Fast Solvers for Models of Fluid Flow

Computational Methods for Fluid Flow
Need to efficiently compute steady flow states to enable
- Implicit time stepping strategies
- Improved stability analysis
- Classification of flow bifurcations

Fluid Models

Incompressible Navier Stokes
\[
\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega.
\]
\[
\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega.
\]

Advection-Diffusion
\[-\nabla^2 \mathbf{u} + (\mathbf{\tilde{u}} \cdot \nabla) \mathbf{u} = g\]
Viscous and Inertial forces occur on disparate scales to lead to sharp flow features which:
- require fine numerical grid resolution
- cause **poorly conditioned** non-symmetric system.

Spatial Discretization

Spectral Element Method
On each element, the solution is expressed via a nodal basis
\[
w_N(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij}(x) \tau_{ij}(y).
\]

Domain Decomposition System

\[
\begin{bmatrix}
\mathbf{P}_1 & 0 & \ldots & 0 \\
0 & \mathbf{P}_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \mathbf{P}_E
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2 \\
\vdots \\
\mathbf{u}_E
\end{bmatrix}
= \begin{bmatrix}
\mathbf{b}_1 \\
\mathbf{b}_2 \\
\vdots \\
\mathbf{b}_E
\end{bmatrix}
\]
\[
\mathbf{P}_E - \sum_{e=1}^{E-1} (\mathbf{P}_e - \mathbf{P}_e^{-1} \mathbf{P}_e) \text{ represents the Schur complement of the system.}
\]

Constant Wind Approximation

When the “wind” \( \tilde{w} \) is constant on each element, then element interiors can be obtained via Fast Diagonalization and \( P^{-1} = A^{-1} \).

Otherwise using a constant wind approximation on each element \( P^{-1} \neq A^{-1} \).

\[
\tilde{P}^{-1} = \tilde{M}(V_y \otimes V_x)(\Lambda_y \otimes I + I \otimes \Lambda_x)^{-1}(V_y^{-1} \otimes V_x^{-1})\tilde{M}
\]

Test Case: Recirculating Wind, \( P_c=400 \)

\[
\tilde{w} = 200(y(1 - x^2), -x(1 - y^2))
\]

Summary & Future Directions

Summary
- Improved simulation efficiency for steady Advection-Diffusion equation

Future Directions
- Improve wind approximation on each element
- Coarse Grid Preconditioner to allow for more elements
- Use Preconditioner in Navier-Stokes simulations
- Apply to realistic fluid simulations

References
- "Matrix-free Block preconditioner for the Navier-Stokes equations", in preparation, 2008