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ASYMPTOTIC APPROXIMATIONS OF INTEGRAL MANIFOLDS*

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Abstract. Multidegree of freedom nonlinear differential equations can often be transformed by means of the method of averaging into equivalent systems with only high-order terms. Under appropriate small-order perturbation conditions these systems have unique surfaces of solutions called integral manifolds. They generalize the notions of periodic and almost periodic solutions for single degree of freedom systems. In parametric form the integral manifolds satisfy a certain system of partial differential equations. Conversely, an Nth-order asymptotic integral manifold is defined as a formal solution of that system of partial differential equations, up to order N in the perturbation parameter. In the main result a system of integral equations is written for the remainder terms. By a contraction argument the system of integral equations has a fixed point, which, added to the Nth-order asymptotic integral manifold, forms an integral manifold for the normal system. By uniqueness this is the integral manifold sought. This implies that the unique integral manifold can be written as a formal series plus high order error terms. As an example a second order asymptotic representation for the periodic solution of a van der Pol oscillator is then developed.

Key words. asymptotic approximation, averaging, differential equations, fixed points, integral manifolds, invariant tori, near-identity transformations, nonlinear differential equations, van der Pol equation

AMS(MOS) subject classifications. 34A34, 34A45, 34C05, 34C15, 34C29, 34C30, 34C45, 34E05, 34E10, 47H10, 47H15

1. Introduction. Multidegree of freedom nonlinear differential equations can often be transformed into systems of the form

$$\begin{aligned} \dot{\theta} &= d(\varepsilon)+\Theta(t, \theta, y, z, \varepsilon), \\ \dot{y} &= Ay+Y(t, \theta, y, z, \varepsilon), \\ \dot{z} &= \varepsilon Cz+\varepsilon Z(t, \theta, y, z, \varepsilon). \end{aligned} \tag{1.1}$$

The existence of integral manifolds for (1.1) is well known (see Bogoliubov and Mitropolsky [1, p. 466] and Hale [9, p. 136], [8]). However, the usual fixed point proofs do not convert easily to direct approximation algorithms. Diliberto and his coauthors [4], [5] present an indirect technique to approximate invariant tori but give no convergence argument. To the author's knowledge no direct approximation method has yet been shown to converge.

In this paper we define a formal integral manifold as an approximate solution to an appropriate system of first order partial differential equations. As we illustrate in an example in § 4, this definition leads to a direct algorithm for generating the formal approximation.

In our main theorem we show that, under certain conditions on (1.1), if a formal approximation to an integral manifold for (1.1) exists, then it converges asymptotically. Using Hale's technique [8] this result extends to integral manifolds for (1.1) an asymptotic result for center stable manifolds in Carr [2, p. 25].

2. Statement of the main theorem. For the rest of this paper we assume that (1.1) satisfies:

(H1) $t\in R$; θ, $d(\varepsilon)\in E^k$; $y\in E^m$; $z\in E^n$; $\varepsilon\in[0, \varepsilon_0]$.

(H2) Θ, Y, Z map $R\times E^k\times E^m\times E^n\times[0, \varepsilon_0]$ to E^k, E^m, E^n, respectively.

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(H3) Θ, Y, Z are periodic in θ with vector period $\omega=(\omega_1,\cdots,\omega_k)$.

(H4) There exists a function $M(\varepsilon)\geqq 0$, $M(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that

$$
\begin{aligned}
&|\Theta(t,\theta,0,0,\varepsilon)|\leqq M(\varepsilon),\\
&|Y(t,\theta,0,0,\varepsilon)|\leqq M(\varepsilon),\\
&|Z(t,\theta,0,0,\varepsilon)|\leqq M(\varepsilon).
\end{aligned}
\tag{2.1}
$$

(H5) There exist continuous functions $\eta(\varepsilon,\sigma,\mu)$, $\lambda(\varepsilon,\sigma,\mu)$, where $(\varepsilon,\sigma,\mu)\in[0,\varepsilon_0]\times[0,\sigma_0]\times[0,\mu_0]$ such that $\eta(\varepsilon,\sigma,\mu)$ and $\lambda(\varepsilon,\sigma,\mu)\to 0$ as $(\varepsilon,\sigma,\mu)\to(0,0,0)$, $\eta(\varepsilon,0,0)=o(\varepsilon)$ and for $(\theta,y,z),(\theta',y',z')\in\{(\theta,y,x)\colon\theta\in E^k,|y|\leqq\sigma,|z|\leqq\mu\}$,

$$
\begin{aligned}
&|\Theta(t,\theta,y,z,\varepsilon)-\Theta(t,\theta',y',z',\varepsilon)|\leqq\eta(\varepsilon,\sigma,\mu)\{|\theta-\theta'|+|y-y'|+|z-z'|\},\\
&|Y(t,\theta,y,z,\varepsilon)-T(t,\theta',y',z',\varepsilon)|\leqq\lambda(\varepsilon,\sigma,\mu)\{|\theta-\theta'|+|y-y'|+|z-z'|\},\\
&|Z(t,\theta,y,z,\varepsilon)-Z(t,\theta',y',z',\varepsilon)|\leqq\lambda(\varepsilon,\sigma,\mu)\{|\theta-\theta'|+|y-y'|+|z-z'|\}.
\end{aligned}
\tag{2.2}
$$

(H6) A is an $m\times m$ matrix, C an $n\times n$ matrix, both with eigenvalues having nonzero real parts.

If (1.1) does not satisfy these conditions, the method of averaging can often be used to transform (1.1) into a similar system satisfying (H1)-(H6) (Hale [8]). We will illustrate this in the example in § 4.

Let D, Δ be fixed positive numbers. We will say that $f\in P_n^\omega(\Delta,D)$ provided that (1) $f\colon R\times E^k\times[0,\varepsilon_0]\to E^n$, (2) f is multiply periodic in θ with vector period $\omega=(\omega_1,\cdots,\omega_k)$, $\omega_i>0$, $i=1,\cdots,k$, and (3) f satisfies $|f(t,\theta,\varepsilon)|\leqq D$ and $|f(t,\theta,\varepsilon)-f(t,\theta',\varepsilon)|\leqq\Delta|\theta-\theta'|$ for $\theta,\theta'\in E^k$.

