcal{N}, \tag{95}$$

where Γ^+ means to integrate counterclockwise about Γ, Δ_{Γ^+} indicates the change in argument over Γ^+ and $\mathcal{N}$ is the number of zeros within Γ. The object in the current case is to show $\mathcal{N} = 0$.

Let $\pm i\omega_c$ be the conjugate roots on the imaginary axis so that $(\pm\omega_c,\ \sigma_c,\ p_c)$ are critical eigen triples on the imaginary axis. Construct an indented Bromwich contour Γ, where $\Gamma = \Gamma_1 + \cdots + \Gamma_7$, with

$$\begin{aligned}
\Gamma_1 &= \{\lambda : \lambda = i\omega,\, A \geq \omega \geq \omega_m + \varepsilon\},\\
\Gamma_2 &= \left\{\lambda : \lambda = i\omega_c + \varepsilon\,\mathrm{e}^{i\theta},\, \frac{\pi}{2} \geq \theta \geq -\frac{\pi}{2}\right\},\\
\Gamma_3 &= \{\lambda : \lambda = i\omega,\, \omega_c - \varepsilon \geq \omega \geq 0\},\\
\Gamma_4 &= \{\lambda : \lambda = i\omega,\, 0 \geq \omega \geq -\omega_c + \varepsilon\},\\
\Gamma_5 &= \left\{\lambda : \lambda = -i\omega_c + \varepsilon e^{i\theta},\, \frac{\pi}{2} \geq \theta \geq -\frac{\pi}{2}\right\},\\
\Gamma_6 &= \{\lambda : \lambda = i\omega,\, -\omega_c - \varepsilon \geq \omega \geq -A\},\\
\Gamma_7 &= \left\{\lambda : \lambda = A\,\mathrm{e}^{i\theta},\, \frac{\pi}{2} \geq \theta \geq -\frac{\pi}{2}\right\},
\end{aligned} \tag{96}$$

and the orientation is counterclockwise.

Since there can be only a finite number of zeros of Equation (19) to the right of the imaginary axis [38], A and ε can be selected so that there are no zeros on Γ_2, Γ_5, and Γ_7. That there are no eigenvalues on Γ_1, Γ_3, Γ_4 and Γ_6 follows from Lemma 3.1 and Corollary 3.1.

One can now finally compute the change of argument as one transverses Γ. On Γ_1, using Equations (51) and (52),

$$\frac{1}{2\pi}\Delta_{\Gamma_1^+} \arg \chi(i\omega) = \frac{1}{2\pi}\left\{\tan^{-1}\left(\frac{\mathcal{I}(\omega_c+\varepsilon)}{\mathcal{R}(\omega_c+\varepsilon)} - \tan^{-1}\left(\frac{\mathcal{I}(A)}{\mathcal{R}(A)}\right)\right)\right\}, \tag{97}$$

where σ_c, p_c are understood. Divide the numerator and denomenator of $\mathcal{I}(A)/\mathcal{R}(A)$ by A^2 and take the limit as $A \to \infty$. This implies that $\tan^{-1}(\mathcal{I}(A)/\mathcal{R}(A)) \approx 0$ for large A in the second term on the right of Equation (97). For the first term on the right of Equation (97) apply the Taylor series to the numerator and denominator for small $\varepsilon > 0$ and take the limit as $\varepsilon \to 0$. Then, in the limit, as $A \to \infty$ and $\varepsilon \to 0$, Equation (97) becomes

$$\frac{1}{2\pi}\Delta_{\Gamma_1^+} \arg \chi(i\omega) = \frac{1}{2\pi}\tan^{-1}\left(\frac{2\xi + p_c\sigma_c\cos\omega_c\sigma_c}{-2\omega_c + p_c\sigma_c\sin\omega_c\sigma_c}\right). \tag{98}$$

On Γ_3, $\mathcal{I}(0) = 0$, $\mathcal{R}(0) = 1$. Apply the Taylor series to $\mathcal{I}(\omega_c - \varepsilon)/\mathcal{R}(\omega_c - \varepsilon)$ and get, for small $\varepsilon > 0$,

$$\frac{1}{2\pi}\Delta_{\Gamma_3^+} \arg \chi(i\omega) = -\frac{1}{2\pi}\Delta_{\Gamma_1^+} \arg \chi(i\omega). \tag{99}$$

By similar arguments

$$\frac{1}{2\pi}\Delta_{\Gamma_4^+}\arg\chi(-i\omega) = -\frac{1}{2\pi}\Delta_{\Gamma_6^+}\arg\chi(-i\omega). \tag{100}$$

On Γ_2 use the integral form of Equation (95) to compute the change in argument. Since $\lambda = i\omega_c$ is a simple root of $\chi(\lambda)$ it is a simple pole of $\chi'(\lambda)/\chi(\lambda)$. The residue at $i\omega_c$ is one so that the Laurent series is

$$\frac{\chi'(\lambda)}{\chi(\lambda)} = \frac{1}{\lambda - i\omega_c} + g(\lambda), \tag{101}$$

where $g(\lambda)$ is analytic and bounded in the neighborhood of $i\omega_c$. Then

$$\frac{1}{2\pi i}\int_{\Gamma_2^+}\frac{\chi'(\lambda)}{\chi(\lambda)}\mathrm{d}\lambda = \frac{1}{2\pi i}\int_{\Gamma_2^+}\frac{\mathrm{d}\lambda}{\lambda - i\omega_c} + \frac{1}{2\pi i}\int_{\Gamma_2^+} g(\lambda)\,\mathrm{d}\lambda. \tag{102}$$

A simple calculation shows that the first integral on the right is $-1/2$ and the second tends to zero as $\varepsilon \to 0$. Therefore

$$\frac{1}{2\pi}\Delta_{\Gamma_2^+}\arg\chi(\lambda) = -\frac{1}{2}. \tag{103}$$

A similar argument applies to Γ_5^+, so that

$$\frac{1}{2\pi}\Delta_{\Gamma_5^+}\arg\chi(\lambda) = -\frac{1}{2}. \tag{104}$$

Finally, on Γ_7, $\lambda = A\exp(i\theta)$, from $\theta = -\pi/2$ to $\pi/2$, and A large, so that

$$\chi(\lambda) \approx A^2(\cos 2\theta + i\sin 2\theta) \tag{105}$$

Then

$$\frac{1}{2\pi}\Delta_{\Gamma_7^+}\arg\chi(\lambda) = \frac{1}{2\pi}\tan^{-1}\left(\frac{\sin 2\theta}{\cos 2\theta}\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}(2\theta)\Bigg|_{-\pi/2}^{\pi/2} = 1. \tag{106}$$

Therefore, adding Equations (99), (100), (103), (104), and (106) yields

$$\frac{1}{2\pi}\Delta_{\Gamma}\arg\chi(\lambda) = 0. \tag{107}$$

This implies that all roots, except $\lambda = \pm i\omega_c$ have negative real parts.

Thus all conditions for the Hopf bifurcation are satisfied for the current problem.

4. Proof of Hopf Bifurcation Theorem

The previous section assures us that the conditions of Theorem 2.1 are satisfied for the class of delay differential equation (3). The results of Theorem 2.1 for this class of equations will be established in this section. It will be done by first converting the delay differential equation into an operator equation, since this becomes a natural formulation of delay differential equations as well as partial differential equations of evolutionary type. Then a geometry will be introduced on the underlying function space by way of a bilinear form that acts like an inner product. The bilinear form is then used to diagonalize the operator equation into two parts, a system of two equations related to the two eigen triples and a third whose eigenvalues have negative real parts. The sytem is then decoupled by projecting the two equations associated with the critical eigen triples onto the center manifold. The projected equations are reduced to normal form. The normal form is shown to have periodic solutions on the center manifold which are transformed to periodic solutions for Equation (3).

4.1. Converting Delay Differential Equations (DDEs) into Operator Equations

The form of Equation (3) for the DDE emphasizes the trajectory nature of the solution in the real space R^2 as a function of the dimensionless time s. However, in order to make use of the results from Ordinary Differential Equations (ODEs) as models for results in DDEs, it is necessary to introduce a new form of equation for (3).

In ODEs the solution of

$$Z' = AZ, \tag{108}$$

$Z \in R^n$, A and $n \times n$ matrix, can be represented as a parametric operator of the form

$$T(s)\phi = Z_s(\phi) = \mathrm{e}^{As}\phi, \tag{109}$$

acting on a vector $\phi \in R^n$. The operator Equation (109) is a prototype of a *Strongly Continuous Semigroup*.

DEFINITION 4.1. A *Strongly Continuous Semigroup* satisfies

$$T(s) \text{ is bounded and linear for } s \geq 0,$$

$$T(0)\phi = \phi \text{ or } T(0) = I,$$

$$\lim_{s \to s_0} \|T(s)\phi - T(s_0)\phi\| \to 0. \tag{110}$$

$\|\cdot\|$ is an appropriate norm, and for any $\phi \in R^n$, the matrix A, in Equation (108), is a prototype of an operator called an *Infinitesimal Generator* [39].

DEFINITION 4.2. An *Infinitesimal Generator* of the semigroup Equation (110) is defined by

$$A\phi = \lim_{s \to 0^+} \frac{1}{s}\left[T(s)\phi - \phi\right]. \tag{111}$$

In general ϕ is not limited to R^n. These properties relate to systems called evolutionary of which the DDE (3) is an example.

At this point consider the linear portion of Equation (3) given by

$$Z'(s) = L(\mu)Z(s) + R(\mu)Z(s - \sigma). \tag{112}$$

The plan is to transform Equation (112) into an equation analogous to Equation (108). In contrast to the fact that the solution of Equation (108) depends on a single vector in R^n, the solution of Equation (112) depends on an entire set of values $s \in [-\sigma, 0]$, which implies that

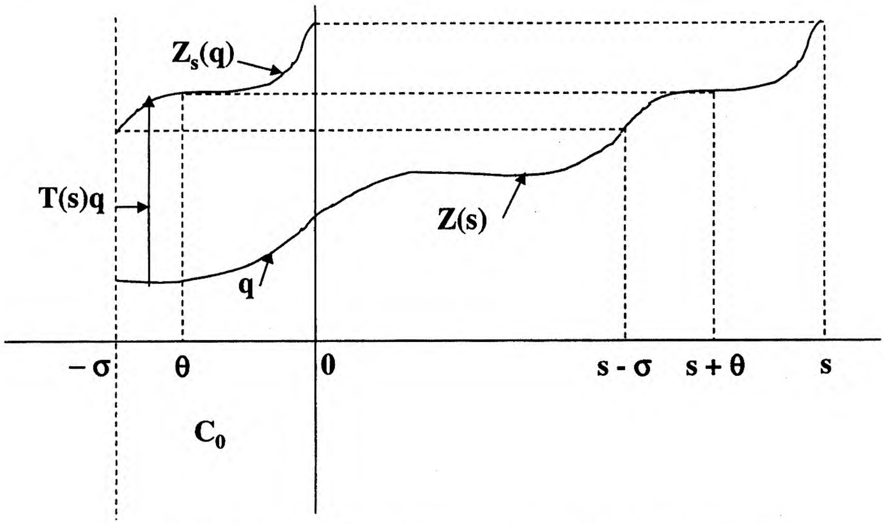

Figure 2. The solution operator maps functions in C_0 to functions in C_0.

the initial space is a function space. Let C^2 be the two-dimensional space of complex numbers and define the infinite-dimensional space $C_0 = C([-\sigma, 0], C^2)$ of continuous functions from $[-\sigma, 0]$ to C^2. Take the norm of C_0 as $\|q\| = \max_{-\sigma \le \theta \le 0} |q(\theta)|$. With this norm C_0 forms a function space, called a *Banach space*. Driver [38], Hale [42] and Hale and Lunel [24] have shown the existence and uniqueness of solutions of Equations (3) and (112) over this space.

Define a family of solution operators of (112) parameterized by s as

$$(T(s)q)(\theta) = (Z_s(q))(\theta) = Z(s+\theta; q) \tag{113}$$

for $\theta \in [-\sigma, 0]$. This is a mapping of a function in C_0 to another function in C_0. Figure 2 shows how this mapping relates to the trajectory representation. It is this mapping in C_0 that allows us to develop analogous results to those in ODEs.

Some of the lemmas below relating to semigroups require proofs from operator theory and will not be given, but the reader is urged to consult the references for the details.

LEMMA 4.1. *The mapping in Equation* (113) *satisfies the semigroup properties of Equation* (110).

Proof. See Yosida [41]. The original observation that Equation (113) satisfies Equation (110) was made by Krasovskii [40]. □

LEMMA 4.2. *The infinitesimal generator, A, defined by Equation* (111) *for Equation* (113), *is given by*

$$(A(\mu)q)\,\theta = \begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}\theta}(\theta), & -\sigma \le \theta < 0, \\ L(\mu)q(0) + R(\mu)q(-\sigma), & \theta = 0, \end{cases} \tag{114}$$

where the parameter μ is included in the definition of A.

Proof. Define an operator on C_0 by

$$\mathcal{L}(q; \mu) = L(\mu)q(0) + R(\mu)q(-\sigma). \tag{115}$$

For $s > 0$ such that $-\sigma \leq s + \theta < 0, \theta \in [-\sigma, 0]$,

$$(T(s)q)(\theta) = (Z_s(q))(\theta) = Z(s+\theta; q) = q(s+\theta). \tag{116}$$

For any $s + \theta > 0$ integrate Equation (112) so that

$$\begin{aligned}(T(s)q)(\theta) - q(0) &= Z(s+\theta; q) - Z(0; q) \\ &= \int_0^{s+\theta} \mathcal{L}(Z_t(q); \mu)\, \mathrm{d}t \\ &= \int_0^{s+\theta} \mathcal{L}(T(t)q; \mu)\, \mathrm{d}t \end{aligned} \tag{117}$$

since $(Z_s(q))(0) = Z(s)$ and $(Z_s(q))(-\sigma) = Z(s-\sigma)$.

