Lie Algebra Contractions and Separation of Variables on Two-Dimensional Hyperboloids. Basis Functions and Interbasis Expansions

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The work was done in collaboration with
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In this talk I would like to present investigation mainly presented in our two articles:

1. G.S.Pogosyan and A.Yakhno. *Lie Algebra Contractions and Separation of Variables on Two-Dimensional Hyperboloids. Coordinate Systems.* ArXiv:1510.03785.

2. E.Kalnins, G.S.Pogosyan and A.Yakhno. *Separation of variables and contractions on two-dimensional hyperboloid* ArXiv:1212.6123v1, SIGMA **8**, 105, 11 pages, 2012

In these articles we reconsider the problem of separation of variables of the Laplace-Beltrami (or Helmholtz) equation

$$\Delta_{LB}\Psi = \lambda\Psi,$$

for the on two-sheeted $H_2^{(+)}$: $u_0^2 - u_1^2 - u_2^2 = R^2, \ R > 0, \ u_0 > 0$, and one-sheeted $H_2^{(0)}$: $u_0^2 - u_1^2 - u_2^2 = -R^2$.

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The Laplace – Beltrami operator in the curvilinear coordinates $(\xi^1,\ \xi^2)$:

$$\Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^i}\sqrt{g}g^{ik}\frac{\partial}{\partial \xi^k},$$

$$ds^2 = g_{ik}d\xi^i d\xi^k, \qquad g = |\det(g_{ik})|, \quad g_{ik}g^{k\mu} = \delta_i^\mu$$

with the following relation between $g_{ik}(\xi)$ and the ambient space metric $G_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$, $(\mu, \nu = 0, 1, 2)$

$$g_{ik}(\xi) = G_{\mu\nu}\frac{\partial u^\mu}{\partial \xi^i}\frac{\partial u^\nu}{\partial \xi^k}.$$

Olevskii (1950) first to show that the Laplace-Beltrami (or Helmholtz) equation **allows separation of the variables in nine orthogonal coordinate systems.**

Thus there exist the nine sets of the wave $\{\Psi^{(\alpha)}\}$ functions such that

$$\Psi^{(\alpha)}_{\lambda_1,\lambda_2}(\xi^1,\xi^2) = N^{(\alpha)}_{\lambda_1,\lambda_2}(R)\,\Psi^{(\alpha)}_1(\xi^1,\lambda_1,\lambda_2)\Psi^{(\alpha)}_2(\xi^2,\lambda_1,\lambda_2),$$

where λ_1, λ_2 are the separation constants and $N_{\lambda,\lambda_2}(R)$ is a normalization constant.

Our main task is by the direct solution of Helmholtz equation in various system of coordinates to construct the corresponding Hilbert space of complete solutions satisfying the normalized condition

$$\int\int \Psi_{\lambda_1,\lambda_2}\Psi^*_{\lambda'_1,\lambda'_2}\sqrt{g}\,d\xi^1 d\xi^2 = \delta(\lambda_1,\lambda'_1)\delta(\lambda_2,\lambda'_2)$$

We use the notation $\delta(\lambda,\lambda')$ for Dirac delta function or Kroneker delta whichever is the constant λ discrete or takes the continuous values.

The third problem have considered is the unitary transformations (interbasis expansions) relating different bases.

Namely if $\Psi^{(I)}_{\rho,\lambda}(\xi^1,\xi^2)$ and $\Psi^{(II)}_{\rho,\mu}(\chi^1,\chi^2)$ two bases corresponding separation of variables in different systems of coordinates, then

$$\Psi^{(I)}_{\rho,\lambda} = \int W^{\mu}_{\rho,\lambda}\Psi^{(II)}_{\lambda_1,\lambda_2}d\mu$$

and vise versa

$$\Psi^{(II)}_{\rho,\mu} = \int \left(W^{\mu}_{\rho,\lambda}\right)^{*}\Psi^{(I)}_{\lambda_1,\lambda_2}d\lambda,$$

Finally we also presented the contraction procedure for the separating systems of coordinate on two-dimensional hyperboloid and corresponding systems on (pseudo)euclidean spaces E_2 and $E_{1,1}$, as the wave functions and interbasis coefficients.

