Matrix-Free Preconditioner for the Steady Advection-Diffusion Equation with Spectral Element Discretization.

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$$-\nabla^2 u + (\vec{w} \cdot \nabla) u = f$$

Viscous and Inertial forces occur on disparate scales lead to sharp flow features which:

- require fine numerical grid resolution
- cause poorly conditioned non-symmetric discrete system.

A Spectral Element Discretization provides

- high order accuracy
- flexible geometric boundaries
- large volume to surface ratio
- low storage requirements

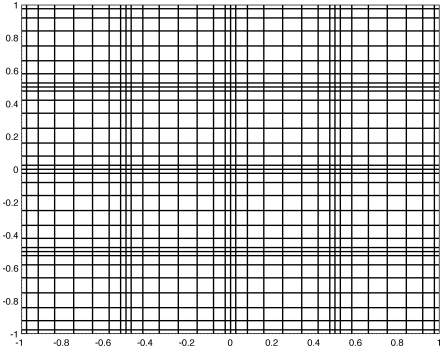


Figure: Example Spectral Element Grid. E=4x4 P=8

The discrete system of advection-diffusion equations are of the form:

$$F(\vec{w})u = Mf$$

F - Sparse Non-Symmetric system with Dense Sub-Blocks

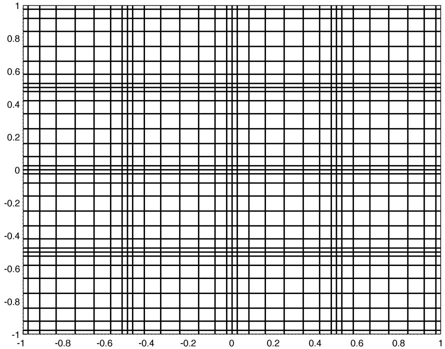
Design Goals:

- Efficient
- Matrix-Free
- Mesh Independent E and P

$$F(\vec{w})u = Mf$$

When $\vec{w}$ is constant in each direction over each element we can use **Domain Decomposition** with **Fast Diagonalization** on each element to efficiently eliminate interior degrees of freedom.

$$\tilde{F} = \hat{M} \otimes \hat{F}(w_x) + \hat{F}(w_y) \otimes \hat{M}$$



When the wind is non-constant, this method can still be used as a Preconditioner to accelerate convergence of GMRES.

$$F(\vec{w})\bar{F}^{-1}\bar{F}u = Mf$$

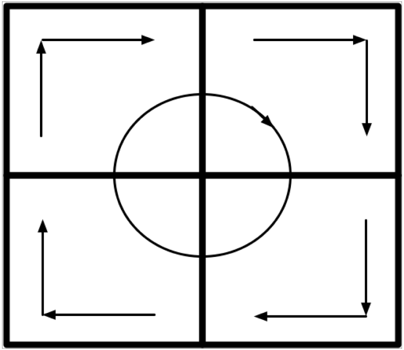


Figure: Depiction of a constant wind approximation

$$F(\vec{w})\bar{F}^{-1}\bar{F}u = Mf$$

- GMRES (outer iteration)
- Domain Decomposition Preconditioner
  - Interior Subdomain Solver - FDM
  - Interface Solver (inner iteration)
    - Inexact Solve
    - Robin-Robin Preconditioner

$$\begin{bmatrix} \bar{F}_{II}^{1} & 0 & \dots & 0 & \bar{F}_{I\Gamma}^{1} \\ 0 & \bar{F}_{II}^{2} & 0 & \dots & \bar{F}_{I\Gamma}^{2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{F}_{II}^{E} & \bar{F}_{I\Gamma}^{E} \\ \bar{F}_{\Gamma I}^{1} & \bar{F}_{\Gamma I}^{2} & \dots & \bar{F}_{I\Gamma}^{E} & \bar{F}_{\Gamma\Gamma} \end{bmatrix}$$