Now, for $-\sigma \leq \theta < 0$, take $s > 0$ such that $-\sigma \leq s + \theta < 0$, then, from Equations (111) and (116)

$$\begin{aligned}(A(\mu)q)(\theta) &= \lim_{s\to 0^+} \frac{1}{s}[(T(s)q)(\theta) - q(\theta)] \\ &= \lim_{s\to 0^+} \frac{1}{s}[q(s+\theta) - q(\theta)] \\ &= \frac{\mathrm{d}q}{\mathrm{d}\theta}(\theta). \end{aligned} \tag{118}$$

For the case $\theta = 0$ use Equation (117) and the mean value theorem of integration to show

$$\begin{aligned}(A(\mu)q)(0) &= \lim_{s\to 0^+} \frac{1}{s}[(T(s)q)(0) - q(0)] \\ &= \lim_{s\to 0^+} \frac{1}{s}\int_0^{s} \mathcal{L}(T(t)q; \mu)\, \mathrm{d}t \\ &= \mathcal{L}(q; \mu) = L(\mu)q(0) + R(\mu)q(-\sigma), \end{aligned} \tag{119}$$

which proves the lemma. □

LEMMA 4.3. *$T(s)q$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}s}T(s)q = AT(s)q, \tag{120}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}s}T(s)q = \lim_{h\to 0} \frac{1}{h}\,(T(s+h) - T(s))\,q. \tag{121}$$

Proof. The proof is given in [41]. □

LEMMA 4.4. *The eigenvalues for* (114) *are given by the* λ *solutions of*

$$\det\left(\lambda I - L(\mu) - \mathrm{e}^{-\lambda\sigma} R(\mu)\right) = 0. \tag{122}$$

Proof. For the proof, see [42]. □

In the case of the linear DDE (112) Equation (122) reduces to

$$\lambda^2 + 2\xi\lambda + (1 + \mu + p_c) - (\mu + p_c)\,\mathrm{e}^{-\lambda\sigma} = 0. \tag{123}$$

Reformulate the nonlinear DDE (3) in operator form by setting, for $q \in C_0$,

$$f(q;\mu) = \begin{pmatrix} 0 \\ (\mu + p_c)\left(E(q(s-\sigma) - q(s))^2 + E(q(s-\sigma) - q(s))^3\right) \end{pmatrix} \tag{124}$$

and then defining a nonlinear operator on C_0 by

$$(\mathrm{F}(q;\mu))(\theta) = \begin{cases} 0, & -\sigma \le \theta < 0, \\ f(q;\mu), & \theta = 0. \end{cases} \tag{125}$$

For $\mu = 0$ write $f(q) = f(q;0)$, $F(q) = F(q;0)$.

LEMMA 4.5. *The operator form for the DDE (3) is given by*

$$\frac{\mathrm{d}}{\mathrm{d}s} Z_s(q) = A(\mu) Z_s(q) + F(Z_s(q);\mu). \tag{126}$$

Proof. This is clear from

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(Z_s(q))(\theta) \;&=\; \begin{cases} \left(\dfrac{\mathrm{d}}{\mathrm{d}s}(Z_s(q))\right)(\theta), & -\sigma \le \theta < 0, \\ L(\mu)Z(s) + RZ(s-\sigma), & \theta = 0, \end{cases} \\ &\quad + \begin{cases} 0, & -\sigma \le \theta < 0, \\ f(Z_s(q);\mu), & \theta = 0, \end{cases} \end{aligned} \tag{127}$$

where

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}(Z_s(q))\right)(\theta) = \frac{\mathrm{d}}{\mathrm{d}s} Z(s+\theta;q) = \frac{\mathrm{d}}{\mathrm{d}\theta} Z(s+\theta;q) = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}(Z_s(q))\right)(\theta). \tag{128}$$

□

4.2. Introducing Geometry by Way of a Formal Adjoint

In ODEs the formal adjoint equation to Equation (108) is

$$Y' = -A^*Y, \tag{129}$$

where $Y \in R^n$ and $A^* = \overline{A}^T$. Equations (108) and (129) are related by the Lagrange identity

$$\overline{Y}^T \Theta Z + \overline{\Theta^* Y}^T Z = \frac{\mathrm{d}}{\mathrm{d}s}\left(\overline{Y}^T Z\right), \tag{130}$$

where $\Theta Z = Z' - AZ$, $\Theta^* Y = Y' + A^* Y$. If Z and Y are solutions of Equations (108) and (129), respectively, then it is easy to show that $(\mathrm{d}/\mathrm{d}s)(\overline{Y}^T Z) = 0$ which implies $\overline{Y}^T Z$ is constant and is the natural inner product of R^n. This property is used to show the well known direct decompositions of R^n

$$
\begin{aligned}
R^n &= \mathcal{R}(A - \lambda I) \oplus \mathcal{N}(\mathcal{A}^* - \lambda \mathcal{I}),\\
R^n &= \mathcal{R}(A^* - \lambda I) \oplus \mathcal{N}(\mathcal{A} - \lambda \mathcal{I}),
\end{aligned}
\tag{131}
$$

where $\mathcal{R}$ represents the range space and $\mathcal{N}$ the null space.

In contrast to R^n, the space C_0 does not have a natural inner product associated with its norm, which can be one difference between a Banach Space and a Hilbert space, although a Hilbert space is a Banach space. The reverse is not true in general. However, following Hale [43], one can introduce a substitute device that acts like an inner product in C_0 and produces a decomposition of C_0 similar to Equation (131).

We make an observation here about notation. The superscript asterisk used here is intended to reference entities, such as operators, eigenvalues, and eigenvectors, associated with adjoints. It is not intended to refer strictly to a conjugate transpose of that entity and follows the notation of Kazarinoff et al. [30].

LEMMA 4.6 (Lagrange Identity). *If*

$$
\begin{aligned}
&\Theta Z(s) = Z'(s) - L(\mu)Z(s) - R(\mu)Z(s - \sigma),\\
&\Theta^* U(s) = U'(s) + L(\mu)^T U(s) + R(\mu)^T U(s + \sigma),
\end{aligned}
\tag{132}
$$

then

$$
\overline{U}^T(s)\Theta Z(s) + \overline{\Theta^* U}^T(s) Z(s) = \frac{\mathrm{d}}{\mathrm{d}s}\langle U, Z\rangle(s), \tag{133}
$$

where

$$
\langle U, Z\rangle(s) = \overline{U}^T(s)Z(s) + \int_{s-\sigma}^{s} \overline{U}^T(t + \sigma)R(\mu)Z(t)\,\mathrm{d}t. \tag{134}
$$

Proof. Integrate by parts the left side of Equation (133). □

This lemma is stated in [43] for more general functional differential equations. Equation (134) is also given in [44]. It seems clear that deriving the natural inner product for R^n from the Lagrange identity (130) motivates the derivation of Equation (134). Again, if Z and U satisfy $\Theta Z(s) = 0$ and $\Theta^* U(s) = 0$ then, from Equation (134), $(\mathrm{d}/\mathrm{d}s)\langle U, Z\rangle(s) = 0$, which implies $\langle U, Z\rangle(s)$ is constant and one can set $s = 0$ in Equation (134) and define the form

$$
\langle U, Z\rangle = \overline{U}^T(0)Z(0) + \int_{-\sigma}^{0} \overline{U}^T(t + \sigma)R(\mu)Z(t)\,\mathrm{d}t. \tag{135}
$$

LEMMA 4.7. *$\langle U, Z\rangle$ is a bilinear form that satisfies*

$$\begin{aligned}
&\langle U, \alpha Z_1 + \beta Z_2\rangle = \alpha\langle U, Z_1\rangle + \beta\langle U, Z_2\rangle,\\
&\langle \alpha U_1 + \beta U_2, Z\rangle = \overline{\alpha}\langle U_1, Z\rangle + \overline{\beta}\langle U_2, Z\rangle,\\
&\langle UM, Z\rangle = \overline{M}^T\langle U, Z\rangle,\\
&\langle U, ZM\rangle = \langle U, Z\rangle M,
\end{aligned} \tag{136}$$

where α,β are complex constants and M a matrix.

Proof. Straightforward using Equation (135). □

One can now construct formal adjoint operators associated with Equations (113) and (114). Let $C_0^* = C([0,\sigma], C^2)$ be the space of continuous functions from $[0,\sigma]$ to C^2 with $\|q^*\| = \max_{0\le\theta\le\sigma}|q^*(\theta)|$ for $q^* \in C_0^*$.

LEMMA 4.8. *Define*

$$\left(T^*(s)q^*\right)(\theta) = \left(U_s(q^*)\right)(\theta) = U(s+\theta; q^*) \tag{137}$$

for $\theta \in [0,\sigma]$, $s \le 0$, then Equation (137) *defines a strongly continuous semigroup with infinitesimal generator*

$$\left(A^*(\mu)q^*\right)\theta = \begin{cases} -\dfrac{dq^*}{d\theta}(\theta), & 0 < \theta \le \sigma,\\[2ex] -\dfrac{dq^*}{d\theta}(0) = L(\mu)^T q^*(0) + R(\mu)^T q^*(\sigma), & \theta = 0.\end{cases} \tag{138}$$

Proof. See Hale [43]. Note that, although the formal infinitesimal generator for Equation (137) is defined as

$$A_0^* q^* = \lim_{s\to 0^-} \frac{1}{s}\left[T^*(s)q^* - q^*\right], \tag{139}$$

Hale [43], for convenience, takes $A^* = -A_0^*$ in Equation (138) as the formal adjoint to Equation (114). □

LEMMA 4.9. *The family of operators* (137) *satisfies*

$$\frac{d}{ds}T^*(s)q^* = -A^*T^*(s)q^*. \tag{140}$$

Proof. The proof is given in [41]. □

One can now state some properties of Equations (114), (138), and (135) that will be useful.

LEMMA 4.10. *For $q \in C_0$, $q^* \in C_0^*$,*

1. *We have*

$$\langle q^*, A(\mu)q\rangle = \langle A^*(\mu)q^*, q\rangle. \tag{141}$$

2. *If q^* is an eigenvector of $A^*(\mu)$ associated with λ^*, i.e. $A^*(\mu)q^* = \lambda^* q^*$, and q an eigenvector of $A(\mu)$ associated with the eigenvalue λ and $\lambda \neq \lambda^*$, then $\langle q^*, q\rangle = 0$.*
3. *If $\beta \in \mathcal{R}(A(\mu) - \lambda I)$ and $q^* \in \mathcal{N}(\mathcal{A}^*(\mu) - \lambda \mathcal{I})$ then $\langle q^*, \beta\rangle = 0$.*
4. *If $q \in \mathcal{N}(A(\mu) - \lambda I)$, $\gamma \in \mathcal{R}(\mathcal{A}^*(\mu) - \lambda \mathcal{I})$ then $\langle \gamma, q\rangle = 0$.*
5. *λ is an eigenvalue of $A(\mu)$ if and only if λ is an eigenvalue of $A^*(\mu)$.*
6. *The dimensions of the eigenspaces of $A(\mu)$ and $A^*(\mu)$ are finite and equal.*
7. *If $q_1^*, \ldots, q_d^*$ is a basis for the eigenspace of $A^*(\mu)$ and $q_1, \ldots, q_d$ is a basis for the eigenspace of $A(\mu)$, construct the matrices $Q^* = (q_1^*, \ldots, q_d^*)$ and $Q = (q_1, \ldots, q_d)$. Define the bilinear form between Q^* and Q by*

$$\langle Q^*, Q\rangle = \begin{pmatrix} \langle q_1^*, q_1\rangle & \ldots & \langle q_1^*, q_d\rangle \\ \vdots & \ddots & \vdots \\ \langle q_d^*, q_1\rangle & \ldots & \langle q_d^*, q_d\rangle \end{pmatrix}. \tag{142}$$

This matrix is non-singular and can be chosen so that $\langle Q^*, Q\rangle = I$.

Proof. Integration by parts proves Equation (141). The proofs of the other properties can be found in [43]. Note that if (142) is not the identity then a change of coordinates can be performed by setting $K = \langle Q^*, Q\rangle^{-1}$ and $Q' = QK$. Then $\langle Q^*, Q'\rangle = \langle Q^*, QK\rangle = \langle Q^*, Q\rangle K = I$. □

Properties (3) and (4) above indicate the orthogonality of the range and null spaces relative to the bilinear form (135).

4.3. Eigenvectors and Eigenvalues of the DDE

In order to simplify calculations one can set $\mu = 0$, since this value will be the only one needed to compute μ_2, τ_2, β_2 in Theorem 2.1, although the results can be shown to be satisfied for $\mu \neq 0$ [11]. Therefore, for the sake of notation, let $A = A(0)$, $A^* = A^*(0)$, $L = L(0)$, $R = R(0)$, $\omega = \omega(0)$.

The basis eigenvectors for A and A^* associated with the eigenvectors $\lambda = i\omega$, $\overline{\lambda} = -i\omega$ will be computed in this section.

LEMMA 4.11. *The two eigenvectors for A, associated with the eigenvalues $\lambda = i\omega$, $\overline{\lambda} = -i\omega$, are given by*

$$\begin{aligned} q(\theta) &= \mathrm{e}^{i\omega\theta}\begin{pmatrix} 1 \\ i\omega \end{pmatrix}, \\ \overline{q}(\theta) &= \mathrm{e}^{-i\omega\theta}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}. \end{aligned} \tag{143}$$

Proof. From Equation (114), if $-\sigma \le \theta < 0$ then $\mathrm{d}q/\mathrm{d}\theta = i\omega q$ implies $q(\theta) = \exp(i\omega\theta)C$ where $C = (c_1, c_2)^T$. For $\theta = 0$, Equation (114) implies $(L + R\exp(-i\omega\sigma))\, C = i\omega C$ or $(L - i\omega I + R\exp(-i\omega\sigma))\, C = 0$. Since $i\omega$ is an eigenvalue, Equations (122) and (123) imply that there is a non-zero solution C. Setting $c_1 = 1$ it is easy to compute $c_2 = i\omega$. □

Define the matrix

$$Q = (q, \overline{q}) \tag{144}$$

for $q, \overline{q}$ from (143). The next lemma shows how to construct the eigenvectors $q^*, \overline{q}^*$ for A^* so that

$$Q^* = (q^*, \overline{q}^*) \tag{145}$$

satisfies $\langle Q^*, Q\rangle = I$.