The talk is structured as follows:

- Some history remarks: symmetries and separation of variables, solutions and contraction
- Description of the general procedure
- "State of the art" for bi-dimensional hyperboloids
- New results: some new relations between coordinates systems, normalization (by inter-basis expansions), contractions of wave functions.

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In "geometrical approach" we say that the D - dimensional Laplace-Beltrami equation

$$\Delta_{LB}\Psi = \lambda\Psi,$$

allows the separation of variables (multiplicative) in a D-dimensional Riemannian space with an orthogonal coordinate system $\vec{\xi} = (\xi^1, \xi^2, \ldots, \xi^D)$ if the substitution

$$\Psi = \prod_{i=1}^{D} \Psi_i(\xi^i, \lambda_1, \lambda_2, ...\lambda_D)$$

split the Laplace-Beltrami equation to the separated equations

$$\frac{1}{f_i}\frac{d}{d\xi_i}\left(f_i\frac{d\psi_i}{d\xi_i}\right) + \sum_k \Phi_{ik}\lambda_k\psi_i = 0.$$

where $f_i \equiv f_i(\xi^i)$, Φ_{ik} is element of Stäckel determinant and $\lambda_1, \lambda_2, ...\lambda_D$ are the separation constants.

In 1966 Smorodinsky and Tugov proven the

Theorem

If the Helmholtz (or Schrödinger) equation admits simple separation of variables in the coordinate system $\vec{\xi}$, then there exists D linearly independent second degree operators I_k, $k = 1, 2, 3, \ldots, D$ (including the Laplace-Beltrami operator) commuting with with each other, and they have the form

$$I_k = -\sum_{i=1}^{D} \left(\Phi^{-1}\right)_{ik} \left[\frac{1}{f_i}\frac{d}{d\xi^i}\left(f_i \frac{d\Psi_i}{d\xi^i}\right)\right], \qquad [I_k, I_l] = 0.$$

The separation constants $\lambda_1, \lambda_2 \cdots \lambda_D$ are the eigenvalues of these operators:

$$I_k \Psi = \lambda_k \Psi.$$

In the algebraic approach [E. Kalnins, W. Miller, Ya. Smorodinsky, P. Winternitz, etc. from 1965 till today] every orthogonal separable coordinate system is characterized by the set of second order commuting operators S_α, ($\alpha = 1, 2, ...D$) (including the Laplace-Beltrami operator) of enveloping algebra of the Lie algebra of the isometry group.

Namely, for our case of two-dimensional hyperboloids the isometry group is $SO(2,1)$. Then we get

$$S_1 = \Delta_{LB} = K_1^2 + K_2^2 - L^2,$$

$$S_2 \in \langle aK_1^2 + b\{K_1, K_2\} + cK_2^2 + d\{K_1, L\} + e\{K_2, L\} + f L^2 \rangle$$

where operators K_1, K_2, L forms the basis of $so(2,1)$ algebra

$$K_1 = u_0\partial_{u_2} + u_2\partial_{u_0}, \quad K_2 = u_0\partial_{u_1} + u_1\partial_{u_0}, \quad L = u_1\partial_{u_2} - u_2\partial_{u_1}$$

and commutation relation are

$$[K_1, K_2] = L, \qquad [K_2, L] = -K_1, \qquad [L, K_1] = -K_2.$$

The irreducible representations are labeled by the eigenvalue of Casimir operator

$$\Delta_{LB}\Psi = \ell(\ell+1)\Psi, \qquad \ell = -1/2 + i\rho, \quad \rho > 0.$$