The norm for $P_n^\omega(\Delta,D)$ will be $\|f\|=\sup_{t,\theta}|f(t,\theta,\varepsilon)|$. For $f,g\in P_n^\omega(\Delta,D)$, $\|f-g\|$ gives the natural metric.

From (H6) we can assume A and C have the form

$$A=\operatorname{diag}(A_+,A_-),\qquad C=\operatorname{diag}(C_+,C_-)\tag{2.3}$$

where the eigenvalues of A_+, C_+ have positive real parts and the eigenvalues of A_-, C_- have negative real parts.

Define the matrices $J(t)$, $K(t)$ by

$$
\begin{aligned}
&J(t)=-\operatorname{diag}(\exp(-A_+t),0),\quad K(t)=-\operatorname{diag}(\exp(-C_+t),0),\quad t>0,\\
&J(t)=\operatorname{diag}(0,\exp(-A_-t)),\qquad K(t)=\operatorname{diag}(0,\exp(-C_-t)),\qquad t<0.
\end{aligned}
\tag{2.4}
$$

They satisfy

$$J(-0)-J(+0)=I,\qquad K(-0)-K(+0)=I,\tag{2.5}$$

$$|J(t)|\leqq\beta\exp(-\alpha|t|),\qquad|K(t)|\leqq\beta\exp(-\alpha|t|),\tag{2.6}$$

for constants $\alpha,\beta>0$, and

$$\dot J=-AJ=-JA,\quad \dot K=-CK=-KC,\quad t\neq 0.\tag{2.7}$$

Let $f\in P_m^\omega(\Delta,D)$, $g\in P_n^\omega(\Delta,D)$ and the unique solution of

$$\dot\theta=d(\varepsilon)+\Theta(t,\theta,f(t,\theta,\varepsilon),g(t,\theta,\varepsilon),\varepsilon),\tag{2.8}$$

with $\theta(t_0)=\theta^0$, be denoted by

$$\theta(t)=T_{x,t_0}^{f,g}(\theta^0),\quad x=t-t_0,\quad\theta(t_0)=\theta^0.\tag{2.9}$$

This exists for all $t \in (-\infty, \infty)$ (see Cronin [3, p. 53]). The solution specified in (2.9) satisfies the following:

$$T^{f,g}_{x,t_0}(\theta^0 + \omega) = T^{f,g}_{x,t_0}(\theta^0) + \omega, \tag{2.10}$$

$$T^{f,g}_{z,t}(T^{f,g}_{t-t_0,t_0}(\theta)) = T^{f,g}_{z+t-t_0,t_0}(\theta). \tag{2.11}$$

The next lemma gives conditions for the existence of a unique integral manifold for (1.1) (see Hale [8, p. 507]) and will be used to show that our formal approximation is asymptotic to a unique integral manifold.

LEMMA 2.1. *Let* (1.1) *satisfy* (H1)-(H6) *and* $(f, g) \in P^{\omega}_m(\Delta, D) \times P^{\omega}_n(\Delta, D)$. *Define the transformation*

$$\begin{aligned} &(T_1(f, g))(t, \theta, \varepsilon) \\ &\quad = \int_{-\infty}^{\infty} J(z) Y[z+t, T^{f,g}_{z,t}(\theta), f(z+t, T^{f,g}_{z,t}(\theta), \varepsilon), g(z+t, T^{f,g}_{z,t}(\theta), \varepsilon), \varepsilon]\, dz, \\ &(T_2(f, g))(t, \theta, \varepsilon) \\ &\quad = \varepsilon \int_{-\infty}^{\infty} K(\varepsilon z) Z[z+t, T^{f,g}_{z,t}(\theta), f(z+t, T^{f,g}_{z,t}(\theta), \varepsilon), g(z+t, T^{f,g}_{z,t}(\theta), \varepsilon), \varepsilon]\, dz. \end{aligned} \tag{2.12}$$

Let $\alpha, \beta > 0$ *be constants such that* (2.6) *holds. Then, if there exists an* $\varepsilon_0 > 0$ *and two functions* $D_0(\varepsilon)$, $\Delta_0(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *such that for* $0 < \varepsilon \leqq \varepsilon_0 < 1$,

$$\eta(\varepsilon, D_0(\varepsilon), D_0(\varepsilon))(1 + 2\Delta_0(\varepsilon)) < \frac{\varepsilon\alpha}{2} < \frac{\alpha}{2}, \tag{2.13a}$$

$$\frac{4\beta}{\alpha} \lambda(\varepsilon, D_0(\varepsilon), D_0(\varepsilon))(1 + 2\Delta_0(\varepsilon)) < \Delta_0(\varepsilon), \tag{2.13b}$$

$$\frac{4\beta}{\alpha} \lambda(\varepsilon, D_0(\varepsilon), D_0(\varepsilon)) < \frac{1}{4}, \tag{2.13c}$$

$$\frac{2\beta}{\alpha} [M(\varepsilon) + 2\lambda(\varepsilon, D_0(\varepsilon), D_0(\varepsilon)) D_0(\varepsilon)] \leqq D_0(\varepsilon), \tag{2.13d}$$

the transformation $T(f, g) = (T_1(f, g), T_2(f, g))$ *has a unique fixed point* $(f^*, g^*) \in P^{\omega}_m(\Delta(\varepsilon), D(\varepsilon)) \times P^{\omega}_n(\Delta(\varepsilon), D(\varepsilon))$, *and* (f^*, g^*) *is a unique integral manifold for* (1.1).

Let $(t, \theta, \varepsilon) \in (-\infty, \infty) \times E^k \times (0, \varepsilon_0]$. $y = F(t, \theta, \varepsilon)$, $z = G(t, \theta, \varepsilon)$, $y \in E^m$, $z \in E^n$, is said to be an *Nth order asymptotic integral manifold* for (1.1) provided

(A1) $F(t, \theta, \varepsilon)$, $G(t, \theta, \varepsilon)$ are periodic in θ, vector period $\omega = (\omega_1, \cdots, \omega_k)$, $\omega_i > 0$, $i = 1, \cdots, k$.

(A2) F, G have continuous partial derivatives of the first order with respect to t, θ.