LEMMA 4.12. *The eigenvectors for A^* associated with the eigenvalues $-i\omega, i\omega$ are given by*

$$\begin{aligned} q^*(\theta) &= \mathrm{e}^{i\omega\theta} D, \\ \overline{q}^*(\theta) &= \mathrm{e}^{-i\omega\theta}\overline{D}, \end{aligned} \tag{146}$$

where $D = (d_1, d_2)^T$ and

$$\begin{aligned} d_1 &= -\left(\frac{p_c}{\omega}\sin\omega\sigma + i\omega\right) d_2, \\ d_2 &= \frac{(\sigma p_c\omega^2\cos\omega\sigma - p_c\omega\sin\omega\sigma) + i(2\omega^3 - \sigma p_c\omega^2\sin\omega\sigma)}{(\sigma p_c\omega\cos\omega\sigma - p_c\sin\omega\sigma)^2 + (2\omega^2 - \sigma p_c\omega\sin\omega\sigma)^2}. \end{aligned} \tag{147}$$

Proof. From Equation (138), for $0 < \theta \le \sigma$,

$$-\frac{\mathrm{d}q^*}{\mathrm{d}\theta} = -i\omega q^* \tag{148}$$

and compute

$$q^*(\theta) = \exp(i\omega\theta)D, \tag{149}$$

where $D = (d_1, d_2)^T$. At $\theta = 0$ one has from Equation (138) that $\left(L^T + R^T\,\mathrm{e}^{i\omega\sigma} + i\omega I\right) D = 0$. The determinant of the matrix on the left is the characteristic equation so that there is a non-zero D. From Equations (143), (144), (145), and (146) one seeks to solve for D so that

$$\langle Q^*, Q\rangle = \begin{pmatrix} \langle q^*, q\rangle & \langle q^*, \overline{q}\rangle \\ \langle \overline{q}^*, q\rangle & \langle \overline{q}^*, \overline{q}\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{150}$$

One needs only to satisfy $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$. From Equations (135), (143), and (146) compute d_1 and d_2 to satisfy

$$\begin{aligned} 1 &= \overline{d}_1 + [\sigma p_c\cos\omega\sigma + i(\omega - \sigma p_c\sin\omega\sigma)]\overline{d}_2, \\ 0 &= \overline{d}_1 + \left[\frac{p_c}{\omega}\sin\omega\sigma - i\omega\right]\overline{d}_2 \end{aligned} \tag{151}$$

from which the result follows. □

LEMMA 4.13. *The matrices B, B^* given by*

$$\begin{aligned} B &= \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}, \\ B^* &= \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix} \end{aligned} \tag{152}$$

satisfy $AQ = QB$, $A^*Q^* = Q^*B^*$ where Q, Q^* are given by (144) *and* (145).

Proof. The proof is based on the calculation $A^*Q^* = A^*(q^*, \overline{q}^*) = (A^*q^*, A^*\overline{q}^*) = (-i\omega q^*, i\omega\overline{q}^*)$. Similarly for AQ. Note that $B = \overline{B}^{*T}$. □

This result says that on eigenspaces the infinitesimal generators can be represented by matrices.

4.4. Diagonalizing the Nonlinear DDE

The nonlinear operator equation (126) will be decomposed in this section into a two-dimensional system with eigenvalues $i\omega$ and $-i\omega$ and another operator equation with eigenvalues having negative real parts. The procedure is based on the work of Hale [43] and depends on the following result.

LEMMA 4.14. *Let $\phi \in C_0$ and $Q,\ Q^*$ given by Equations* (144) *and* (145) *so that* $\langle Q^*,\ Q\rangle = I$. *Then there is a vector $b \in C^2$ and a function $\overline{\phi} \in C_0$ such that*

$$\phi = Qb + \overline{\phi} \tag{153}$$

and $\langle Q^*, \overline{\phi}\rangle = 0$.

Proof. Let $b = \langle Q^*, \phi\rangle$ and define $\overline{\phi} = \phi - Qb$. Then $\langle Q^*, \phi\rangle = \langle Q^*, \phi - Qb\rangle = \langle Q^*, \phi\rangle - \langle Q^*, Q\rangle b = 0$. □

One can apply this representation to decompose the nonlinear system (126) with $\mu = 0$. Let $Z_s \in C_0$ be the unique family of solutions of Equation (126), where the initial function notation has been dropped for simplicity. Define

$$Y(s) = \langle Q^*,\ Z_s\rangle = \begin{pmatrix} \langle q^*,\ Z_s\rangle \\ \langle \overline{q}^*,\ Z_s\rangle \end{pmatrix}, \tag{154}$$

where $Y(s) \in C^2$ for $s \geq 0$, and set $Y(s) = (y_1(s),\ y_2(s))^T$.

LEMMA 4.15. *If one defines*

$$W_s = Z_s - QY(s), \tag{155}$$

where Q is given by Equation (144) *and $Y(s)$ is given by Equation* (154), *then* $\langle Q^*,\ W_s\rangle = 0$, *where Q^* is given by Equation* (145).

Proof. For the proof, see Lemma 4.14. □

LEMMA 4.16.

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle Q^*,\ Z_s\rangle = \left\langle Q^*,\ \frac{\mathrm{d}Z_s}{\mathrm{d}s}\right\rangle. \tag{156}$$

Proof. Differentiate Equation (135) and use $Z_s(0) = Z(s),\ Z_s(t) = Z(s+t)$. □

One can now decompose Equation (126) as

LEMMA 4.17.

$$\frac{\mathrm{d}}{\mathrm{d}s}y_1(s) = i\omega y_1(s) + \overline{q}^{*T}(0)f(Z_s),$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\overline{y}_1(s) = -i\omega\overline{y}_1(s) + q^{*T}(0)f(Z_s),$$

$$\frac{\mathrm{d}}{\mathrm{d}s}W_s(\theta) = \begin{cases} (AW_s)(\theta) - 2\mathrm{Re}\{\overline{q}^{*T}(0)f(Z_s)q(\theta)\}, & -\sigma \le \theta < 0, \\ (AW_s)(0) - 2\mathrm{Re}\{\overline{q}^{*T}(0)f(Z_s)q(0)\} + f(Z_s), & \theta = 0. \end{cases} \tag{157}$$

Proof. Use Equations (126), (136), Lemma 4.3 and Equation (156) to write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}Y(s) &= \left\langle Q^*,\ \frac{\mathrm{d}Z_s}{\mathrm{d}s}\right\rangle \\ &= \langle Q^*,\ AZ_s\rangle + \langle Q^*,\ F(Z_s)\rangle \\ &= \langle A^*Q^*,\ Z_s\rangle + \langle Q^*,\ F(Z_s)\rangle \\ &= BY(s) + \langle Q^*,\ F(Z_s)\rangle. \end{aligned} \tag{158}$$

Using Equations (125) and (135) compute $\langle q^*,\ F(Z_s)\rangle = \overline{q}^{*T}(0)(F(Z_s))(0) = \overline{q}^{*T}(0)f(Z_s)$. Similarly $\langle \overline{q}^*,\ F(Z_s)\rangle = q^{*T}(0)f(Z_s)$. Then

$$\langle Q^*,\ F(Z_s)\rangle = \begin{pmatrix} \langle q^*,\ F(Z_s)\rangle \\ \langle \overline{q}^*,\ F(Z_s)\rangle \end{pmatrix} = \begin{pmatrix} \overline{q}^{*T}(0)f(Z_s) \\ q^{*T}(0)f(Z_s) \end{pmatrix} \tag{159}$$

which yields the first two equations in Equation (157) if $y_2(s) = \overline{y}_1(s)$. But this is clear from $y_2(s) = \langle \overline{q}^*,\ Z_s\rangle = \overline{\langle q^*,\ Z_s\rangle} = \overline{y}_1(s)$ since Z_s is real by definition. One can now construct the third equation. From (155), with $\theta = 0$,

$$W(s) = W_s(0) = Z_s(0) - Q(0)T(s) = Z(s) - Q(0)Y(s). \tag{160}$$

From Equations (114) and (126)

$$\frac{\mathrm{d}}{\mathrm{d}s}Z(s) = \frac{\mathrm{d}Z_s}{\mathrm{d}s}(0) = (AZ_s)(0) + (F(Z_s))(0) = LZ(s) + RZ(s-\sigma) + f(Z_s). \tag{161}$$

Differentiate Equation (160) and combine it with Equations (158) and (161) to give

$$\frac{\mathrm{d}}{\mathrm{d}s}W(s) = LZ(s) + RZ(s-\sigma) + f(Z_s) - Q(0)BY(s) - Q(0)\langle Q^*, F(Z_s)\rangle. \tag{162}$$

Now apply the infinitesimal generator (114) to

$$Z_s = W_s + QY(s) \tag{163}$$

to get

$$(AZ_s)(\theta) = (AW_s)(\theta) + (AQ)(\theta)Y(s) = (AW_s)(\theta) + QBY(s). \tag{164}$$

If $\theta = 0$ in Equation (164) then

$$LZ(s) + RZ(s-\sigma) = LW(s) + RW(s-\sigma) + Q(0)BY(s). \tag{165}$$

Substitute (165) into (162) to get

$$\frac{\mathrm{d}}{\mathrm{d}s}W(s) = LW(s) + RW(s-\sigma) + f(Z_s) - Q(0)\langle Q^*, F(Z_s)\rangle. \tag{166}$$

From (163)

$$Z_s = qy_1(s) + \overline{q}y_2(s) + W_s = qy_1(s) + \overline{qy}_1(s) + W_s. \tag{167}$$

One can now determine the operator equation for W_s by starting with Equation (167) in the form

$$W_s = Z_s - qy_1(s) - \overline{qy}_1(s). \tag{168}$$

Use the fact that $i\omega,\ -i\omega$ are eigenvalues of A associated with the eigenvectors $q,\ \overline{q}$ to write

$$\begin{aligned}
\frac{\mathrm{d}W_s}{\mathrm{d}s} &= \frac{\mathrm{d}Z_s}{\mathrm{d}s} - q\frac{\mathrm{d}y_1}{\mathrm{d}s}(s) - \overline{q}\frac{\mathrm{d}\overline{y}_1}{\mathrm{d}s}(s)\\
&= AZ_s + F(Z_s) - q\frac{\mathrm{d}y_1}{\mathrm{d}s}(s) - \overline{q}\frac{\mathrm{d}\overline{y}_1}{\mathrm{d}s}(s)\\
&= A\{W_s + qy_1(s) + \overline{qy}_1(s)\} + F(Z_s)\\
&\quad - q\{i\omega y_1(s) + \overline{q}^{*T}(0)f(Z_s)\} - \overline{q}\{-i\omega\overline{y}_1(s) + q^{*T}(0)f(Z_s)\}\\
&= AW_s - 2Re\{\overline{q}^{*T}(0)f(Z_s)q\} + F(Z_s).
\end{aligned} \tag{169}$$

Use Equation (125) to complete the lemma. □

In order to simplify the notation write Equation (157) in the form

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}s}(s) &= i\omega y_1(s) + F_1(Y,\ W_s),\\
\frac{\mathrm{d}\overline{y}_1}{\mathrm{d}s}(s) &= -i\omega\overline{y}_1(s) + \overline{F}_1(Y,\ W_s),\\
\frac{\mathrm{d}W_s}{\mathrm{d}s} &= AW_s + F_2(Y,\ W_s),
\end{aligned} \tag{170}$$

where

$$\begin{aligned}
F_1(Y,\ W_s) &= \overline{q}^{*T}(0)f(Z_s),\\
F_2(Y,\ W_s) &= \begin{cases} -2\mathrm{Re}\{\overline{q}^{*T}(0)f(Z_s)q(\theta)\}, & -\sigma \le \theta < 0,\\ -2\mathrm{Re}\{\overline{q}^{*T}(0)f(Z_s)q(0)\} + f(Z_s), & \theta = 0.\end{cases}
\end{aligned} \tag{171}$$

Note that Equation (170) is a coupled system. In the next section the center manifold will be used as a tool to partially decouple it.

4.5. Reduction to Normal Form on the Center Manifold

In Section 4.4 the operator equation (126) was decomposed into system (170). In this section the argument of Kazarinoff et al. [30] will be used to look for a center manifold [46] $w(y, \overline{y})$ that approximately solves

$$\mathcal{D}_y w(y, \overline{y})\{i\omega y + F_1(y, w(y, \overline{y}))\} + \mathcal{D}_{\overline{y}} w(y, \overline{y})\{-i\omega\overline{y} + \overline{F}_1(y, w(y, \overline{y}))\}$$
$$= Aw(y, \overline{y}) + F_2(y, w(y, \overline{y})), \tag{172}$$

where $\mathcal{D}$ represents the derivative with respect to the subscripted variable. The subscript 1 of y has been dropped for simplicity. The projected equation on the center manifold will be shown to satisfy

$$\begin{aligned}\frac{dy}{ds} &= i\omega y + F_1(y, w(y, \overline{y})),\\ \frac{d\overline{y}}{ds} &= -i\omega\overline{y} + \overline{F}_1(y, w(y, \overline{y})).\end{aligned} \tag{173}$$

Then Equation (173) will be reduced to a normal form by a transformation of variables, $y \to v$, so that the new system will take the form

$$\begin{aligned}\frac{dv}{ds} &= i\omega v + c_{21}v^2\overline{v},\\ \frac{d\overline{v}}{ds} &= -i\omega\overline{v} + c_{21}\overline{v}^2 v,\end{aligned} \tag{174}$$

where the other higher-order terms have been dropped.