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Classifying the quadratic form, corresponding to the second order operators

$$M = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

with respect to the transformations, induced by the group of the inner automorphisms

$$\overline{M}_{K_1} = A_{K_1}^T M A_{K_1} =$$
$$= \begin{pmatrix} a & b\cosh a_1 + d\sinh a_1 & b\sinh a_1 + d\cosh a_1 \\ b\cosh a_1 + d\sinh a_1 & c\cosh^2 a_1 + e\sinh 2a_1 + f\sinh^2 a_1 & (c+f)/2\sinh 2a_1 + e\cosh 2a_1 \\ b\sinh a_1 + d\cosh a_1 & (c+f)/2\sinh 2a_1 + e\cosh 2a_1 & c\sinh^2 a_1 + e\sinh 2a_1 + f\cosh^2 a_1 \end{pmatrix},$$
$$\overline{M}_{K_2} = A_{K_2}^T M A_{K_2}, \ \overline{M}_L = A_L^T M A_L,$$

including reflections and linear combination with Casimir operator one obtain the complete set of symmetry operators.

Writing the quadratic polynomial $Q = a^{ik} p_i p_k$ corresponding to the second-order operator $S = a^{ik} \partial_{\xi_i} \partial_{\xi_k}$, we'll obtain the quadratic form $Q = Ap_1^2 + 2Bp_1 p_2 + Cp_2^2$. Diagonalizing this form by finding the characteristic numbers from the equation

$$\det(a_{ik} - \rho g_{ik}) = 0,$$

$$\begin{vmatrix} C/\Delta - \rho g_{11} & -B/\Delta \\ -B/\Delta & A/\Delta - \rho g_{22} \end{vmatrix} = 0, \ \Delta = AC - B^2,$$

taking the real roots of characteristic equation $\lambda_1 = 1/\rho_1$, $\lambda_2 = 1/\rho_2$ as a new independent variables, one can determine the corresponding separable coordinate system, resolving the following equations

$$\lambda_1 + \lambda_2 = Ag_{11} + Cg_{22}, \quad \lambda_1 \lambda_2 = (AC - B^2) g_{11} g_{22}. \tag{1}$$

Note, that in the case of sub-group operators, the diagonalization means "canonic" variables (where operator takes the form of translation).

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The concept of Lie algebra contractions were introduced firstly by [Inönü, Wigner, 1953]: Inhomogeneous Lorentz group $ISO(3,1) \rightarrow$ Galilei one $G(3)$, as a limit with respect to speed of light $c \rightarrow \infty$

1. Noncosmological limit: We let the radius of the universe $R \rightarrow \infty$ in such a way that $c/R \rightarrow 0$:

$$\begin{array}{l} SO(4,1) \searrow \\ \qquad\qquad ISO(3,1) \\ SO(3,2) \nearrow \end{array}$$

2. Nonrelativistic limit: We let the velocity of light $c \rightarrow \infty$:

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Contraction of algebra can be considered as a basis change that becomes singular in a limit. Nevertheless, the Lie bracket exists and is well defined in this singular limit.
The original and contracted algebras are not isomorphic, but are of the same dimension.

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Enveloping algebra:
$$aK_1^2 + b\{K_1, K_2\} + cK_2^2 + d\{K_1, L\} + e\{K_2, L\} + f L^2$$

⇓ classification by ⇓
inner automorphisms

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⇓ diagonalization ⇓

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⇓ solution of LB equation ⇓

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Set of normalized basis functions

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taking contraction limit: $R^{-1} \to 0$

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$aK_1^2 + b\{K_1, K_2\} + cK_2^2 + d\{K_1, L\} + e\{K_2, L\} + f L^2$

$\Downarrow$ classification by $\Downarrow$ inner automorphisms

Set of operators on hyperboloids

$\Downarrow$ diagonalization $\Downarrow$

Coordinate systems admitting separation of variables

$\Downarrow$ solution of LB equation $\Downarrow$

Set of solutions

interbasis expansions $\Downarrow$ normalization $\Downarrow$

Set of normalized basis functions

introducing Beltrami coordinates:

H_2	H_1
$x_{1,2} = R\dfrac{u_{1,2}}{\sqrt{R^2+u_1^2+u_2^2}}$	$y_{0,1} = R\dfrac{u_{0,1}}{\sqrt{R^2+u_0^2-u_1^2}}$

taking contraction limit: $R^{-1} \to 0$

$o(2,1) \to e(2)$	$o(2,1) \to e(1,1)$
operators on $H \to E(2)$	operators on $H \to E(1,1)$
9 systems on $H_2 \to$ 4 on $E(2)$	9 sys. on $H_1 \to$ 9 on $E(1,1)$
Δ_{LB} on $H_2 \to \Delta_{LB}$ on $E(2)$	Δ_{LB} on $H_1 \to \Delta_{LB}$ on $E(1,1)$

$\to$ basis functions on the planes

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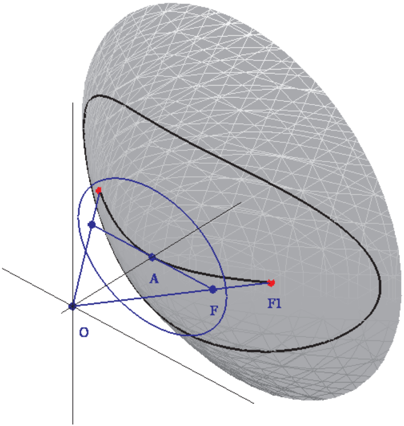
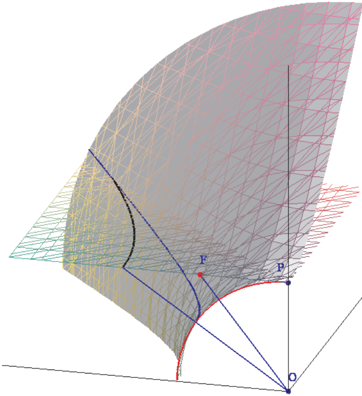
$\to$ basis functions on the planes

H_2 the Beltrami coordinates are the coordinates (x_1, x_2) on the projective plane $u_0 = R$ in the interior of the circle $x_1^2 + x_2^2 = R^2$

On H_1 the Beltrami coordinates are the coordinates (y_0, y_1) on the projective plane $u_2 = R$ between hyperbolas $y_0^2 - y_1^2 = R^2$.





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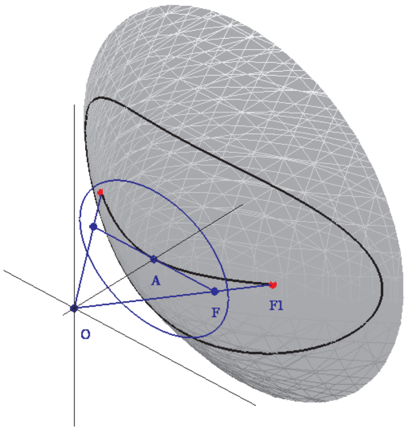

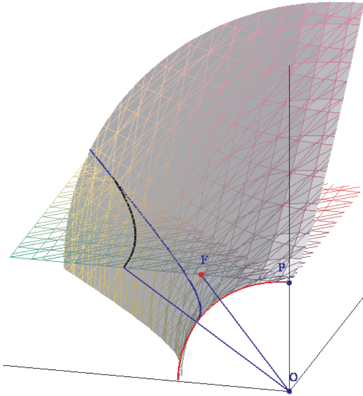

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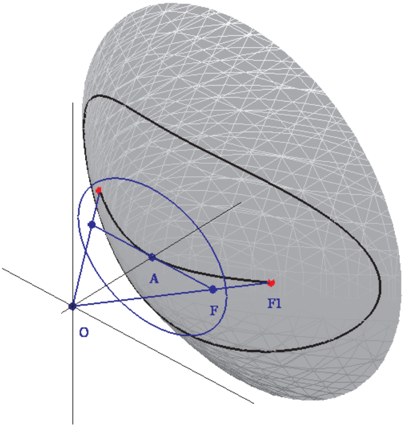