(A3) There exist two functions $U(\varepsilon)$, $W(\varepsilon)$ defined for $\varepsilon \in (0, \varepsilon_0]$ such that $U(\varepsilon)$ and $W(\varepsilon) \to 0$ as $\varepsilon \to 0$. Furthermore, $W(\varepsilon) = o(\varepsilon)$, and, if D_1, D_2 represent the matrices of partial derivatives with respect to t, θ, then for all $(t, \theta) \in (-\infty, \infty) \times E^k$

$$\begin{aligned} &|F(t, \theta, \varepsilon)| \leqq U(\varepsilon), \quad |D_1 F(t, \theta, \varepsilon)| \leqq U(\varepsilon), \quad |D_2 F(t, \theta, \varepsilon)| \leqq U(\varepsilon), \\ &|G(t, \theta, \varepsilon)| \leqq W(\varepsilon), \quad |D_1 G(t, \theta, \varepsilon)| \leqq W(\varepsilon), \quad |D_2 G(t, \theta, \varepsilon)| \leqq W(\varepsilon). \end{aligned} \tag{2.14}$$

(A4) There exist two functions $\gamma(\varepsilon)$, $\delta(\varepsilon)$, defined for $\varepsilon \in (0, \varepsilon_0]$ such that

$\gamma(\varepsilon), \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\delta(\varepsilon) = o(\varepsilon)$. Furthermore, if $\theta, \theta' \in E^k$, then

$$(2.15)\qquad \begin{aligned} &|F(t,\theta,\varepsilon) - F(t,\theta',\varepsilon)| \leqq \gamma(\varepsilon)|\theta - \theta'|, \\ &|D_1F(t,\theta,\varepsilon) - D_1F(t,\theta',\varepsilon)| \leqq \gamma(\varepsilon)|\theta - \theta'|, \qquad i = 1, 2, \\ &|G(t,\theta,\varepsilon) - G(t,\theta',\varepsilon)| \leqq \delta(\varepsilon)|\theta - \theta'|, \\ &|D_1G(t,\theta,\varepsilon) - D_1G(t,\theta',\varepsilon)| \leqq \delta(\varepsilon)|\theta - \theta'|, \qquad i = 1, 2. \end{aligned}$$

(A5) If $R(F, G)$, $S(F, G)$ are defined by

$$(2.16)\qquad \begin{aligned} (R(F,G))(t,\theta,\varepsilon) = {} & D_1F(t,\theta,\varepsilon) + D_2F(t,\theta,\varepsilon) \\ & \cdot [d(\varepsilon) + \Theta(t,\theta,F(t,\theta,\varepsilon),G(t,\theta,\varepsilon),\varepsilon)] \\ & - AF(t,\theta,\varepsilon) - Y(t,\theta,F(t,\theta,\varepsilon),G(t,\theta,\varepsilon),\varepsilon), \\ (S(F,G))(t,\theta,\varepsilon) = {} & D_1G(t,\theta,\varepsilon) + D_2G(t,\theta,\varepsilon) \\ & \cdot [d(\varepsilon) + \Theta(t,\theta,F(t,\theta,\varepsilon),G(t,\theta,\varepsilon),\varepsilon)] \\ & - \varepsilon CG(t,\theta,\varepsilon) - \varepsilon Z(t,\theta,F(t,\theta,\varepsilon),G(t,\theta,\varepsilon),\varepsilon), \end{aligned}$$

then there exists a constant $B > 0$ such that

$$(2.17)\qquad |(R(F,G))(t,\theta,\varepsilon)| \leqq B\varepsilon^N, \qquad |(S(F,G))(t,\theta,\varepsilon)| \leqq B\varepsilon^{N+1}.$$

We can now state the main theorem as follows:

THEOREM 2.1. *Let* (1.1) *satisfy* (H1)-(H6), *and* (F, G) *be an Nth order asymptotic integral manifold for the given system* (1.1). *Then there exist two functions* $D_0(\varepsilon)$, $\Delta_0(\varepsilon) \to 0$, *as* $\varepsilon \to 0$, *and an* $\varepsilon^* > 0$ *such that for* $0 < \varepsilon \leqq \varepsilon^* < 1$ *there exists a unique integral manifold for* (1.1) *defined by* $y = f(t,\theta,\varepsilon)$, $z = g(t,\theta,\varepsilon)$, *periodic in* θ *with vector period* ω, $r = r(t,\theta,\varepsilon)$, $s = s(t,\theta,\varepsilon)$, *periodic in* θ *with vector period* ω, *and a constant* $K > 0$ *such that* $(r, s) \in P_m^\omega(\Delta_0(\varepsilon), K\varepsilon^N) \times P_n^\omega(\Delta_0(\varepsilon), K\varepsilon^N)$ *and*

$$(2.18)\qquad f(t,\theta,\varepsilon) = F(t,\theta,\varepsilon) + r(t,\theta,\varepsilon), \qquad g(t,\theta,\varepsilon) = G(t,\theta,\varepsilon) + s(t,\theta,\varepsilon).$$

3. Proof of the main result. Let α, β be given in (2.7), B in (2.17). Then pick K as the fixed value

$$(3.1)\qquad K = 4B\beta\alpha^{-1}.$$

Let Δ, ε, k be fixed and positive, $N \geqq 1$. Then $P_m^\omega(\Delta, K\varepsilon^N) \times P_n^\omega(\Delta, K\varepsilon^N)$ is a complete metric space with the metric

$$(3.2)\qquad \|(r_1, s_1), (r_2, s_2)\| = \|r_1 - r_2\| + \|s_1 - s_2\|,$$

for $r_1, r_2 \in P_m^\omega(\Delta, K\varepsilon^N)$ and $s_1, s_2 \in P_n^\omega(\Delta, K\varepsilon^N)$. Furthermore, let (F, G) be the Nth order asymptotic integral manifold of (1.1), and $(r, s) \in P_m^\omega(\Delta, K\varepsilon^N) \times P_n^\omega(\Delta, K\varepsilon^N)$. From the definition of (F, G) the following is always true:

$$(3.3)\qquad (F + r, G + s) \in P_m^\omega(\gamma(\varepsilon) + \Delta, U(\varepsilon) + K\varepsilon^N) \times P_n^\omega(\delta(\varepsilon) + \Delta, W(\varepsilon) + K\varepsilon^N).$$

Let $M(\varepsilon)$ be given in (2.1). Then select the functions $D_0(\varepsilon)$ and $D(\varepsilon)$ so that

$$(3.4)\qquad D_0(\varepsilon) = D(\varepsilon) = \max\{U(\varepsilon) + K\varepsilon^N, W(\varepsilon) + K\varepsilon^N, 4\beta\alpha^{-1}M(\varepsilon)\}.$$

We can assume $D(\varepsilon) \leqq \min\{\sigma_0, \mu_0\}$ for $\varepsilon \in [0, \varepsilon_0]$, where σ_0, μ_0 come from (H5).