4.5.1. *Center Manifold Projection*

The main result of this section is the following lemma:

LEMMA 4.18. *Let an approximate center manifold, satisfying Equation* (172), *be given as a quadratic form in y and $\overline{y}$ with coefficients as functions of θ*

$$w(y, \overline{y})(\theta) = w_{20}(\theta)\frac{y^2}{2} + w_{11}(\theta)y\overline{y} + w_{02}(\theta)\frac{\overline{y}^2}{2}. \tag{175}$$

Then the projected equation (173) *on the center manifold takes the form*

$$\begin{aligned}\frac{dy}{ds} &= i\omega y + g(y, \overline{y}),\\ \frac{d\overline{y}}{ds} &= -i\omega\overline{y} + \overline{g}(y, \overline{y}),\end{aligned} \tag{176}$$

where the $g(y, \overline{y})$ is given by

$$g(y, \overline{y}) = g_{20}\frac{y^2}{2} + g_{11}y\overline{y} + g_{02}\frac{\overline{y}^2}{2} + g_{21}\frac{y^2\overline{y}}{2}. \tag{177}$$

The coefficients g_{ij} are given by

$$\begin{aligned} g_{20} &= 2E\gamma^2\overline{d}_2 p_c, \\ g_{11} &= 2E\gamma\overline{\gamma}\overline{d}_2 p_c, \\ g_{02} &= 2E\overline{\gamma}^2\overline{d}_2 p_c, \\ g_{21} &= 2p_c\left\{E\left[-\frac{g_{20}\gamma}{i\omega} - \frac{\overline{g}_{02}}{3i\omega} + \frac{g_{20}(\mathrm{e}^{-2i\omega\sigma}-1)}{\overline{d}_2\Delta}\right]\overline{\gamma}\right. \\ &\quad \left. +2E\left[\frac{g_{11}\gamma}{i\omega} - \frac{\overline{g}_{11}\overline{\gamma}}{i\omega}\right]\gamma + 3E\gamma^2\overline{\gamma}\right\}\overline{d}_2, \end{aligned} \tag{178}$$

where

$$\begin{aligned} \gamma &= \mathrm{e}^{-i\omega\sigma} - 1, \\ \Delta &= -4\omega^2 + 4i\xi\omega + p_c\left(1 - \mathrm{e}^{2i\omega\sigma}\right). \end{aligned} \tag{179}$$

Proof. If one assumes a center manifold of the form (175) then, from Equations (157), (167), (170), and (173), on the center manifold

$$\frac{\mathrm{d}y}{\mathrm{d}s} = i\omega y + \overline{q}^{*T}(0) f(w(y,\ \overline{y}) + qy + \overline{qy}). \tag{180}$$

Define

$$g(y,\ \overline{y}) = \overline{q}^{*T}(0) f(w(y,\ \overline{y}) + qy + \overline{qy}), \tag{181}$$

where $\overline{q}^{*T}(0) = \left(\overline{d}_1,\ \overline{d}_2\right)$ and

$$f(w(y,\ \overline{y}) + qy + \overline{qy}) = \begin{pmatrix} 0 \\ p_c\left(E\left[w(y,\ \overline{y})_1(-\sigma) + yq_1(-\sigma) + \overline{yq}_1(\sigma)\right.\right. \\ \left. -w(y,\ \overline{y})_1(0) - yq_1(0) - \overline{yq}_1(0)\right]^2 \\ +E\left[w(y,\ \overline{y})_1(-\sigma) + yq_1(-\sigma) + \overline{yq}_1(\sigma)\right. \\ \left.\left. -w(y,\ \overline{y})_1(0) - yq_1(0) - \overline{yq}_1(0)\right]^3\right) \end{pmatrix}. \tag{182}$$

From (175)

$$\begin{aligned} w(y,\ \overline{y})(0) &= w_{20}(0)\frac{y^2}{2} + w_{11}(0)y\overline{y} + w_{02}(0)\frac{\overline{y}^2}{2}, \\ w(y,\ \overline{y})(-\sigma) &= w_{20}(-\sigma)\frac{y^2}{2} + w_{11}(-\sigma)y\overline{y} + w_{02}(-\sigma)\frac{\overline{y}^2}{2}, \end{aligned} \tag{183}$$

where $w_{ij}(\theta) = (w_{ij}^1(\theta),\ w_{ij}^2(\theta))^T$.

Note here that in order to compute $\mu_2,\ \tau_2,\ \beta_2$ one need only determine $g(y,\ \overline{y})$ in the form (177). To find the coefficients (178) begin by expanding the nonlinear terms of Equation (182) up to cubic order, keeping only the cubic term $y^2\overline{y}$. To help simplify the notation let

$$\gamma = \mathrm{e}^{-i\omega\sigma} - 1. \tag{184}$$

Then, using Equations (143), (183), and (184),

$$
\begin{aligned}
&E[w(y,\ \overline{y})_1(-\sigma)+yq_1(-\sigma)+\overline{y}\overline{q}_1(\sigma)-w(y,\ \overline{y})_1(0)-yq_1(0)-\overline{y}\overline{q}_1(0)]^2\\
&\quad = Ey^2\gamma^2+2Ey\overline{y}\gamma\overline{\gamma}+E\overline{y}^2\overline{\gamma}^2+E\{[w^1_{20}(-\sigma)-w^1_{20}(0)]\overline{\gamma}\\
&\qquad +2[w^1_{11}(-\sigma)-w^1_{11}(0)]\gamma\}y^2\overline{y},\\
&E[w(y,\ \overline{y})_1(-\sigma)+yq_1(-\sigma)+\overline{y}\overline{q}_1(\sigma)-w(y,\ \overline{y})_1(0)-yq_1(0)-\overline{y}\overline{q}_1(0)]^3\\
&\quad = 3E\gamma^2\overline{\gamma}y^2\overline{y}.
\end{aligned}
\tag{185}
$$

From Equations (177) and (181) through (185) one can compute the coefficients for Equation (178), except at this point one only has

$$
g_{21}=2p_c\left\{E\left[w^1_{20}(-\sigma)-w^1_{20}(0)\right]\overline{\gamma}+2E\left[w^1_{11}(-\sigma)-w^1_{11}(0)\right]\gamma+3E\gamma^2\overline{\gamma}\right\}\overline{d}_2. \tag{186}
$$

In order to complete the computation of g_{21} one needs to compute the center manifold coefficients $w_{20},\ w_{11}$. To do this substitute g_{20}, g_{11}, g_{02} from Equations (178), (185), and (186) into (182) to get

$$
f(w(y,\ \overline{y})+qy+\overline{q}\overline{y})=\left\{\frac{g_{20}y^2}{2\overline{d}_2}+\frac{g_{11}y\overline{y}}{\overline{d}_2}+\frac{g_{02}\overline{y}^2}{2\overline{d}_2}+\frac{g_{21}y^2\overline{y}}{\overline{d}_2}\right\}\begin{pmatrix}0\\1\end{pmatrix}. \tag{187}
$$

From the definition of F_2 in Equations (171), (177), and (181) write F_2 as

$$
\begin{aligned}
F_2(y,\ \overline{y})(\theta) &= -\{g_{20}q(\theta)+\overline{g}_{02}\overline{q}(\theta)\}\frac{y^2}{2}\\
&\quad -\{g_{11}q(\theta)+\overline{g}_{11}\overline{q}(\theta)\}y\overline{y}\\
&\quad -\{g_{02}q(\theta)+\overline{g}_{20}\overline{q}(\theta)\}\frac{\overline{y}^2}{2}
\end{aligned}
\tag{188}
$$

for $-\sigma\le\theta<0$ and for $\theta=0$

$$
\begin{aligned}
F_2(y,\ \overline{y})(0) &= -\left\{g_{20}q(0)+\overline{g}_{02}\overline{q}(0)-\frac{g_{20}}{\overline{d}_2}\begin{pmatrix}0\\1\end{pmatrix}\right\}\frac{y^2}{2}\\
&\quad -\left\{g_{11}q(0)+\overline{g}_{11}\overline{q}(0)-\frac{g_{11}}{\overline{d}_2}\begin{pmatrix}0\\1\end{pmatrix}\right\}y\overline{y}\\
&\quad -\left\{g_{02}q(0)+\overline{g}_{20}\overline{q}(0)-\frac{g_{02}}{\overline{d}_2}\begin{pmatrix}0\\1\end{pmatrix}\right\}\frac{\overline{y}^2}{2}.
\end{aligned}
\tag{189}
$$

Note that to compute the coefficients of the center manifold one needs only to work to the second order.

Since $g_{02}/\overline{d}_2=\overline{g}_{20}/d_2$ write the coefficients of $F_2(y,\ \overline{y})$ as

$$
F^2_{20}(\theta) = \begin{cases} -\left(g_{20}q(\theta)+\overline{g}_{02}\overline{q}(\theta)\right), & -\sigma\le\theta<0,\\ -\left(g_{20}q(0)+\overline{g}_{02}\overline{q}(0)-\dfrac{g_{20}}{\overline{d}_2}\begin{pmatrix}0\\1\end{pmatrix}\right), & \theta=0,\end{cases}
$$

$$F_{11}^2(\theta) = \begin{cases} -\left(g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta)\right), & -\sigma \le \theta < 0, \\ -\left(g_{11}q(0) + \overline{g}_{11}\overline{q}(0) - \dfrac{g_{11}}{\overline{d}_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), & \theta = 0, \end{cases}$$

$$F_{02}^2(\theta) = \overline{F}_{20}^2(\theta). \tag{190}$$

One can now set up equation Equation (172) to approximate the center manifold. On this manifold one must have

$$W(s) = w(y(s),\ \overline{y}(s)). \tag{191}$$

By taking derivatives, the equation for the manifold becomes

$$w_y(y,\ \overline{y})y'(s) + w_{\overline{y}}(y,\ \overline{y})\overline{y}'(s) = Aw(y(s),\ \overline{y}(s)) + F_2(y(s),\ \overline{y}(s)), \tag{192}$$

where $w(y,\ \overline{y})$ is given by Equation (175). The partial derivatives are given by

$$\begin{aligned} w_y(y,\ \overline{y}) &= w_{20}y + w_{11}\overline{y} \\ w_{\overline{y}}(y,\ \overline{y}) &= w_{11}y + w_{02}\overline{y}. \end{aligned} \tag{193}$$

Using Equations (176) and (193) expand the terms of Equation (192) to second order as

$$\begin{aligned} & w_y(y,\ \overline{y})(\theta)y'(s) = i\omega w_{20}(\theta)y^2(s) + i\omega w_{11}(\theta)\overline{y}(s)y(s), \\ & w_{\overline{y}}(y,\ \overline{y})(\theta)\overline{y}'(s) = -i\omega w_{11}(\theta)y(s)\overline{y}(s) - i\omega w_{02}(\theta)\overline{y}^2(s), \\ & Aw(y,\ \overline{y})(\theta) = (Aw_{20})(\theta)\frac{y^2}{2} + (Aw_{11})(\theta)y\overline{y} + (Aw_{02})(\theta)\frac{\overline{y}^2}{2}, \\ & F_2(y,\ \overline{y})(\theta) = F_{20}^2(\theta)\frac{y^2}{2} + F_{11}^2(\theta)y\overline{y} + F_{20}^2(\theta)\frac{\overline{y}^2}{2}, \end{aligned} \tag{194}$$

Substitute Equation (194) into Equation (192) and equate coefficients to get

$$\begin{aligned} & 2i\omega w_{20}(\theta) - Aw_{20} = F_{20}^2(\theta), \\ & -Aw_{11} = F_{11}^2(\theta), \\ & -2i\omega w_{02}(\theta) - Aw_{02} = F_{02}^2(\theta). \end{aligned} \tag{195}$$

Since $F_{02}^2 = \overline{F}_{20}^2$ and $w_{02} = \overline{w}_{20}$, one only needs to solve for w_{20} and w_{11}.
Using Equation (195) it will be shown that $w_{20},\ w_{11}$ take the form

$$\begin{aligned} w_{20}(\theta) &= c_1q(\theta) + c_2\overline{q}(\theta) + M\,\mathrm{e}^{2i\omega\theta}, \\ w_{11}(\theta) &= c_3q(\theta) + c_4\overline{q}(\theta) + N, \end{aligned} \tag{196}$$

where $c_i, i = 1, \ldots, 4$ are constants and $M,\ N$ are vectors.
To compute $c_3,\ c_4,\ N$, use the second equation in (195), the definition of A in Equations (114) and (190). Then for $-\sigma \le \theta < 0$

$$\frac{\mathrm{d}w_{11}}{\mathrm{d}\theta}(\theta) = g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta). \tag{197}$$

Integrate Equation (197) and use Equation (143) to get

$$w_{11}(\theta) = \frac{g_{11}}{i\omega} q(\theta) - \frac{\overline{g}_{11}}{i\omega} \overline{q}(\theta) + N. \tag{198}$$

Clearly

$$\begin{aligned} c_3 &= \frac{g_{11}}{i\omega}, \\ c_4 &= -\frac{\overline{g}_{11}}{i\omega}. \end{aligned} \tag{199}$$

To determine N use Equation (114) for $\theta = 0$ and the fact that $q(0),\ \overline{q}(0)$ are eigenvectors of A with eigenvalues $i\omega,\ -i\omega$ at $\theta = 0$ along with (190) to show that

$$(L + R)N = -\frac{g_{11}}{\overline{d}_2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{200}$$

which can be solved for

$$N = -\frac{g_{11}}{\overline{d}_2} \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \tag{201}$$

To solve for $c_1,\ c_2,\ M$, use the definition of A in Equations (114) for $-\sigma \leq \theta < 0$, (190), (195) to get

$$\frac{\mathrm{d}w_{20}}{\mathrm{d}\theta} = 2i\omega w_{20}(\theta) + g_{20} q(\theta) + \overline{g}_{02} \overline{q}(\theta). \tag{202}$$

This non-homogeneous system has the solution

$$w_{20}(\theta) = -\frac{g_{20}}{i\omega} q(\theta) - \frac{\overline{g}_{02}}{3i\omega} \overline{q}(\theta) + M\, \mathrm{e}^{2i\omega\theta}. \tag{203}$$

Again, clearly

$$\begin{aligned} c_1 &= -\frac{g_{20}}{i\omega}, \\ c_2 &= -\frac{\overline{g}_{02}}{3i\omega}. \end{aligned} \tag{204}$$

To solve for M use the definition of Equations (114) for $\theta = 0$, (190) and the fact that $q(0)$, $\overline{q}(0)$ are eigenvectors of A with eigenvalues $i\omega,\ -i\omega$ at $\theta = 0$ to show that

$$(2i\omega I - L - \mathrm{R}\mathrm{e}^{-2i\omega\sigma})M = \frac{g_{20}}{\overline{d}_2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{205}$$

which can be solved for M as

$$M = \frac{g_{20}}{\overline{d}_2 \Delta} \begin{pmatrix} 1 \\ 2i\omega \end{pmatrix}, \tag{206}$$

where

$$\Delta = -4\omega^2 + 4i\xi\omega + p_c(1 - \mathrm{e}^{2i\omega\sigma}). \tag{207}$$

One can now return to Equation (186) and use Equations (198) through (201) and (203) through (207) to construct g_{21} in Equation (178), which concludes the construction of Equation (177) and thus the projected equation (176) on the center manifold. □

4.5.2. *Normal Form on the Center Manifold*
Normal form theory, following the argument of Wiggins [45] (see also [47]), will be used to reduce Equation (176) to the simpler form (174) on the center manifold.