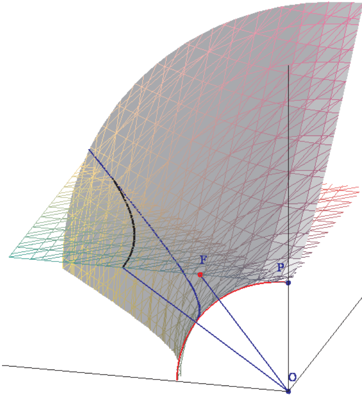

Contraction of Lie algebras

$o(2,1) \to e(2)$

$$
\begin{aligned}
-\frac{K_1}{R} &\equiv \pi_2 = \partial_{x_2} - \frac{x_2}{R^2}(x_1\partial_{x_1} + x_2\partial_{x_2}), \\
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L &= x_1\partial_{x_2} - x_2\partial_{x_1} = x_1\pi_2 - x_2\pi_1
\end{aligned}
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commutator relations of $o(2,1)$ take the form

$$[\pi_1, \pi_2] = \frac{L}{R^2},\ [\pi_1, L] = \pi_2,\ [L, \pi_2] = \pi_1 .$$

Let us take the basis of $e(2)$ in the form

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as $R^{-1} \to 0$: $\pi_1 \to p_1,\ \pi_2 \to p_2,\ L \to -M$
The Laplace-Beltrami operator contracts:

$$\Delta_{LB} = \pi_1^2 + \pi_2^2 - \frac{M^2}{R^2} \to \Delta = p_1^2 + p_2^2 .$$

$o(2,1) \to e(1,2)$

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$$[\pi_1,\pi_2] = \frac{L}{R^2},\ [\pi_1, L] = \pi_2,\ [L,\pi_2] = \pi_1.$$

Let us take the basis of $e(2)$ in the form

$$p_1 = \partial_{x_1}, \qquad p_2 = \partial_{x_2}, \qquad M = x_2\partial_{x_1} - x_1\partial_{x_2},$$

$$[p_1,p_2] = 0,\ [p_1,M] = -p_2,\ [M,p_2] = -p_1.$$

as $R^{-1} \to 0$: $\pi_1 \to p_1,\ \pi_2 \to p_2,\ L \to -M$
The Laplace-Beltrami operator contracts:

$$\Delta_{LB} = \pi_1^2 + \pi_2^2 - \frac{M^2}{R^2} \to \Delta = p_1^2 + p_2^2.$$

$o(2,1) \to e(1,2)$

$$
\begin{aligned}
-K_1/R &\equiv \pi_0 = \partial_{y_0} - \frac{y_0}{R^2}(y_0\partial_{y_0} + y_1\partial_{y_1}), \\
-L/R &\equiv \pi_1 = \partial_{y_1} + \frac{y_1}{R^2}(y_0\partial_{y_0} + y_1\partial_{y_1}), \\
-K_2 &= y_1\partial_{y_0} + y_0\partial_{y_1} = y_1\pi_0 + y_0\pi_1
\end{aligned}
$$

commutator relations of $o(2,1)$ take the form

$$[\pi_0,\pi_1] = -\frac{K_2}{R^2},\ [\pi_0, K_2] = -\pi_1,\ [K_2,\pi_1] = \pi_0.$$

Let us take the basis of $e(1,1)$

$$p_0 = \partial_{y_0}, \qquad p_1 = \partial_{y_1}, \qquad M = y_0\partial_{y_1} + y_1\partial_{y_0},$$

$$[p_0,p_1] = 0,\ [p_0,M] = p_1,\ [M,p_1] = -p_0.$$

As $R^{-1} \to 0$: $\pi_0 \to p_0,\ \pi_1 \to p_1,\ K_2 \to -M$
Laplace-Beltrami operator $\to$ the $e(1,1)$ one:

$$\Delta_{LB} = \pi_0^2 + \frac{M^2}{R^2} - \pi_1^2 \to \Delta = p_0^2 - p_1^2.$$

Contraction of Lie algebras

$o(2,1) \to e(2)$

$$
\begin{aligned}
-\frac{K_1}{R} &\equiv \pi_2 = \partial_{x_2} - \frac{x_2}{R^2}(x_1\partial_{x_1} + x_2\partial_{x_2}),\\
-\frac{K_2}{R} &\equiv \pi_1 = \partial_{x_1} - \frac{x_1}{R^2}(x_1\partial_{x_1} + x_2\partial_{x_2}),\\
L &= x_1\partial_{x_2} - x_2\partial_{x_1} = x_1\pi_2 - x_2\pi_1
\end{aligned}
$$

commutator relations of $o(2,1)$ take the form

$$[\pi_1, \pi_2] = \frac{L}{R^2}, \; [\pi_1, L] = \pi_2, \; [L, \pi_2] = \pi_1.$$