After formal LU the DD system can be re-written as:

$$\begin{bmatrix} \bar{F}_{II}^1 & 0 & \dots & 0 & \bar{F}_{I\Gamma}^1 \\ 0 & \bar{F}_{II}^2 & 0 & \dots & \bar{F}_{I\Gamma}^2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{F}_{II}^E & \bar{F}_{I\Gamma}^E \\ 0 & 0 & \dots & 0 & \bar{F}_0 \end{bmatrix} \begin{pmatrix} u_{I^1} \\ u_{I^2} \\ \vdots \\ u_{I^E} \\ u_\Gamma \end{pmatrix} = \begin{pmatrix} \hat{b}_{I^1} \\ \hat{b}_{I^2} \\ \vdots \\ \hat{b}_{I^E} \\ g_\Gamma \end{pmatrix}$$

$\bar{F}_0 = \sum_{e=1}^{E}(\bar{F}_{\Gamma\Gamma}^e - \bar{F}_{\Gamma I}^e \, \bar{F}_{II}^{e\;-1} \, \bar{F}_{I\Gamma}^e)$ represents the Schur complement of the system.

$$\bar{F}^{e\,-1} = \tilde{M}(V_y \otimes V_x)(\Lambda_y \otimes I + I \otimes \Lambda_x)^{-1}(V_y^{-1} \otimes V_x^{-1})\tilde{M}$$

- All based on Tensor Products of 1D operators
- Easily extended to 3D problems

$$\tilde{M} = (\hat{M}^{\frac{-1}{2}} \otimes \hat{M}^{\frac{-1}{2}})$$

The interface problem can be solved via GMRES, and Matrix-Vector products can be applied element-wise

$$\bar{F}_0 u_\Gamma = g_\Gamma$$

$$\sum_{e=1}^{E}(\bar{F}^e_{\Gamma\Gamma} - \bar{F}^e_{\Gamma I}\,{\bar{F}^e_{II}}^{-1}\,\bar{F}^e_{I\Gamma})u_\Gamma = \sum_{e=1}^{E}(\hat{b}_{\Gamma^e} - \bar{F}^e_{\Gamma I}\,{\bar{F}^e_{II}}^{-1}\,\hat{b}_{I^e})$$

Recirculating Wind Test Case

$$\vec{w} = 200(y(1-x^2), -x(1-y^2))$$

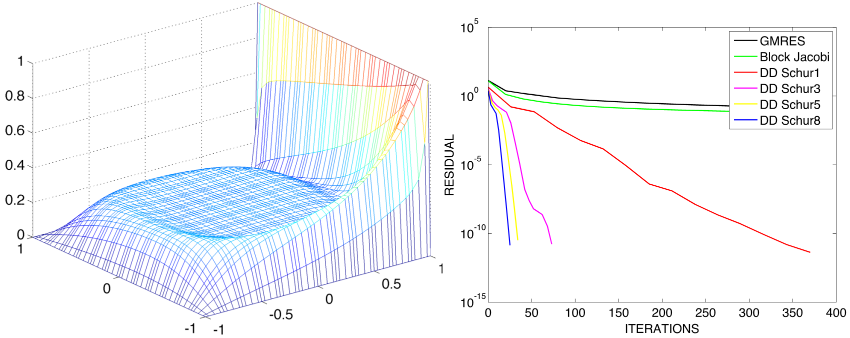


Figure: Solution and convergence comparison of outer GMRES iterations needed depending on number of interface solve steps taken. Peclet number= 400 P=4, E=12x12

Table: H Refinement (N=4)

E	Inner Its	Outer Its
10×10	5	37
11×11	5	38
12×12	5	38
13×13	5	38

Table: P Refinement ($E=4 \times 4$)

N	Inner Its	Outer Its
5	8	52
7	8	56
9	8	55
11	8	53
13	8	51

Summary

- We have developed a Domain Decomposition Based Preconditioner for the Spectral Element Discretization of the Advection Diffusion Equation
  - We use an element-wise constant wind approximation to exploit the Fast Diagonalization for interior solves
  - We use an inexact Solve to obtain interface
- Our method works well for changes in the mesh, both E & P

Future Directions

- Precondition Interface Solve with Robin-Robin preconditioner
- Add Coarse Grid preconditioner to progagate flow information between elements efficiently for large E
- Use this method in Block Preconditioner for Navier-Stokes