LEMMA 4.19. *System* (176) *can be reduced by a near identity transformation to* (174) *where*

$$c_{21} = \frac{i}{2\omega}\left\{g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right\} + \frac{g_{21}}{2}. \tag{208}$$

Proof. Write Equation (176) as

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}s} &= i\omega y + g_2(y,\ \overline{y}) + g_3(y,\ \overline{y}),\\ \frac{\mathrm{d}\overline{y}}{\mathrm{d}s} &= -i\omega\overline{y} + \overline{g}_2(y,\ \overline{y}) + \overline{g}_3(y,\ \overline{y}),\end{aligned} \tag{209}$$

where g_2 and g_3 are the second and third-order terms in g respectively. If one introduces the near identity transformation

$$\begin{aligned}y &= u + h_2(u,\ \overline{u}),\\ \overline{y} &= \overline{u} + \overline{h}_2(u,\ \overline{u}),\end{aligned} \tag{210}$$

Equation (209) is reduced to

$$\frac{\mathrm{d}u}{\mathrm{d}s} = i\omega u + \{\mathcal{L}_2(h_2) + g_2(u,\ \overline{u})\} + O(3) \tag{211}$$

along with its conjugate, where $O(3)$ represents third-order terms and

$$\mathcal{L}_2(h_2) = i\omega h_2 - \left(i\omega\frac{\partial h_2}{\partial u}u - i\omega\frac{\partial h_2}{\partial \overline{u}}\overline{u}\right), \tag{212}$$

Let

$$S_2 = \mathrm{span}\{u^2,\ u\overline{u},\ \overline{u}^2\} \tag{213}$$

be the space of quadratic polynomials in $u,\ \overline{u}$. $\mathcal{L}_2$ is a linear map from $S_2 \to S_2$. One can examine what $\mathcal{L}_2$ does to the basis elements. A simple calculation shows that

$$\begin{aligned}\mathcal{L}_2(u^2) &= -i\omega u^2,\\ \mathcal{L}_2(u\overline{u}) &= i\omega u\overline{u},\\ \mathcal{L}_2(\overline{u}^2) &= 3i\omega\overline{u}^2,\end{aligned} \tag{214}$$

so that $\mathcal{L}_2$ is invertible and one can solve for h_2 so that

$$\mathcal{L}_2(h_2) + g_2(u,\ \overline{u}) = 0, \tag{215}$$

which means that the quadratic terms can be eliminated. Repeat the argument for

$$\frac{\mathrm{d}u}{\mathrm{d}s} = i\omega u + g_3(u, \overline{u}). \tag{216}$$

Again take a near identity transformation

$$u = v + h_3(v, \overline{v}) \tag{217}$$

and reduce (216) to

$$\frac{\mathrm{d}v}{\mathrm{d}s} = i\omega v + \{\mathcal{L}_3(h_3) + g_3(v, \overline{v})\}, \tag{218}$$

where

$$\mathcal{L}_3(h_3) = i\omega h_3 - \left(i\omega \frac{\partial h_3}{\partial v} v - i\omega \frac{\partial h_3}{\partial \overline{v}} \overline{v} \right). \tag{219}$$

Let

$$S_3 = \mathrm{span}\{v^3, \ v^2\overline{v}, \ v\overline{v}^2, \ \overline{v}^3\} \tag{220}$$

be the space of cubic polynomials in $v, \ \overline{v}$. Apply $\mathcal{L}_3$ to the basis set to get

$$\begin{aligned}
&\mathcal{L}_3(v^3) = -2i\omega v^3, \\
&\mathcal{L}_3(v^2\overline{v}) = 0, \\
&\mathcal{L}_3(v\overline{v}^2) = 2i\omega v\overline{v}^2, \\
&\mathcal{L}_3(\overline{v}^3) = 4i\omega\overline{v}^3,
\end{aligned} \tag{221}$$

which means that terms of the form $v^2\overline{v}$ cannot be eliminated. Thus the normal form, up to cubic terms, is given by Equation (174). This motivates why, in the construction of $g(y, \ \overline{y})$ in Equation (177), only cubic terms of the form $y^2\overline{y}$ are maintained.

Now that the normal form is known one can compute c_{21} by applying

$$y = v + h(v, \overline{v}) \tag{222}$$

to

$$\frac{\mathrm{d}y}{\mathrm{d}s} = i\omega y + g(y, \overline{y}), \tag{223}$$

where g is given by Equation (177), and reduce Equation (223) directly. In order to eliminate quadratic terms take

$$h(v, \overline{v}) = h_{20}\frac{v^2}{2} + h_{11}v\overline{v} + h_{02}\frac{\overline{v}^2}{2}. \tag{224}$$

The derivatives are

$$\begin{aligned}
D_v h &= h_{20}v + h_{11}\overline{v}, \\
D_{\overline{v}} h &= h_{11}v + h_{02}\overline{v}.
\end{aligned} \tag{225}$$

Now, insert the near identity transformation (222) into Equation (223), use the normal form (174), and the definition of $g(y,\ \overline{y})$ in Equation (177) to write

$$
\begin{aligned}
&i\omega v + c_{21}v^2\overline{v} + D_v h(i\omega + c_{21}v^2\overline{v}) + D_{\overline{v}}h(-i\omega\overline{v} + \overline{c}_{21}\overline{v}^2 v),\\
&\quad = i\omega(v+h) + \frac{g_{20}}{2}(v+h)^2 + g_{11}(v+h)(\overline{v}+\overline{h})\\
&\qquad + \frac{g_{02}}{2}(\overline{v}+\overline{h})^2 + \frac{g_{21}}{2}(v+h)^2(\overline{v}+\overline{h}).
\end{aligned} \tag{226}
$$

Next, insert Equation (224), expand and equate all powers of $v,\ \overline{v}$, drop all cubic terms, other than $v^2\overline{v}$, and all higher-order terms in order to compute the coefficients of h that eliminate quadratic terms as

$$
\begin{aligned}
h_{20} &= \frac{g_{20}}{i\omega},\\
h_{11} &= -\frac{g_{11}}{i\omega},\\
h_{02} &= -\frac{g_{02}}{3i\omega},
\end{aligned} \tag{227}
$$

and finally compute

$$
c_{21} = g_{20}h_{11} + g_{11}\overline{h}_{11} + \frac{g_{11}h_{20}}{2} + \frac{g_{02}\overline{h}_{02}}{2} + \frac{g_{21}}{2}, \tag{228}
$$

which yields Equation (208) when Equation (227) is substituted. □

For a more general discussion of the normal form on the center manifold when $\mu \neq 0$, see [11]. Up to this point one only needed $\mu = 0$. But, to compute the periodic solution, reintroduce $\mu \neq 0$.

4.6. The Periodic Solution on the Center Manifold and Its Stability

As shown in [11] the general normal form for the case $\mu \neq 0$ is given by

$$
\frac{dv}{ds} = \lambda(\mu)v + c_{21}(\mu)v^2\overline{v}, \tag{229}
$$

where $\lambda(0) = i\omega$ and $c_{21}(0)$ is given by Equation (208). The periodic solutions of Equation (229) will first be computed, their stability determined, and then they will be transformed to the periodic solutions of Equation (3). The argument is based on that of Hassard et al. [11].

LEMMA 4.20. *Let $\varepsilon > 0$ and an initial condition for a periodic solution of Equation* (229) *be given as*

$$
v(0;\ \varepsilon) = \varepsilon. \tag{230}
$$

Then there exists a family of periodic solutions $v(s;\ \mu(\varepsilon))$ of Equation (229) *with*

$$
\begin{aligned}
\mu(\varepsilon) &= \mu_2\varepsilon^2 + \cdots,\\
\beta(\varepsilon) &= \beta_2\varepsilon^2 + \cdots,\\
T(\varepsilon) &= T_0(1 + \tau_2\varepsilon^2 + \cdots),
\end{aligned} \tag{231}
$$

where $T(\varepsilon)$ is the period of $v(s,\ \mu(\varepsilon))$, $\beta(\varepsilon)$ is the non-zero characteristic exponent, and

$$\begin{aligned}
\mu_2 &= -\frac{\mathrm{Re}\,\{c_{21}(0)\}}{\alpha'(0)}, \\
\beta_2 &= 2\mathrm{Re}\,\{c_{21}(0)\}, \\
\tau_2 &= -\frac{1}{\omega}(\mu_2\omega'(0) + \mathrm{Im}\{c_{21}(0)\}), \\
T_0 &= \frac{2\pi}{\omega}.
\end{aligned} \tag{232}$$

Furthermore, $v(s,\ \mu(\varepsilon))$ can be transformed into a family of periodic solutions for Equation (3) given by

$$Z(s) = \mathcal{P}(s, \mu(\varepsilon)) = 2\varepsilon \mathrm{Re}\{q(0)\,\mathrm{e}^{i\omega s}\} + \varepsilon^2 \mathrm{Re}\{M\,\mathrm{e}^{2i\omega s} + N\}, \tag{233}$$

with $\varepsilon = (\mu/\mu_2)^{1/2}$. For $\mu_2 > 0$ the Hopf bifurcation is called supercritical and for $\mu_2 < 0$ it is called subcritical.

Proof. One can begin looking for a periodic solution for Equation (229) with initial condition (230) by changing variables

$$\eta = \frac{v}{\varepsilon}. \tag{234}$$

Then Equation (229) becomes

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = \lambda(\mu)\eta + \varepsilon^2 c_{21}(\mu)\eta^2\overline{\eta}, \tag{235}$$

with $\eta(0,\ \mu) = 1$. One first looks for the general form of solution for Equation (235). For fixed μ assume an expansion

$$\eta(s,\ \varepsilon,\ \mu) = \eta_0(s,\ \mu) + \varepsilon\eta_1(s,\ \mu) + \varepsilon^2\eta_2(s,\ \mu) + \cdots, \tag{236}$$

with

$$\begin{aligned}
\eta_0(0,\ \mu) &= 1, \\
\eta_i(0,\ \mu) &= 0, \quad i = 1,\ 2, \ldots.
\end{aligned} \tag{237}$$

Substitute Equation (236) into Equation (235) and use Equation (237) to show

$$\begin{aligned}
&\eta_0(s,\ \mu) = \mathrm{e}^{\lambda(\mu)s}, \\
&\eta_1(s,\ \mu) = 0, \\
&\frac{\mathrm{d}\eta_2}{\mathrm{d}s}(s,\ \mu) = \lambda(\mu)\eta_2(s,\ \mu) + c_{21}(\mu)\,\mathrm{e}^{(2\alpha(\mu)+\lambda(\mu))s},
\end{aligned} \tag{238}$$

where $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$. Note that $\eta_2(0,\ \mu) = 0$ from Equation (237). Furthermore, for $\mu = 0,\ \alpha(0) = 0,\ \omega(0) = \omega$ and for $\mu \neq 0,\ \alpha(\mu) \neq 0$. Therefore, there are two cases for $\eta_2(s,\ \mu)$ in Equation (238)

$$\eta_2(s,\ \mu) = \begin{cases} c_{21}(0)s\,\mathrm{e}^{i\omega s}, & \mu = 0, \\ \dfrac{c_{21}(\mu)}{2\alpha(\mu)}(\mathrm{e}^{2\alpha(\mu)s} - 1)\,\mathrm{e}^{\lambda(\mu)s}, & \mu \neq 0. \end{cases} \tag{239}$$

If one defines

$$e(s,\ \mu) = \begin{cases} s, & \mu = 0, \\ \dfrac{\mathrm{e}^{2\alpha(\mu)s} - 1}{2\alpha(\mu)}, & \mu \neq 0, \end{cases} \tag{240}$$

and sets

$$\eta_2(s,\ \mu) = c_{21}(\mu)e(s,\ \mu)\,\mathrm{e}^{\lambda(\mu)s}, \tag{241}$$

then, combining Equations (236), (238) and (241), the solution of Equation (235), to second order in ε, can be written

$$\eta(s,\ \varepsilon,\ \mu) = \mathrm{e}^{\lambda(\mu)s} + \varepsilon^2 c_{21}(\mu)e(s,\ \mu)\,\mathrm{e}^{\lambda(\mu)s}. \tag{242}$$

Now look for functions $\mu(\varepsilon)$ and $T(\varepsilon)$ in the form

$$\begin{aligned} \mu(\varepsilon) &= \mu_1\varepsilon + \mu_2\varepsilon^2 + \cdots, \\ T(\varepsilon) &= T_0(1 + \tau_1\varepsilon + \tau_2\varepsilon^2 + \cdots), \end{aligned} \tag{243}$$

where $T(\varepsilon)$ will be the period of the family of periodic solutions $\eta(s,\ \varepsilon,\ \mu(\varepsilon))$. In order not to lose powers of ε, write Equation (242) as

$$\mathrm{e}^{-\lambda(\mu)s}\eta(s,\ \varepsilon,\ \mu) = 1 + \varepsilon^2 c_{21}(\mu)e(s,\ \mu). \tag{244}$$

From the periodicity requirement and the initial condition for Equation (235) one must have

$$\eta(0,\ \varepsilon,\ \mu(\varepsilon)) = \eta(T(\varepsilon),\ \varepsilon,\ \mu(\varepsilon)) = 1. \tag{245}$$

This implies, setting $s = T(\varepsilon)$ and $\mu = \mu(\varepsilon)$ in Equation (244), that

$$\mathrm{e}^{-\lambda(\mu(\varepsilon))T(\varepsilon)} = 1 + \varepsilon^2 c_{21}(\mu(\varepsilon))e(T(\varepsilon),\ \mu(\varepsilon)). \tag{246}$$

Expand both sides of Equation (246) in perturbation series. On the left, expand $\lambda(\mu(\varepsilon)$ by Taylor series up to second order in ε as

$$\lambda(\mu(\varepsilon)) = \lambda_0 + \lambda_1\varepsilon + \lambda_2\varepsilon^2 + \cdots, \tag{247}$$

where

$$\begin{aligned} \lambda_0 &= \lambda(0) = i\omega, \\ \lambda_1 &= \frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}(0)\mu_1, \\ \lambda_2 &= \frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}(0)\mu_2 + \frac{\mu_1{}^2}{2}\frac{\mathrm{d}^2\lambda}{\mathrm{d}\varepsilon^2}(0). \end{aligned} \tag{248}$$