Let us take the basis of $e(2)$ in the form

$$p_1 = \partial_{x_1}, \qquad p_2 = \partial_{x_2}, \qquad M = x_2\partial_{x_1} - x_1\partial_{x_2},$$

$$[p_1, p_2] = 0, \; [p_1, M] = -p_2, \; [M, p_2] = -p_1.$$

as $R^{-1} \to 0$: $\pi_1 \to p_1, \; \pi_2 \to p_2, \; L \to -M$
The Laplace-Beltrami operator contracts:

$$\Delta_{LB} = \pi_1^2 + \pi_2^2 - \frac{M^2}{R^2} \to \Delta = p_1^2 + p_2^2.$$

$o(2,1) \to e(1,2)$

$$
\begin{aligned}
-K_1/R &\equiv \pi_0 = \partial_{y_0} - \frac{y_0}{R^2}(y_0\partial_{y_0} + y_1\partial_{y_1}),\\
-L/R &\equiv \pi_1 = \partial_{y_1} + \frac{y_1}{R^2}(y_0\partial_{y_0} + y_1\partial_{y_1}),\\
-K_2 &= y_1\partial_{y_0} + y_0\partial_{y_1} = y_1\pi_0 + y_0\pi_1
\end{aligned}
$$

commutator relations of $o(2,1)$ take the form

$$[\pi_0, \pi_1] = -\frac{K_2}{R^2}, \; [\pi_0, K_2] = -\pi_1, \; [K_2, \pi_1] = \pi_0.$$

Let us take the basis of $e(1,1)$

$$p_0 = \partial_{y_0}, \qquad p_1 = \partial_{y_1}, \qquad M = y_0\partial_{y_1} + y_1\partial_{y_0},$$

$$[p_0, p_1] = 0, \; [p_0, M] = p_1, \; [M, p_1] = -p_0.$$

As $R^{-1} \to 0 : \pi_0 \to p_0, \; \pi_1 \to p_1, \; K_2 \to -M$
Laplace-Beltrami operator $\to$ the $e(1,1)$ one:

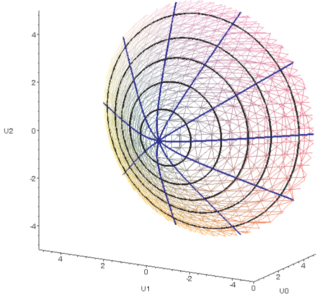
$$\Delta_{LB} = \pi_0^2 + \frac{M^2}{R^2} - \pi_1^2 \to \Delta = p_0^2 - p_1^2.$$

Example H_2: **Pseudo-spherical to polar**

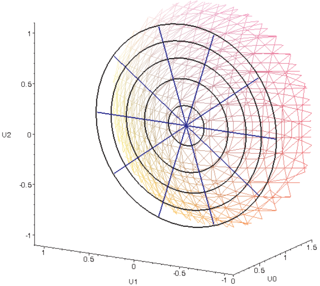
$u_0 = R\cosh\xi,\ u_1 = R\sinh\xi\cos\eta,\ u_2 = R\sinh\xi\sin\eta,\ \xi > 0, \eta \in [0, 2\pi)$

we fix the geodesic parameter $r = \xi R$. As $R^{-1} \to 0$, then $\tanh\xi \simeq r/R$ and $\xi \to 0$. In the limit, for Beltrami coordinates we have

$$x_1 = R\frac{u_1}{u_0} \to x = r\cos\varphi, \qquad x_2 = R\frac{u_2}{u_0} \to y = r\sin\varphi.$$



Spherical system



Projective plane

Operator $L^2 \to M^2 = X_S$ that corresponds to polar coordinates on E_2.