From (2.9) the unique solution of

$$(3.5)\qquad \dot\theta = d(\varepsilon) + \Theta(t,\theta,F(t,\theta,\varepsilon) + r(t,\theta,\varepsilon), G(t,\theta,\varepsilon) + s(t,\theta,\varepsilon),\varepsilon).$$

$\theta(t_0)=\theta^0$, is given for $t\in(-\infty,\infty)$ by

$$\theta(t)=T^{F+r,G+s}_{x,t_0}(\theta^0),\quad x=t-t_0,\quad \theta(t_0)=\theta^0, \tag{3.6}$$

and satisfies

$$T^{F+r,G+s}_{x,t_0}(\theta^0+\omega)=T^{F+r,G+s}_{x,t_0}(\theta^0)+\omega,\qquad T^{F+r,G+s}_{z,t}(T_{t-t_0,t_0}(\theta))=T^{F+r,G+s}_{z+t-t_0,t_0}(\theta). \tag{3.7}$$

In order to simplify the notation slightly, let

$$\hat\theta(x)=\theta(x+t_0)=\theta(t). \tag{3.8}$$

Two solutions of (5.9) can be compared. In fact, if (r,s), $(r^*,s^*)\in P^\omega_m(\Delta,K\varepsilon^N)\times P^\omega_m(\Delta,K\varepsilon^N)$ and $\theta^*(t)$, $\theta(t)$ are the unique solutions of

$$\dot\theta^*=d(\varepsilon)+\Theta(t,\theta^*,F(t,\theta^*,\varepsilon)+r^*(t,\theta^*,\varepsilon),G(t,\theta^*,\varepsilon)+s^*(t,\theta^*,\varepsilon),\varepsilon),$$

$$\dot\theta=d(\varepsilon)+\Theta(t,\theta,F(t,\theta,\varepsilon)+r(t,\theta,\varepsilon),G(t,\theta,\varepsilon)+s(t,\theta,\varepsilon),\varepsilon),$$

with $\theta^*(t_0)=\theta^0$, $\theta(t_0)=\theta^0$, then (see Bogoliubov and Mitropolsky [1, pp. 469-470])

$$\begin{aligned}|\theta^*(t)-\theta(t)|&\\ \le|\theta^*-\theta^0|&\exp(\eta(\varepsilon,D(\varepsilon),D(\varepsilon))(1+\gamma(\varepsilon)+\delta(\varepsilon)+2\Delta)|x|)\\ &+\frac{\|(r^*,s^*),(r,s)\|}{1+\gamma(\varepsilon)+\delta(\varepsilon)+2\Delta}[\exp(\eta(\varepsilon,D(\varepsilon),D(\varepsilon))(1+\gamma(\varepsilon)+\delta(\varepsilon)+2\Delta)|x|)-1]\end{aligned} \tag{3.9}$$

where $x=t-t_0$.

Before constructing a mapping on $P^\omega_m(\Delta,K\varepsilon^N)\times P^\omega_n(\Delta,K\varepsilon^N)$ we state some identities. Let J, K be the matrices defined in (2.4). Then, using (3.8),

$$F(t,\theta,\varepsilon)=\int_{-\infty}^{\infty}\frac{d}{dz}[J(z)F(z+t,\hat\theta(z),\varepsilon)]\,dz, \tag{3.10}$$

$$G(t,\theta,\varepsilon)=\int_{-\infty}^{\infty}\frac{d}{dz}[K(\varepsilon z)G(z+t,\hat\theta(z),\varepsilon)]\,dz, \tag{3.11}$$

$$\frac{d}{dz}[J(z)F(z+t,\hat\theta(z),\varepsilon)]=-J(z)\left[AF(z+t,\hat\theta(z),\varepsilon)-\frac{d}{dz}F(z+t,\hat\theta(z),\varepsilon)\right], \tag{3.12}$$

$$\begin{aligned}&\frac{d}{dz}[K(\varepsilon z)G(z+t,\hat\theta(z),\varepsilon)]\\ &\quad=-K(\varepsilon z)\left[\varepsilon CG(z+t,\hat\theta(z),\varepsilon)-\frac{d}{dz}G(z+t,\hat\theta(z),\varepsilon)\right],\end{aligned} \tag{3.13}$$

$$\frac{d}{dz}F(z+t,\hat\theta(z),\varepsilon)=D_1F(z+t,\hat\theta(z),\varepsilon)+D_2F(z+t,\hat\theta(z),\varepsilon)\frac{d}{dz}\hat\theta(z), \tag{3.14}$$

$$\frac{d}{dz}G(z+t,\hat\theta(z),\varepsilon)=D_1G(z+t,\hat\theta(z),\varepsilon)+D_2G(z+t,\hat\theta(z),\varepsilon)\frac{d}{dz}\hat\theta(z) \tag{3.15}$$

where

$$\frac{d}{dz}\hat{\theta}(z)=d(\varepsilon)+\Theta(z+t,\hat{\theta}(z),F(z+t,\hat{\theta}(z),\varepsilon)+r(z+t,\hat{\theta}(z),\varepsilon),G(z+t,\hat{\theta}(z),\varepsilon)+s(z+t,\hat{\theta}(z),\varepsilon),\varepsilon). \tag{3.16}$$

Identities (3.10) and (3.11) follow by breaking the integrals at 0 and using (2.5). The others follow by differentiation.