Then, from Equations (243, (247), and (248)

$$\lambda(\mu(\varepsilon)T(\varepsilon) = (T_0\lambda_0) + (T_0(\lambda_1 + \lambda_0\tau_1))\varepsilon + (T_0(\lambda_2 + \lambda_1\tau_1 + \lambda_0\tau_2))\varepsilon^2 + \cdots. \tag{249}$$

For the sake of notation set

$$\begin{aligned}(T\lambda)_0 &= T_0\lambda_0,\\ (T\lambda)_1 &= T_0(\lambda_1 + \lambda_0\tau_1),\\ (T\lambda)_2 &= T_0(\lambda_2 + \lambda_1\tau_1 + \lambda_0\tau_2).\end{aligned} \tag{250}$$

Then, Equations (249) and (250) can be used to expand the left-hand side of Equation (246) in Taylor series to second order in ε as

$$\mathrm{e}^{-\lambda(\mu(\varepsilon))T(\varepsilon)} = \mathrm{e}^{-(T\lambda)_0}\left[1 - (T\lambda)_1\varepsilon + \left\{\frac{{(T\lambda)_1}^2}{2} - (T\lambda)_2\right\}\varepsilon^2\right]. \tag{251}$$

To expand the right-hand side of Equation (246) to second order in ε all that is needed is to expand the coefficient of the second term to zero order, using Equation (240), as

$$c_{21}(\mu(\varepsilon))e(T(\varepsilon),\ \mu(\varepsilon)) = c_{21}(0)T_0 + \cdots. \tag{252}$$

Now substitute Equations (251) and (252) into (246) and equate both sides to second order in ε to get

$$\begin{aligned}&\mathrm{e}^{-(T\lambda)_0} = 1,\\ &(T\lambda)_1 = 0,\\ &-(T\lambda)_2 = c_{21}(0)T_0.\end{aligned} \tag{253}$$

From Equations (248), (250), (253) this implies

$$\begin{aligned}&T_0 = \frac{2\pi}{\omega},\\ &T_0\left[\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\mu}(0) + i\frac{\mathrm{d}\omega}{\mathrm{d}\mu}(0)\right)\mu_1 + i\omega\tau_1\right] = 0.\end{aligned} \tag{254}$$

Since $(\mathrm{d}\alpha/\mathrm{d}\mu)(0) \neq 0$ Equation (254) implies $\mu_1 = 0$ and $\tau_1 = 0$. Then from Equations (248), (250), and (253)

$$-T_0\left(\mu_2\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(0) + i\omega\tau_2\right) = c_{21}(0)T_0. \tag{255}$$

Equating the real and imaginary parts of Equation (255) one has Equation (243) up to the second order in ε with

$$\begin{aligned}&\mu_1 = \tau_1 = 0,\\ &\mu_2 = -\frac{\mathrm{Re}\{c_{21}(0)\}}{\frac{\mathrm{d}\alpha}{\mathrm{d}\mu}(0)},\\ &\tau_2 = -\frac{1}{\omega}\left(\mu_2\frac{\mathrm{d}\omega}{\mathrm{d}\mu}(0) + \mathrm{Im}\{c_{21}(0)\}\right),\\ &T_0 = \frac{2\pi}{\omega}.\end{aligned} \tag{256}$$

This constructs the family of periodic solutions $\eta(s,\ \varepsilon,\ \mu(\varepsilon))$ with period $T(\varepsilon)$ for Equation (235).

In order to determine the stability of this family of periodic solutions one must compute the characteristic exponents of the variational equation of Equation (235) about the family of periodic solutions $\eta(s,\ \varepsilon,\ \mu(\varepsilon))$. To do this introduce a perturbed solution $\eta + \psi$ into Equation (235). For fixed μ the variational equation is given by

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \lambda(\mu)\psi + \varepsilon^2 c_{21}(\mu)(2\eta\bar{\eta}\psi + \eta^2\bar{\psi}) \tag{257}$$

along with its conjugate. Since η is periodic with period $T(\varepsilon)$ the coefficients of Equation (257) are periodic with the same period. Expand the solution as

$$\psi(s,\ \varepsilon,\ \mu) = \psi_0(s,\ \mu) + \varepsilon\psi_1(s,\ \mu) + \varepsilon^2\psi_2(s,\ \mu) + \cdots, \tag{258}$$

where the initial conditions are

$$\begin{aligned} &\psi(0,\ \varepsilon,\ \mu) = \psi_0(0,\ \mu), \\ &\psi_i(0,\ \mu) = 0, \quad i = 1,\ 2, \ldots. \end{aligned} \tag{259}$$

Substituting Equations (242) and (258) into Equation (257) and equating coefficients up to second order in ε gives

$$\begin{aligned} \frac{\psi_0}{\mathrm{d}s} &= \lambda(\mu)\psi_0, \\ \frac{\psi_1}{\mathrm{d}s} &= \lambda(\mu)\psi_1, \\ \frac{\psi_2}{\mathrm{d}s} &= \lambda(\mu)\psi_2 + c_{21}(\mu)\,\mathrm{e}^{\lambda(\mu)s}[2\,\mathrm{e}^{\bar{\lambda}(\mu)s}\psi_0 + \mathrm{e}^{\lambda(\mu)s}\bar{\psi}_0]. \end{aligned} \tag{260}$$

From the initial conditions (259) compute

$$\begin{aligned} \psi_0(s,\ \mu) &= \psi_0(0,\ \mu)\,\mathrm{e}^{\lambda(\mu)s}, \\ \psi_1(s,\ \mu) &= 0, \\ \psi_2(s,\ \mu) &= c_{21}(\mu)e(s,\ \mu)[2\psi_0(0,\ \mu) + \bar{\psi}_0(0,\ \mu)]\,\mathrm{e}^{\lambda(\mu)s}, \end{aligned} \tag{261}$$

where Equation (240) has been used with $\mu \neq 0$. Therefore, to second order in ε, the general form of the solution of the variational equation (257) is

$$\psi(s,\ \varepsilon,\ \mu) = \psi_0(0,\ \mu)\,\mathrm{e}^{\lambda(\mu)s} + \varepsilon^2 c_{21}(\mu)[2\psi_0(0,\ \mu) + \bar{\psi}_0(0,\ \mu)]e(s,\ \mu)\,\mathrm{e}^{\lambda(\mu)s}. \tag{262}$$

Now, setting $s = T(\varepsilon)$ and $\mu = \mu(\varepsilon)$, rewrite Equation (262) as

$$\begin{aligned} &\psi(T(\varepsilon),\ \varepsilon,\ \mu(\varepsilon)) \\ &\quad = [\mathrm{e}^{\lambda(\mu(\varepsilon))T(\varepsilon)} + \varepsilon^2 c_{21}(\mu(\varepsilon))2e(T(\varepsilon),\ \mu(\varepsilon))\,\mathrm{e}^{\lambda(\mu(\varepsilon))T(\varepsilon)}]\psi(0,\ \varepsilon,\ \mu(\varepsilon)) \\ &\qquad + [\varepsilon^2 c_{21}(\mu(\varepsilon))e(T(\varepsilon),\ \mu(\varepsilon))\,\mathrm{e}^{\lambda(\mu(\varepsilon))T(\varepsilon)}]\bar{\psi}(0,\ \varepsilon,\ \mu(\varepsilon)) \end{aligned} \tag{263}$$

along with its conjugate

$$
\begin{aligned}
&\overline{\psi}(T(\varepsilon),\ \varepsilon,\ \mu(\varepsilon)) \\
&\quad = [\varepsilon^2\overline{c}_{21}(\mu(\varepsilon))e(T(\varepsilon),\ \mu(\varepsilon))\,\mathrm{e}^{\overline{\lambda}(\mu(\varepsilon))T(\varepsilon)}]\psi(0,\ \varepsilon,\ \mu(\varepsilon)) \\
&\qquad + [\mathrm{e}^{\overline{\lambda}(\mu(\varepsilon))T(\varepsilon)} + \varepsilon^2\overline{c}_{21}(\mu(\varepsilon))2e(T(\varepsilon),\ \mu(\varepsilon))\,\mathrm{e}^{\overline{\lambda}(\mu(\varepsilon))T(\varepsilon)}]\overline{\psi}(0,\ \varepsilon,\ \mu(\varepsilon)).
\end{aligned} \tag{264}
$$

Set

$$
\begin{aligned}
a_{11} &= \mathrm{e}^{\lambda(\mu(\varepsilon))T(\varepsilon)} + \varepsilon^2 c_{21}(\mu(\varepsilon))2e(T(\varepsilon),\ \mu(\varepsilon))\,\mathrm{e}^{\lambda(\mu(\varepsilon))T(\varepsilon)}, \\
a_{12} &= \varepsilon^2 c_{21}(\mu(\varepsilon))e(T(\varepsilon),\ \mu(\varepsilon))\,\mathrm{e}^{\lambda(\mu(\varepsilon))T(\varepsilon)}, \\
a_{21} &= \overline{a}_{12}, \\
a_{22} &= \overline{a}_{11},
\end{aligned} \tag{265}
$$

then write Equations (263) and (264) in matrix form

$$
\begin{pmatrix} \psi(T(\varepsilon),\ \varepsilon,\ \mu(\varepsilon)) \\ \overline{\psi}(T(\varepsilon),\ \varepsilon,\ \mu(\varepsilon)) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \psi(0,\ \varepsilon,\ \mu(\varepsilon)) \\ \overline{\psi}(0,\ \varepsilon,\ \mu(\varepsilon)) \end{pmatrix}. \tag{266}
$$

From Floquet theory the eigenvalues of the monodromy matrix with elements a_{ij} are of the form 1 and $\mathrm{e}^{\beta(\varepsilon)T(\varepsilon)}$. The sum of these eigenvalues is the trace of the monodromy matrix or

$$
1 + \mathrm{e}^{\beta(\varepsilon)T(\varepsilon)} = a_{11} + a_{22} = 2\mathrm{Re}\{a_{11}\}. \tag{267}
$$

Set

$$
\begin{aligned}
\beta(\varepsilon) &= \beta_0 + \beta_1\varepsilon + \beta_2\varepsilon^2 + \cdots, \\
T(\varepsilon) &= \frac{2\pi}{\omega}(1 + \tau_2\varepsilon^2 + \cdots).
\end{aligned} \tag{268}
$$

In order to compute the coefficients of $\beta(\varepsilon)$ use Equation (267) by first expanding a_{11} from (265) to second order in ε. From Equations (251) and (253) through (255)

$$
\mathrm{e}^{-\lambda(\mu(\varepsilon))T(\varepsilon)} = 1 + \frac{2\pi}{\omega}c_{21}(0)\varepsilon^2, \tag{269}
$$

so that by the geometric series

$$
\mathrm{e}^{\lambda(\mu(\varepsilon))T(\varepsilon)} = 1 - \frac{2\pi}{\omega}c_{21}(0)\varepsilon^2. \tag{270}
$$

Expanding to order zero $c_{21}(\mu(\varepsilon)) = c_{21}(0)$ and from Equations (240) and (256), again expanding to zero order, $e(T(\varepsilon),\ \mu(\varepsilon)) = e(T(0),\ 0) = T(0) = 2\pi/\omega$. Then, using Equations (265) and (270), compute

$$
a_{11} = \left(1 - \frac{2\pi}{\omega}c_{21}(0)\varepsilon^2\right)\left(1 + 2\frac{2\pi}{\omega}c_{21}(0)\varepsilon^2\right) = 1 + \frac{2\pi}{\omega}c_{21}(0)\varepsilon^2. \tag{271}
$$

Inserting Equation (271) into Equation (267) yields

$$
1 + \mathrm{e}^{\beta(\varepsilon)T(\varepsilon)} = 2 + \varepsilon^2\frac{2\pi}{\omega}(2\mathrm{Re}\{c_{21}(0)\}) \tag{272}
$$

or

$$\mathrm{e}^{\beta(\varepsilon)T(\varepsilon)} = 1 + \varepsilon^2 \frac{2\pi}{\omega}(2\mathrm{Re}\{c_{21}(0)\}). \tag{273}$$

Using Equation (268), expand the left-hand side of Equation (273) by Taylor series to second order in ε and get

$$\mathrm{e}^{\beta(\varepsilon)T(\varepsilon)} = \mathrm{e}^{(2\pi/\omega)\beta_0}\left(1 + \frac{2\pi}{\omega}\beta_1\varepsilon + \left[\frac{2\pi}{\omega}(\beta_2 + \tau_2\beta_0) + \left(\frac{2\pi}{\omega}\right)^2 {\beta_1}^2\right]\varepsilon^2\right). \tag{274}$$

Now, equating powers of ε in Equations (273) and (274)

$$\begin{aligned} &\mathrm{e}^{(2\pi/\omega)\beta_0} = 1, \\ &\frac{2\pi}{\omega}\beta_1 = 0, \end{aligned} \tag{275}$$

which implies $\beta_0 = \beta_1 = 0$ since they are real. Finally one has

$$\beta_2 = 2\mathrm{Re}\{c_{21}(0)\}. \tag{276}$$

From Floquet theory the sign of β_2 determines the nature of stability.