Let T_1, T_2 be the transformation defined by (2.12). Then, for $(r,s)\in P_m^\omega(\Delta,K\varepsilon^N)\times P_n^\omega(\Delta,K\varepsilon^N)$ consider the mapping

$$E(r,s)=(E_1(r,s),E_2(r,s)) \tag{3.17}$$

where

$$\begin{aligned}(E_1(r,s))(t,\theta,\varepsilon)&=(T_1(F+r,G+s))(t,\theta,\varepsilon)-F(t,\theta,\varepsilon),\\(E_2(r,s))(t,\theta,\varepsilon)&=(T_2(F+r,G+s))(t,\theta,\varepsilon)-G(t,\theta,\varepsilon).\end{aligned} \tag{3.18}$$

E is well defined and can be written in integral form by combining (2.12), (2.16) and (3.10)-(3.18) to get

$$\begin{aligned}&(E_1(r,s))(t,\theta,\varepsilon)\\&\quad=\int_{-\infty}^{\infty}J(z)\Psi_1(z+t,\hat{\theta}(z),r(z+t,\hat{\theta}(z),\varepsilon),s(z+t,\hat{\theta}(z),\varepsilon),\varepsilon)\,dz,\\&(E_2(r,s))(t,\theta,\varepsilon)\\&\quad=\int_{-\infty}^{\infty}K(\varepsilon z)\Psi_2(z+t,\hat{\theta}(z),r(z+t,\hat{\theta}(z),\varepsilon),s(z+t,\hat{\theta}(z),\varepsilon),\varepsilon)\,dz\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\Psi_1(t,\theta,r,s,\varepsilon)&=\{Y(t,\theta,F+r,G+s,\varepsilon)-Y(t,\theta,F,G,\varepsilon)\}-(R(F,G))(t,\theta,\varepsilon)\\&\quad-\{D_2F(t,\theta,\varepsilon)[\Theta(t,\theta,F+r,G+s,\varepsilon)-\Theta(t,\theta,F,G,\varepsilon)]\},\\\Psi_2(t,\theta,r,s,\varepsilon)&=\varepsilon\{Z(t,\theta,F+r,G+s,\varepsilon)-Z(t,\theta,F,G,\varepsilon)\}-(S(F,G))(t,\theta,\varepsilon)\\&\quad-\{D_2G(t,\theta,\varepsilon)[\Theta(t,\theta,F+r,G+s,\varepsilon)-\Theta(t,\theta,F,G,\varepsilon)]\}.\end{aligned} \tag{3.20}$$

Equations (3.7) imply that E maps pairs of functions, periodic in θ with vector period ω, to pairs of functions with the same periodicity property.

From (H1)-(H6) and (A1)-(A5) and the definition (3.20) it is algebraically cumbersome but not hard to show that there exist functions $L_1(\varepsilon)$, $L_2(\varepsilon)$, $L_3(\varepsilon)$, $L_4(\varepsilon)\to 0$ as $\varepsilon\to 0$, with $L_3(\varepsilon)$, $L_4(\varepsilon)=o(\varepsilon)$ as $\varepsilon\to 0$, such that

$$2\lambda(\varepsilon,D(\varepsilon),D(\varepsilon))\leqq L_1(\varepsilon), \tag{3.21a}$$

$$2\varepsilon\lambda(\varepsilon,D(\varepsilon),D(\varepsilon))\leqq L_3(\varepsilon), \tag{3.21b}$$

$$\begin{aligned}&|\Psi_1(t,\theta^*,r^*,s^*,\varepsilon)-\Psi_1(t,\theta^0,r,s,\varepsilon)|\\&\quad\leqq(L_1(\varepsilon)+\Delta L_2(\varepsilon))|\theta^*-\theta^0|+L_2(\varepsilon)\|(r^*,s^*),(r,s)\|,\end{aligned} \tag{3.21c}$$

$$\begin{aligned}&|\Psi_2(t,\theta^*,r^*,s^*,\varepsilon)-\Psi_2(t,\theta^0,r,s,\varepsilon)|\\&\quad\leqq(L_3(\varepsilon)+\Delta L_2(\varepsilon))|\theta^*-\theta^0|+L_4(\varepsilon)\|(r^*,s^*),(r,s)\|.\end{aligned} \tag{3.21d}$$

From (3.20) and conditions (H5) and (A2)-(A5) there exist two functions $B_1(\varepsilon)$, $B_2(\varepsilon)\to 0$ as $\varepsilon\to 0$, $B_2(\varepsilon)=o(\varepsilon)$, such that

(3.22) $$|\Psi_1(t,\theta,r,s,\varepsilon)|\leqq(B_1(\varepsilon)+B)\varepsilon^N, \qquad |\Psi_2(t,\theta,r,s,\varepsilon)|\leqq(B_2(\varepsilon)+\varepsilon B)\varepsilon^N$$

where $B>0$ is the constant in (A5).

Let $\eta(\varepsilon,\sigma,\mu)$ be given in (H5) and $\alpha>0$ from (2.6). By uniform continuity of $\eta(\varepsilon,\sigma,\mu)$ on $[0,\varepsilon_0]\times[0,\sigma_0]\times[0,\mu_0]$ there exists $\delta_0>0$ such that

(3.23) $$|\eta(\varepsilon,\sigma,\mu)-\eta(\varepsilon,0,0)|\leqq\frac{\varepsilon\alpha}{8}$$

for $(\sigma,\mu)\in[0,\delta_0]\times[0,\delta_0]$.