At this point one can construct the family of periodic solutions for Equation (229). Begin with Equation (242), the Taylor series for $\lambda(\mu(\varepsilon))$, given by Equations (247) and (248), and the fact that $\mu_1 = 0$, shows that

$$\mathrm{e}^{\lambda(\mu(\varepsilon))s} = \mathrm{e}^{i\omega s}\left[1 + \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(0)\mu_2 s\varepsilon^2\right]. \tag{277}$$

Substitute for μ_2 from Equation (256) and use the fact that

$$\mathrm{d}\lambda/\mathrm{d}\mu(0) = \mathrm{d}\alpha/\mathrm{d}\mu(0) + i\,\mathrm{d}\omega/\mathrm{d}\mu(0)$$

to get

$$\mathrm{e}^{\lambda(\mu(\varepsilon))s} = \mathrm{e}^{i\omega s}\left(1 + i\varepsilon^2 s\left[-\mathrm{Re}\{c_{21}(0)\} - \frac{\mathrm{Re}\{c_{21}(0)\}\frac{\mathrm{d}\omega}{\mathrm{d}\mu}(0)}{\frac{\mathrm{d}\alpha}{\mathrm{d}\mu}(0)}\right]\right). \tag{278}$$

Using Equation (240), expand to the zeroth order in ε

$$\begin{aligned} &c_{21}(\mu(\varepsilon)) = c_{21}(0) + \cdots, \\ &e(s,\ \mu(\varepsilon)) = s + \cdots. \end{aligned} \tag{279}$$

Therefore from Equations (242) and (278), up to second order in ε,

$$\eta(s,\ \varepsilon,\ \mu(\varepsilon)) = \mathrm{e}^{i\omega s}\left(1 + i\varepsilon^2 s\left[\mathrm{Im}\{c_{21}(0)\} - \frac{\mathrm{Re}\{c_{21}(0)\}\frac{\mathrm{d}\omega}{\mathrm{d}\mu}(0)}{\frac{\mathrm{d}\alpha}{\mathrm{d}\mu}(0)}\right]\right). \tag{280}$$

One can also show that, to the second order in ε, one gets the right-hand side of Equation (280) when one substitutes for τ_2 from Equation (256) in

$$\mathrm{e}^{(2\pi/T(\varepsilon))is} = \mathrm{e}^{i\omega(1-\tau_2\varepsilon^2)s} = \mathrm{e}^{i\omega s}(1 - i\omega\tau_2 s\varepsilon^2). \tag{281}$$

Then Equations (280) and (281) imply

$$\eta(s,\ \varepsilon,\ \mu(\varepsilon)) = \mathrm{e}^{(2\pi/T(\varepsilon))is}. \tag{282}$$

With the change of variable (234) one obtains the solution for Equation (229) as

$$v(s,\ \varepsilon,\ \mu(\varepsilon)) = \varepsilon\,\mathrm{e}^{\frac{2\pi}{T(\varepsilon)}is} = \varepsilon\,\mathrm{e}^{i\omega s}. \tag{283}$$

Finally one can construct the bifurcating periodic solution of Equation (3). From the near identity transformation (222), and Equations (224), and (227)

$$\begin{aligned} y(s,\ \varepsilon) &= v(s,\ \varepsilon,\ \mu(\varepsilon)) + \frac{g_{20}}{2i\omega}v(s,\ \varepsilon,\ \mu(\varepsilon))^2 \\ &\quad - \frac{g_{11}}{i\omega}v(s,\ \varepsilon,\ \mu(\varepsilon))\overline{v}(s,\ \varepsilon,\ \mu(\varepsilon)) - \frac{g_{02}}{6i\omega}\overline{v}(s,\ \varepsilon,\ \mu(\varepsilon))^2. \end{aligned} \tag{284}$$

From Equations (283) and (284), to second order in ε,

$$y(s,\ \varepsilon) = \varepsilon\,\mathrm{e}^{i\omega s} + \frac{g_{20}}{2i\omega}\varepsilon^2\,\mathrm{e}^{2i\omega s} - \frac{g_{11}}{i\omega}\varepsilon^2 - \frac{g_{02}}{6i\omega}\varepsilon^2\,\mathrm{e}^{-2i\omega s}. \tag{285}$$

One can now relate $Z(s)$ and $y(s)$ by the transformation (167), with $\theta = 0$, in the form

$$Z(s) = q(0)y(s) + \overline{q}(0)\overline{y}(s) + W_s(0), \tag{286}$$

where $W_s(0) = w(y(s),\ \overline{y}(s))$. From Equations (175) and (285), up to second order in ε,

$$\begin{aligned} W_s(0) &= \frac{\varepsilon^2}{2}w_{20}(0)\,\mathrm{e}^{2i\omega s} + \varepsilon^2 w_{11}(0) + \frac{\varepsilon^2}{2}w_{02}(0)\,\mathrm{e}^{-2i\omega s} \\ &= \varepsilon^2\mathrm{Re}\{w_{20}(0)\,\mathrm{e}^{2i\omega s} + w_{11}(0)\}. \end{aligned} \tag{287}$$

Now from Equations (285) through (287) write

$$\begin{aligned} Z(s,\varepsilon) &= 2\mathrm{Re}\,\{q(0)y(s)\} + W_s(0) \\ &= 2\varepsilon\mathrm{Re}\{q(0)\,\mathrm{e}^{i\omega s}\} + 2\varepsilon^2\mathrm{Re}\left\{q(0)\left[\frac{g_{20}}{2i\omega}\,\mathrm{e}^{2i\omega s} - \frac{g_{11}}{i\omega} - \frac{g_{02}}{6i\omega}\,\mathrm{e}^{-2i\omega s}\right]\right\} \\ &\quad + \varepsilon^2\mathrm{Re}\{w_{20}(0)\,\mathrm{e}^{2i\omega s} + w_{11}(0)\} \end{aligned} \tag{288}$$

for $0 \le s \le T(\varepsilon)$. By using Equation (196) with $\theta = 0$, and using Equations (199), (201), (204), (206), (207), and (167) one can write with some lengthy but straightforward calculations

$$Z(s,\varepsilon) = \mathcal{P}(s,\mu(\varepsilon)) = 2\varepsilon\mathrm{Re}\{q(0)\,\mathrm{e}^{i\omega s}\} + \varepsilon^2\mathrm{Re}\{M\,\mathrm{e}^{2i\omega s} + N\} \tag{289}$$

since N is real. This is the specific form of Equation (9). Finally, note that since $\mu \approx \mu_2\varepsilon^2$ one can take $\varepsilon = (\mu/\mu_2)^{1/2}$ which allows one to associate $Z(s)$ with the parameter $p = \mu + p_c$.

□

5. Application to Machine Tool Chatter

The machining tool model used only for illustration in this paper is taken from Kalmár-Nagy et al. [12] and can be written as

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{kf_0}{m\alpha}\left(1 - \left(\frac{f}{f_0}\right)^{\alpha}\right), \tag{290}$$

where $\omega_n = \sqrt{r/m}$ is the natural frequency of the undamped free oscillating system and $\xi = c/2m\omega_n$ is the relative damping factor and k is the cutting force coefficient that is related to the slope of the power-law curve used to define the right-hand side of Equation (290). The parameters m, r, c and α are taken as $m = 10$ kg, $r = 3.35$ MN/m, $c = 156$ kg/s, and $\alpha = 0.41$ and were obtained from measurements of the machine-tool response function [12]. α was obtained from a cutting force model due to Taylor [49]. These then imply that $\omega_n = 578.79$ 1/s, $\xi = 0.0135$. f_0 is the nominal chip width and

$$f = f_0 + x(t) - x(t - \tau), \tag{291}$$

where the delay $\tau = 2\pi/\Omega_\tau$ is the time for one revolution of the turning center spindle (or $\tau = 60/\Omega_\tau$ if Ω_τ is in RPM). k will be taken as the bifurcation parameter. The displacement $x(t)$ is directed positively into the workpiece and the tool is assumed not to leave the workpiece.

The model is simplified by introducing a non-dimensional time s and displacement z by

$$\begin{aligned} s &= \omega_n t, \\ z &= \frac{x}{A}, \end{aligned} \tag{292}$$

where the length scale is computed as

$$A = \frac{3f_0}{2-\alpha}, \tag{293}$$

a new bifurcation parameter p is set to

$$p = \frac{k}{m\omega_n^2}, \tag{294}$$

and the delay parameter becomes

$$\sigma = \omega_n \tau. \tag{295}$$

The dimensionless model then becomes, after expanding the right-hand side of Equation (290) to the third order,

$$\frac{d^2z}{ds^2} + 2\xi\frac{dz}{ds} + z = p(\Delta z + E(\Delta z^2 + \Delta z^3)), \tag{296}$$

where

$$\begin{aligned} \Delta z &= z(s - \sigma) - z(s), \\ E &= \frac{3(1-\alpha)}{2(2-\alpha)}. \end{aligned} \tag{297}$$

Since the Hopf bifurcation studied in this paper is local, the bifurcation parameter will be written as

$$p = \mu + p_c, \tag{298}$$

where p_c is a critical value at which bifurcation occurs. Then Equation (296) can be put into vector form (3) As shown in Equation (123) the characteristic equation for Equation (1) is given by

$$\lambda^2 + 2\xi\lambda + (1 + p) - p\,\mathrm{e}^{-\lambda\sigma} = 0. \tag{299}$$

By Lemma 3.1, for $\omega > 1$, there is a uniquely defined sequence $\sigma_r = \sigma_r(\omega)$, $r = 0, 1, 2, \dots$ and a uniquely defined $p = p(\omega)$ such that (ω, σ_r, p), $r = 0, 1, 2, \dots$, are critical eigen triples for the characteristic Equation (299). These are given by Equations (32) and (33), using Equations (23) and (28).

One can plot p against $\Omega_r = 1/\sigma_r$, where Ω_r is the rotation rate of the turning center spindle, for $r = 0, 1, 2, \dots$. Since p must be positive, select any set of values $\omega > 1$ such that $G(\omega) < 0$. Next select $r = 0, 1, 2, \dots, N$ for some N. For each r compute the pairs (Ω_r, p) for each ω. When these families of pairs are plotted they form a family of N lobes as shown in Figure 3 for the case $N = 5$, where the plots are based on the value of $\xi = 0.0135$. Each lobe is parameterized by the same vector of ω's so that each point on a lobe boundary represents an eigenvalue of Equation (19) for a given p and $\sigma_r = 1/\Omega_r$. This plot is called a *stability chart* and was introduced by Tobias and Fishwick [50]. The minimum of each lobe is asymptotic to a line often called the *stability limit*. The significance of the stability chart in the linear case is that the lobe boundaries divide the plane into regions of stable and unstable response. In particular, the regions below the lobes are stable and those above are unstable. In the nonlinear case the Hopf bifurcation at the lobe boundaries allow for the possibility of unstable oscillations below the stability boundary. This would be the case in a subcritical Hopf bifurcation. Since the parameter p is proportional to material removal, the regions between lobes represent areas that can be exploited for material removal above the stability limit line. This property is currently being exploited in high speed machining [6].

To simplify the calculations in the following sections only the bifurcation at the minimum point of the lobes, p_m, given by Equations (60), (61) will be examined. Any other point on a lobe would involve more complicated expressions for any p greater than p_m and obscure the essential arguments. As was previously said a discussion of the bifurcation that occurs when two or more lobes cross is beyond the scope of the current paper and will be considered at a later time. From Equations (61) and (28)

$$\begin{aligned} \psi_m &= -\pi + \tan^{-1}\left(\sqrt{1 + 2\xi}\right), \\ \Omega_m &= \frac{1}{\sigma_m} = \frac{\omega_m}{2(\psi_m + r\pi) + 3\pi}, \end{aligned} \tag{300}$$

for $r = 0, 1, 2, \dots$. For the case of $\xi = 0.0135$ one has that $\psi_m = -2.3495$. When $r = 0$, $\Omega_m = 0.2144$ ($\sigma_r = 4.6642$), which is the dimensionless rotation rate at the minimum of the first lobe to the right in Figure 3. This point is selected purely in order to illustrate the calculations. The stability limit in Figure 3 is given by Equation (60) as

$$p_m = 2\xi(\xi + 1) = 0.027365. \tag{301}$$

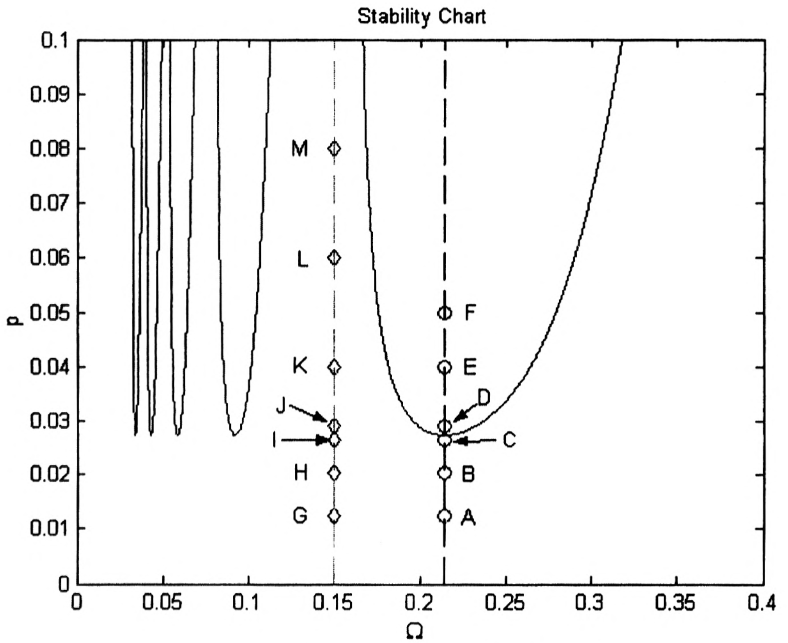

Figure 3. Locations of sample simulated solutions.

The frequency at this limit is given by Equation (61) as

$$\omega_m = \sqrt{1+2\xi} = 1.01341. \tag{302}$$

From Equation (90)

$$\begin{aligned} \alpha'(0) &= \frac{1}{2(1+\xi)^2(1+\xi\sigma_m)}, \\ \omega'(0) &= \frac{\sqrt{1+2\xi}}{2(1+\xi)^2(1+\xi\sigma_m)}. \end{aligned} \tag{303}$$

From Equation (147) the value of $\overline{d}_2$ can be calculated as

$$\overline{d}_2 = \frac{-\xi - i\sqrt{1+2\xi}}{2(1+\xi\sigma_c)(1+\xi)^2}, \tag{304}$$

where $\sigma = \sigma_m,\ p_c = p_m,\ \omega = \omega_m$ in Equation (147).