Select the functions

(3.24) $$\Delta(\varepsilon)=8\beta\alpha^{-1}\max\{L_1(\varepsilon),\varepsilon^{-1}L_3(\varepsilon)\}, \qquad \Delta_0(\varepsilon)=\Delta(\varepsilon)+\gamma(\varepsilon)+\delta(\varepsilon),$$

and select $\varepsilon^*\in(0,1)$ so that if $\varepsilon\in(0,\varepsilon^*)$, then

(3.25a) $$D(\varepsilon)\leqq\delta_0,$$

(3.25b) $$\eta(\varepsilon,0,0)<\frac{\varepsilon\alpha}{8}<\frac{\alpha}{8},$$

(3.25c) $$1+2(\gamma(\varepsilon)+\delta(\varepsilon)+2\Delta(\varepsilon))\leqq 2,$$

(3.25d) $$4\beta\alpha^{-1}\max\{L_2(\varepsilon),\varepsilon^{-1}L_4(\varepsilon)\}\leqq\tfrac{1}{2},$$

(3.25e) $$\max\{B_1(\varepsilon),\varepsilon^{-1}B_2(\varepsilon)\}\leqq B,$$

(3.25f) $$2\beta\alpha^{-1}\{L_1(\varepsilon)+(1+\Delta(\varepsilon))L_2(\varepsilon)\}\leqq\tfrac{1}{4},$$

(3.25g) $$\varepsilon^{-1}2\beta\alpha^{-1}\{L_3(\varepsilon)+(1+\Delta(\varepsilon))L_4(\varepsilon)\}\leqq\tfrac{1}{4},$$

(3.25h) $$\lambda(\varepsilon,y,z)\leqq\frac{\alpha}{16\beta}.$$

From (3.23) and (3.25a)-(3.25c) we have

(3.26a) $$-\alpha+\eta(\varepsilon,D(\varepsilon),D(\varepsilon))(1+2(\Delta(\varepsilon)+\gamma(\varepsilon)+\delta(\varepsilon)))\leqq-\frac{\alpha}{2},$$

(3.26b) $$-\varepsilon\alpha+\eta(\varepsilon,D(\varepsilon),D(\varepsilon))(1+2(\Delta(\varepsilon)+\gamma(\varepsilon)+\delta(\varepsilon)))\leqq-\frac{\varepsilon\alpha}{2}.$$

The conditions of Lemma 2.1 are now satisfied. In fact, (3.24), (3.25a) and (3.25b) imply (2.13a). Formulae (3.21a), (3.24) and (3.25h) imply that

(3.27) $$\left[\frac{4\beta}{\alpha}\lambda(\varepsilon,D(\varepsilon),D(\varepsilon))\right]\left[1-\frac{8\beta}{\alpha}\lambda(\varepsilon,D(\varepsilon),D(\varepsilon))\right]^{-1}\leqq\Delta(\varepsilon).$$

Furthermore, (3.25h) implies that

(3.28) $$\frac{8\beta}{\alpha}\lambda(\varepsilon,D(\varepsilon),D(\varepsilon))(\gamma(\varepsilon)+\delta(\varepsilon))<\gamma(\varepsilon)+\delta(\varepsilon).$$

Then, (2.13b) follows from (3.27) and (3.28). Formula (2.13c) is equivalent to (3.25h). Finally (3.25h) and (3.4) imply (2.13d).

Lemma 2.1 states that there is a unique integral manifold for (1.1) in $P_m^\omega(\Delta_0(\varepsilon), D_0(\varepsilon)) \times P_n^\omega(\Delta_0(\varepsilon), D_0(\varepsilon))$, where $\Delta_0(\varepsilon)$ and $D_0(\varepsilon)$ are defined in (3.24) and (3.4), respectively. We must now show that this is the integral manifold for which (F, G) is an asymptotic approximation.

We show that E has a fixed point. First, E_1 and E_2 map $O(\varepsilon^N)$ functions to $O(\varepsilon^N)$ functions. From (3.1), (3.19), (3.22) and (3.25e) we have

$$|(E_1(r, s))(t, \theta, \varepsilon)| \leqq 2\beta\alpha^{-1}[B_1(\varepsilon) + B]\varepsilon^N \leqq K\varepsilon^N, \tag{3.29}$$

$$|(E_2(r, s))(t, \theta, \varepsilon)| \leqq 2\beta\alpha^{-1}[\varepsilon^{-1}B_2(\varepsilon) + B]\varepsilon^N \leqq K\varepsilon^N. \tag{3.30}$$

E_1 and E_2 satisfy the Lipschitz and contraction conditions. From (2.6), (3.9), (3.19), (3.21c) and (3.26a) we have

$$\begin{aligned} &|(E_1(r^*, s^*))(t, \theta^*, \varepsilon) - (E_1(r, s))(t, \theta^0, \varepsilon)| \\ &\qquad \leqq 4\beta\alpha^{-1}(L_1(\varepsilon) + \Delta(\varepsilon)L_2(\varepsilon))|\theta^* - \theta^0| \\ &\qquad\quad + 2\beta\alpha^{-1}\{L_1(\varepsilon) + (1 + \Delta(\varepsilon))L_2(\varepsilon)\}\|(r^*, s^*), (r, s)\|. \end{aligned} \tag{3.31}$$

Similarly, using (3.26b), we have

$$\begin{aligned} &|(E_2(r^*, s^*))(t, \theta^*, \varepsilon) - (E_2(r, s))(t, \theta^0, \varepsilon)| \\ &\quad \leqq 4\beta\alpha^{-1}\{\varepsilon^{-1}L_3(\varepsilon) + \varepsilon^{-1}\Delta(\varepsilon)L_4(\varepsilon)\}|\theta^* - \theta^0| \\ &\quad\quad + 2\beta\alpha^{-1}\{\varepsilon^{-1}L_3(\varepsilon) + \varepsilon^{-1}(1 + \Delta(\varepsilon))L_4(\varepsilon)\}\|(r^*, s^*), (r, s)\|. \end{aligned} \tag{3.32}$$

Formulae (3.24), (3.25d), (3.25f), (3.25g), (3.31) and (3.32) imply that

$$\begin{aligned} |(E_1(r^*, s^*))(t, \theta^*, \varepsilon) - (E_1(r, s))(t, \theta^0, \varepsilon)| \leqq \Delta(\varepsilon)|\theta^* - \theta^0| + \tfrac{1}{4}\|(r^*, s^*), (r, s)\|, \\ |(E_2(r^*, s^*))(t, \theta^*, \varepsilon) - (E_2(r, s))(t, \theta^0, \varepsilon)| \leqq \Delta(\varepsilon)|\theta^* - \theta^0| + \tfrac{1}{4}\|(r^*, s^*), (r, s)\|. \end{aligned} \tag{3.33}$$

Then, if we set $r^* = r$ and $s^* = s$ in (3.33), we have, with (3.29), that E maps $P_m^\omega(\Delta(\varepsilon), K\varepsilon^N) \times P_n^\omega(\Delta(\varepsilon), K\varepsilon^N)$ to itself. If we set $\theta^* = \theta^0 = \theta$ in (3.33) we have that E is a contraction. In particular,

$$\|E(r^*, s^*), E(r, s)\| \leqq \tfrac{1}{2}\|(r^*, s^*), (r, s)\|.$$

Therefore E has a unique fixed point (r_0, s_0) in $P_m^\omega(\Delta(\varepsilon), K\varepsilon^N) \times P_n^\omega(\Delta(\varepsilon), K\varepsilon^N)$. (r_0, s_0) is also in $P_m^\omega(\Delta_0(\varepsilon), K\varepsilon^N) \times P_n^\omega(\Delta_0(\varepsilon), K\varepsilon^N)$ since $\Delta(\varepsilon) \leqq \Delta_0(\varepsilon)$.

To complete the proof we must show that $(F + r_0, G + s_0)$ is an integral manifold in $P_m^\omega(\Delta_0(\varepsilon), D_0(\varepsilon)) \times P_n^\omega(\Delta_0(\varepsilon), D_0(\varepsilon))$. Then by uniqueness $f = F + r_0$, $g = G + s_0$ and the theorem is complete.

First of all (3.3), (3.4) and (3.24) imply that $(F + r_0, G + s_0) \in P_m^\omega(\Delta_0(\varepsilon), D_0(\varepsilon)) \times P_n^\omega(\Delta_0(\varepsilon), D_0(\varepsilon))$. Next, $(F + r_0, G + s_0)$ is an integral manifold, if we can show, setting

$$\theta(t) = T_{t-t_0, t_0}^{F+r_0, G+s_0}(\theta),$$

that the functions $r_0 = r_0(t, \theta(t), \varepsilon)$, $s_0 = s_0(t, \theta(t), \varepsilon)$ solve

$$\begin{aligned} \frac{d}{dt}(F + r_0) &= A(F + r_0) + Y(t, \theta, F + r_0, G + s_0, \varepsilon), \\ \frac{d}{dt}(G + s_0) &= \varepsilon C(G + s_0) + \varepsilon Z(t, \theta, F + r_0, G + s_0, \varepsilon). \end{aligned} \tag{3.34}$$

From (3.7) and the fact that (r_0, s_0) is the unique fixed point of (3.17) using (3.19), we can write

$$(3.35)\qquad \begin{aligned} r_0(t) &= \int_{-\infty}^{\infty} J(\tau - t)\Psi_1(\tau, \Theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau, \\ s_0(t) &= \int_{-\infty}^{\infty} K(\varepsilon(\tau - t))\Psi_2(\tau, \theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau. \end{aligned}$$

Rewrite (3.35) as follows:

$$(3.36)\qquad \begin{aligned} r_0(t) &= \int_{-\infty}^{t} J(\tau - t)\Psi_1(\tau, \theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau \\ &\quad + \int_{t}^{\infty} J(\tau - t)\Psi_1(\tau, \theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau, \\ s_0(t) &= \int_{-\infty}^{t} K(\varepsilon(\tau - t))\Psi_2(\tau, \theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau \\ &\quad + \int_{t}^{\infty} K(\varepsilon(\tau - t))\Psi_2(\tau, \theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau. \end{aligned}$$

Differentiate the first equation in (3.36) with respect to t and use (2.5), (2.7) to get

$$(3.37)\qquad \begin{aligned} \frac{dr_0}{dt} &= J(-0)\Psi_1(t, \theta(t), r_0(t), s_0(t), \varepsilon) \\ &\quad + \int_{-\infty}^{t} \left(\frac{d}{dt} J(\tau - t)\right) \Psi_1(\tau, \theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau \\ &\quad - J(+0)\Psi_1(t, \theta(t), r_0(t), s_0(t), \varepsilon) \\ &\quad - \int_{\infty}^{t} \left(\frac{d}{dt} J(\tau - t)\right) \Psi_1(\tau, \theta(\tau), r_0(\tau), s_0(\tau), \varepsilon)\, d\tau \\ &= Ar_0(t) + \Psi_1(t, \theta(t), r_0(t), s_0(t), \varepsilon). \end{aligned}$$

By a similar argument,

$$(3.38)\qquad \frac{ds_0}{dt} = \varepsilon C s_0(t) + \Psi_2(t, \theta(t), r_0(t), s_0(t), \varepsilon).$$

The result then follows by substituting the definitions of Ψ_1, Ψ_2 from (3.20) and the definitions of R, S from (2.16) into (3.37) and (3.38) and rearranging terms.

4. Example. Consider the van der Pol oscillator

$$(4.1)\qquad \ddot{X} - \varepsilon(1 - X^2)\dot{X} + X = 0,$$

which is equivalent to

$$(4.2)\qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2.$$

If we change coordinates, using $x_1 = \rho \cos \theta$, $x_2 = \rho \sin \theta$, $\rho \geqq 0$, (4.2) becomes

$$(4.3)\qquad \dot{\theta} = 1 + \varepsilon\Theta(\theta, \rho), \qquad \dot{\rho} = \varepsilon X(\theta, \rho)$$

where

$$(4.4)\qquad \Theta(\theta, \rho) = (1-\rho^2\cos^2\theta)\sin\theta\cos\theta, \qquad X(\theta, \rho) = (1-\rho^2\cos^2\theta)\rho\sin^2\theta.$$

In order to apply Theorem 2.1 we need to transform (4.3) into (1.1) in such a way that (H1)-(H6) are satisfied. We will do this in two steps. First we apply the method of averaging (see e.g. Hale [10, p. 183], Gilsinn [6], [7]) and seek a transformation of coordinates

$$(4.5)\qquad \theta = \phi + \varepsilon u(\phi, r), \qquad \rho = r + \varepsilon w(\phi, r)$$

so that (4.3) is equivalent to a system of the form

$$(4.6)\qquad \dot{\phi} = 1 + \varepsilon^2\Phi(\phi, r, \varepsilon), \qquad \dot{r} = \varepsilon X_0(r) + \varepsilon^2 R(\phi, r, \varepsilon)$$

where $X_0(r)$ is the integral average

$$(4.7)\qquad X_0(r) = \frac{1}{2\pi}\int_0^{2\pi} X(\phi, r)\, d\phi = \frac{r}{2}\left(1-\frac{r^2}{4}\right).$$