One can now compute $g_{20},\ g_{11},\ g_{02}$, and g_{21} from Equations (178) and (179). Then from Equation (228) one computes c_{21} and finally, from Equation (232), one can compute $\mu_2,\ \tau_2,\ \beta_2$ as

$$\begin{aligned} \mu_2 &= -0.09244, \\ \tau_2 &= 0.002330, \\ \beta_2 &= 0.08466. \end{aligned} \tag{305}$$

This implies that at the lobe boundary the DDE bifurcates into a family of unstable periodic solutions in a subcritical manner. Using Equation (289), one can compute the form of the bifurcating solutions for Equation (3) as

$$Z(s) = \begin{pmatrix} 2\varepsilon \cos 1.01341s + \varepsilon^2 ((-2.5968e-6) \cos 2.02682s - \\ \quad - 0.014632 \sin 2.02682s + 0.060113) \\ -2\varepsilon \sin 1.01341s + \varepsilon^2 (-0.02966 \cos 2.02682s + \\ \quad + (5.2632e-6) \sin 2.02682s) \end{pmatrix}. \tag{306}$$

As noted at the end of Section 4 one can take as an approximation

$$\varepsilon = \left(\frac{\mu}{\mu_2}\right)^{1/2}. \tag{307}$$

It is clear from Equation (305) that μ must be negative. Thus select

$$\varepsilon = \frac{(-\mu)^{1/2}}{0.3040395}. \tag{308}$$

The period of the solution can be computed as

$$T(\varepsilon) = \frac{2\pi}{\omega_m}\left(1 + \tau_2\varepsilon^2\right) = 6.2000421\left(1 + 0.002330\,\varepsilon^2\right) \tag{309}$$

and the characteristic exponent is given by

$$\beta = 0.08466\,\varepsilon^2. \tag{310}$$

The first of two sets of simulations was initialized at the points A through F in Figure 3 along the line $\Omega = 0.2144$ (selected for ease of calculation only). This line crosses the minimum of the first lobe in Figure 3. The simulations numerically demonstrate that there are three branches of periodic solutions emanating from the critical bifurcation point $p_m = 0.027365$. The three branches are shown in Figure 4. Two are unstable and one is stable in the following sense. The amplitude of the unstable subcritical branch is computed as two times Equation (308) (see Equation (306)). Solutions initialized below the subcritical branch converge to the zero solution. Those initialized above the subcritical branch grow in amplitude. The solutions initialized above zero for bifurcation parameter values greater than the critical value grow in amplitude. Similar results would be obtained along lines crossing at other critical points on the lobes.

The work of Kalmár-Nagy et al. [12] shows experimental evidence of a jump phenomena at the stability boundary that leads to a hysteresis effect with multivaludeness in the amplitude response curve. Simulations with the current model do not support this effect. The author conjectures that one possibility is that this may be due to the lack of structural nonlinearities in the model similar to those included in the model of Hanna and Tobias [15]. Another possibility is contact loss between the tool and the surface. This question requires further study.

The Hopf bifurcation result is very local around the boundary and only for very small initial amplitudes is it possible to track the unstable limit cycles along the branching amplitude curve. This is shown in Figure 5 where initial simulation functions were selected as constants with values along the approximate subcritical curve and the delay differential equation was integrated forward over five delay intervals. Note that nearer the critical bifurcation point the

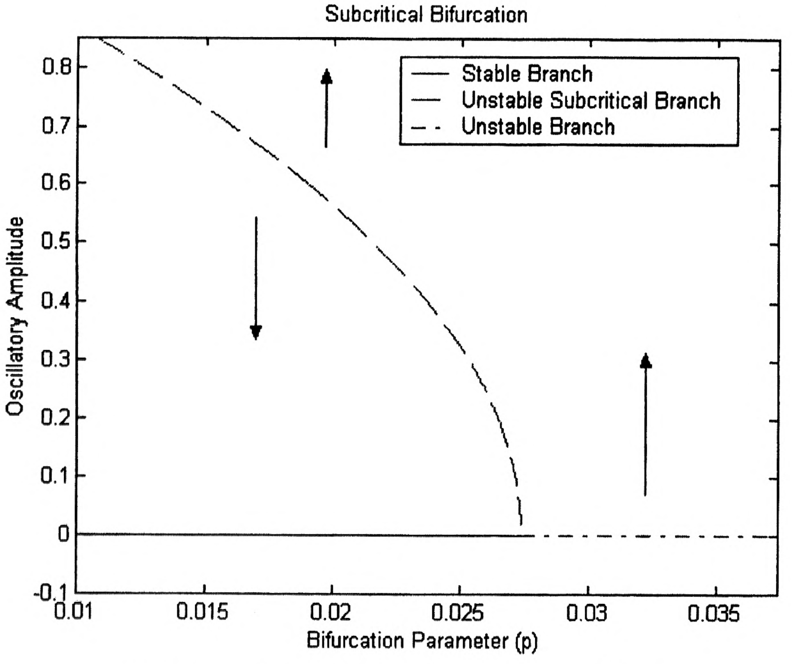

Figure 4. Amplitudes of bifurcating branches of solutions.

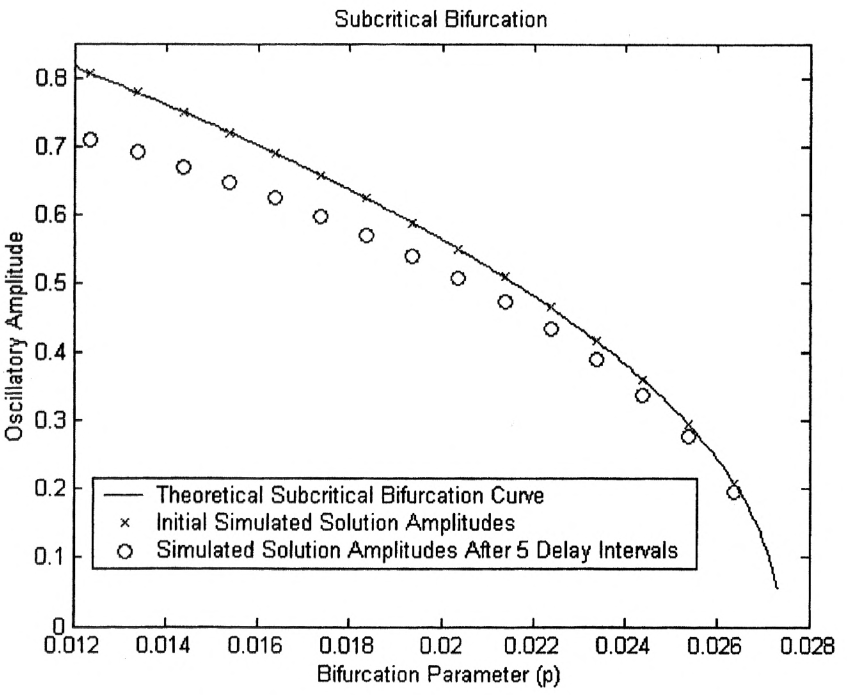

Figure 5. Theoretical and simulated subcritical solution amplitudes.

solution amplitude remains near the initial value, whereas further along the curve the solution amplitude drops away significantly from the subcritical curve.

Since the subcritical bifurcation curve in Figure 4 is an approximation to the true subcritical curve given by twice (308), solutions initialized on the curve, in this case, tend to decay to zero. This occurs at points A, B, and C in Figure 3. Some initial function values were selected just above the approximate subcritical curve in Figure 5 and the oscillation amplitudes also decayed, but when initial amplitudes were take significantly above the approximate curve the oscillations became unstable as expected. This result only points out that from the computational point of view the true subcritical curve falls somewhat above the approximate curve in Figure 5. However, on another choice of approximation curve the oscillations might grow in amplitude rather than decay, which shows the difficulty of finding the exact subcritical curve.

The decay at point B, when initialized on the subcritical is similar to the result at point A, but with a less rapid decay. However, when a solution is initialized above the curve at point B, given by $(\Omega, p) = (0.2144, 0.020365)$, the solution amplitude grows when initialized at an amplitude of 0.8 (Figure 5). At point C, when the solution is initialized on the subcritical bifurcation curve, the phase plot remains very close to a periodic orbit, indicating that the Hopf results are very local in being able to predict the unstable periodic solution. The behavior at points D, E, and F of Figure 3 are similar in that all of the solutions initialized above zero experience growth and eventually explode numerically.

The second set of simulations, initialized at points G through M along the line $\Omega = 0.15$, in Figure 3, shows the stability of solutions for parameters falling between lobes, in that the solutions of the delay differential equation (296) all decay to zero. The gaps between lobes are significant for machining. Since the parameter p is proportional to chip width, the larger the p value for which the system is stable the more material can be removed without chatter, where chatter can destroy the surface finish of the workpiece. These large gaps tend to appear between the lobes in high-speed machining with spindle rotation rates of the order of 20,000 RPM or greater.

Appendix: An Integration Method for DDE

The current initial value problem takes the general form

$$\begin{aligned}
&\dot{x} = f(x, x_\tau), \quad t \geq 0, \\
&x(0) = x_0, \\
&x(t) = \phi(t), \quad -\tau \leq t \leq 0,
\end{aligned} \tag{311}$$

where $x \in R^2$, $\phi(t)$ is the initial value function, which does not have to solve the equation but is assumed to be piecewise continuous and $x_\tau(t) = x(t - \tau)$. It also is not necessary that $\phi(0) = x_0$.

The general method used to integrate the differential equations is referred to as the 'method of steps' [51]. In this method the integration process begins with an initial function defined over an interval $[-\tau,0]$, where τ is the delay. Since the initial function is defined here the system

$$\dot{x}(t) = f(x(t), \phi(t - \tau)) \tag{312}$$

can be integrated by a standard method, such as a Runge–Kutta–Fehlberg (RKF) method on the interval $[0,\tau]$ [52]. The discrete values generated are saved and used to interpolate the delay terms during the integration over $[\tau, 2\tau]$. The process then continues for as many intervals as desired.

For simple Euler-type integrators the values of the solution can be stored at fixed step mesh points. These can easily be looked up when evaluating the delay term. However, the Euler methods are low precision algorithms. A high precision RKF, adaptive stepping algorithm, requires not only that the delay term be evaluated between previous mesh points, but that it must also be interpolated in case the value of the solution at a delay falls between mesh points. This happens both because of step length changes and because of the algorithm itself.

The one step method used in this study is based on the following algorithm

$$\begin{aligned} x_{n+1} &= x_n + h\Phi(t_n, x_n, z(t_n), h), \\ t_{n+1} &= t_n + h, \\ \dot{x}_{n+1} &= f(x(t_{n+1}), x(t_{n+1})), \end{aligned} \tag{313}$$

where

$$z(t) = \begin{cases} \phi(t-\tau), & 0 \le t < \tau, \\ x_0, & t = \tau, \\ P_q(t-\tau; (x_i); (\dot{x}_i)), & t > \tau, \end{cases} \tag{314}$$

where $P_q(t; (x_i); (\dot{x}_i))$ denotes the Hermite interpolation polynomial [52], of odd degree q, over the support points $(t_i, x_i, \dot{x}_i)$. The number of support points depends on the degree q. Φ will be defined below.

Introducing the interpolation changes the order of convergence of the RKF algorithm. It can be shown that the order of convergence of the resulting algorithm is the minimum of p, the order of the RKF algorithm, and q, the Hermite interpolation polynomial degree [53]. For example, for a fourth-order RKF algorithm the Hermite interpolation polynomial should be fourth degree or higher in order to maintain the proper global order of convergence.

For the fourth-order RKF algorithm the function Φ in Equation (313) is specified as

$$\Phi(t_n, x_n; z(t_n); h) = \sum_{j=1}^{5} \gamma_j k_j, \tag{315}$$

where

$$\begin{aligned} k_1 &= f(x_n, z(t_n)) \\ k_j &= f\left(x_n + h\sum_{i-1}^{j-1} \beta_{ji} k_i, z(t_n + \alpha_j h)\right) \end{aligned} \tag{316}$$

for $j = 2, \ldots, 6$. The truncation error in x_{n+1} is approximately

$$\text{truncerr} = h\sum_{j=1}^{6} c_j k_j. \tag{317}$$

The coefficients γ_j, α_j, c_j and β_j can be found in [52]. The interpolation polynomial for the current program is taken as a degree nine. This was done in order to be sure that no significant error accumulated while integrating over a large number of time delay intervals.

The Hermite interpolation polynomials can be written in terms of Newton divided difference coefficients [52]. Since both the value of the polynomial and its derivative are specified at n points this means that $2n$ conditions are imposed and one can look for an at most $2n-1$ degree polynomial. For a five point interpolation this means an at most ninth-order polynomial. To write the polynomial suppose that t falls within the mesh of points $t_{j-2}, t_{j-1}, t_j, t_{j+1}, t_{j+2}$. Let x_k and $\dot{x}_k$ be specified at $k = j-2, \ldots, j+2$. Then the unique polynomial of degree nine interpolating these points is given by

$$\begin{aligned} P_9(t) &= x(t_{j-2}) + (t - t_{j-2})x[t_{j-2}, t_{j-2}] \\ &\quad + (t - t_{j-2})^2 x[t_{j-2}, t_{j-2}, t_{j-1}] + \cdots \\ &\quad + (t - t_{j-2})^2 \ldots (t - t_{j+2})x[t_{j-2}, t_{j-2}, \ldots, t_{j+2}, t_{j+2}], \end{aligned} \tag{318}$$

where $x[t_{j-2}, t_{j-2}], \ldots, x[t_{j-2}, t_{j-2}, \ldots, t_{j+2}, t_{j+2}]$ are the divided difference coefficients.

The program developed for this study was based on the program RKF45 by Shampine and Watts [54]. A listing of an early version of RKF45 is given in [55]. The adaptive stepwise control of the program is based on selecting a step that ensures the truncation error for x_{n+1} satisfies

$$|\text{truncerr}| \leq \text{ABSERR} + \text{RELERR}|x_n|, \tag{319}$$

where ABSERR and RELERR are the desired absolute and relative error tolerances specified by the program user. The version of RKF45 in [55] was modified for this study in order to include the capability of saving state and derivative values at each of the time step mesh points during the current interval of integration. These are then stored for later look up. For each time step a fast search program finds the past interval in which the delay term falls. The program then uses the high-order Hermite interpolation routine to compute the state value at the delay time.

Acknowledgements

The author wishes to thank Drs. Tim Burns and Jon Pratt of the National Institute of Standards and Technology for their careful reading and criticisim of early versions of the manuscript. Thanks are also due to Professor Balakumar Balachandran of the University of Maryland, College Park, for the many helpful discussions he had with the author on the paper. Thanks are also due to the referees for their many helpful suggestions.

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