We note that the integral average $\Theta_0(r) = 0$. To obtain u, w for (4.5) we solve, according to the method of averaging,

$$(4.8)\qquad \frac{\partial u}{\partial \phi} = \Theta(\phi, \rho), \qquad \frac{\partial w}{\partial \phi} = X(\phi, \rho) - X_0(\rho).$$

The right-hand sides have zero integral average. This yields

$$(4.9)\qquad u(\phi, \rho) = \frac{\sin^2\phi}{2} + \rho^2\left(\frac{\cos^4\phi}{4}\right), \qquad w(\phi, \rho) = \frac{\rho^3}{8}\sin 4\phi - \frac{\rho}{4}\sin 2\phi.$$

Substitute (4.5) and (4.9) into (4.3) and solve for $\dot{\phi}$ and $\dot{r}$. This can be done very generally by letting

$$(4.10)\qquad A = \begin{pmatrix} \dfrac{\partial u}{\partial \phi} & \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial \phi} & \dfrac{\partial w}{\partial r} \end{pmatrix}$$

and noting that

$$(4.11)\qquad (I + \varepsilon A)^{-1} = I - \varepsilon A + \varepsilon^2 A^2 + O(\varepsilon^3).$$

Use (4.8), (4.11) and the Taylor series for $\Theta(\phi + \varepsilon u, r + \varepsilon w)$, $X(\phi + \varepsilon u, r + \varepsilon w)$ to transform (4.3) to

$$(4.12)\qquad \begin{aligned} \dot{\phi} &= 1 - \varepsilon^2\frac{\partial u}{\partial r}X_0(r) + \varepsilon^2\left(\frac{\partial \Theta}{\partial \phi}u + \frac{\partial \Theta}{\partial r}w\right) + O(\varepsilon^3), \\ \dot{r} &= \varepsilon X_0(r) - \varepsilon^2\frac{\partial w}{\partial r}X_0(r) + \varepsilon^2\left(\frac{\partial X}{\partial \phi}u + \frac{\partial X}{\partial r}w\right) + O(\varepsilon^3), \end{aligned}$$

which is in the form (4.6). Next we can put (4.12) into the form (1.1) by introducing a translation. We observe that $X_0(r) = 0$ for $r = 0$ and 2. $r = 0$ corresponds to the zero solution of (4.1) and is not of interest. Introduce $r=2+s$ into (4.12) and get

$$
\begin{aligned}
\dot\phi &= 1-\varepsilon^2\frac{\partial u}{\partial r}(\phi, s+2)X_0(s+2)\\
&\quad+\varepsilon^2\left(\frac{\partial\Theta}{\partial\phi}(\phi, s+2)u(\phi, s+2)+\frac{\partial\Theta}{\partial r}(\phi, s+2)w(\phi, s+2)\right)+O(\varepsilon^3),\\
\dot s &= \varepsilon X_0'(2)s+\varepsilon\{X_0(s+2)-X_0'\}-\varepsilon^2\frac{\partial w}{\partial r}(\phi, s+2)X_0(s+2)\\
&\quad+\varepsilon^2\left(\frac{\partial X}{\partial\phi}(\phi, s+2)u(\phi, s+2)+\frac{\partial X}{\partial r}(\phi, s+2)w(\phi, s+2)\right)+O(\varepsilon^3),
\end{aligned}
\tag{4.13}
$$

which is in the form (1.1) without the middle equation and satisfies (H1)-(H6).

We can now construct a second order asymptotic integral manifold for (4.13) by seeking a function $G(\phi, \varepsilon)$ that satisfies the second condition of (2.17). The first condition in (2.17) does not apply in our case. This we do by assuming the form

$$s=G(\phi, \varepsilon)=\varepsilon g_1(\phi)+\varepsilon^2 g_2(\phi)+O(\varepsilon^3) \tag{4.14}$$

where $g_1(\phi)$ and $g_2(\phi)$ are to be determined. The constant term is zero since condition (2.14) must be satisfied. But we must also satisfy the fact that $G(\phi, \varepsilon)$ must be dominated by a function $W(\varepsilon)$ such that $W(\varepsilon)=o(\varepsilon)$ as $\varepsilon\to 0$. Therefore we must have $g_1(\phi)=0$ and

$$s=\varepsilon^2 g_2(\phi)+O(\varepsilon^3). \tag{4.15}$$

We need only to solve for $g_2(\phi)$. Insert (4.15) into (4.13), equate powers of ε, and get

$$g_2'(\phi)=\frac{\partial X}{\partial\phi}(\phi, 2)u(\phi, 2)+\frac{\partial X}{\partial r}(\phi, 2)w(\phi, 2) \tag{4.16}$$

where

$$\frac{\partial X}{\partial\phi}=2r\sin\phi\cos\phi-r^3\sin 2\phi\cos 2\phi, \qquad \frac{\partial X}{\partial r}=\sin^2\phi-3r^2\cos^2\phi\sin^2\phi. \tag{4.17}$$

Solving (4.16) amounts to approximating the solution of the second partial differential equation in (2.16). For our case this yields

$$g_2(\phi)=-\tfrac{5}{8}\cos 2\phi+\tfrac{25}{32}\cos 4\phi+\tfrac{1}{6}\cos 6\phi-\tfrac{1}{16}\cos 8\phi. \tag{4.18}$$

Theorem 2.1 then implies that (4.13) has a unique integral manifold and

$$G(\phi, \varepsilon)=\varepsilon^2(-\tfrac{5}{8}\cos 2\phi+\tfrac{25}{32}\cos 4\phi+\tfrac{1}{6}\cos 6\phi-\tfrac{1}{16}\cos 8\phi)+O(\varepsilon^3) \tag{4.19}$$

is a second order asymptotic approximation to it. Transformating back, $2+G(\phi, \varepsilon)$ is an approximate integral manifold for (4.12). This in turn transforms through (4.5) to an approximate integral manifold for (4.3) which corresponds to the known limit cycle for (4.1).

As a final note this procedure extends to multidegree of freedom systems where θ, ρ in (4.3) are vectors. The algebra becomes more complex but the argument remains the same.

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