Example H_1: **Equidistant (I, II) to pseudo-polar**

Type I: $u_0 = R \sinh \tau_1 \cosh \tau_2, \; u_1 = R \sinh \tau_1 \sinh \tau_2, \; u_2 = \pm R \cosh \tau_1.$ we fix the geodesic parameter r. As $R^{-1} \to 0$, $\tanh \tau_1 \simeq \frac{r}{R}$ and $\tau_1 \to 0$. When ($|t| \geq |x|$):

$$y_0 = R \frac{u_0}{u_2} = R \tanh \tau_1 \cosh \tau_2 \to r \cosh \tau_2 \equiv t,$$

$$y_1 = R \frac{u_1}{u_2} = R \tanh \tau_1 \sinh \tau_2 \to r \sinh \tau_2 \equiv x.$$

Type II: $u_0 = R \sin \phi \sinh \tau, \quad u_1 = R \sin \phi \cosh \tau, \quad u_2 = R \cos \phi$. for the fixed geodesic parameter r, as $R^{-1} \to 0$: $\tan \phi \simeq \frac{r}{R}$. For $|\tilde{x}| \geq |\tilde{t}|$:

$$y_0 = R \frac{u_0}{u_2} = R \tan \phi \sinh \tau \to r \sinh \tau \equiv \tilde{t},$$

$$y_1 = R \frac{u_1}{u_2} = R \tan \phi \cosh \tau \to r \cosh \tau \equiv \tilde{x}.$$

$S_{EQ} = K_2^2 \to M^2 = X_S$ that corresponds to polar coordinates in the pseudo-Euclidean plane.

Example H_1: **Equidistant (I, II) to pseudo-polar**

Type I: $u_0 = R\sinh\tau_1\cosh\tau_2,\ u_1 = R\sinh\tau_1\sinh\tau_2,\ u_2 = \pm R\cosh\tau_1$. we fix the geodesic parameter r. As $R^{-1}\to 0$, $\tanh\tau_1 \simeq \frac{r}{R}$ and $\tau_1 \to 0$. When $(|t| \geq |x|)$:

$$y_0 = R\frac{u_0}{u_2} = R\tanh\tau_1\cosh\tau_2 \to r\cosh\tau_2 \equiv t,$$

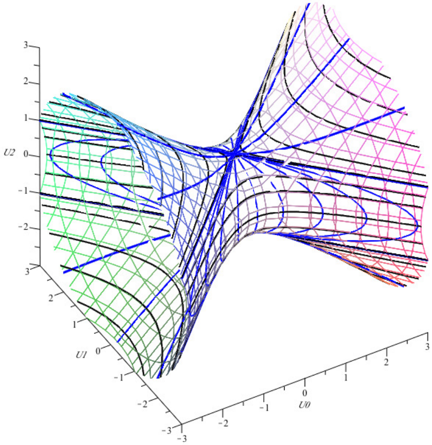
$$y_1 = R\frac{u_1}{u_2} = R\tanh\tau_1\sinh\tau_2 \to r\sinh\tau_2 \equiv x.$$

Type II: $u_0 = R\sin\phi\sinh\tau, \quad u_1 = R\sin\phi\cosh\tau, \quad u_2 = R\cos\phi$. for the fixed geodesic parameter r, as $R^{-1}\to 0$: $\tan\phi \simeq \frac{r}{R}$. For $|\tilde{x}| \geq |\tilde{t}|$:

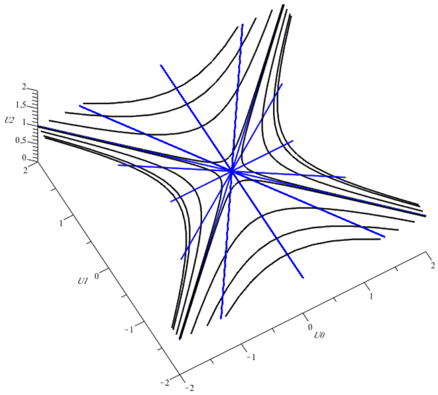
$$y_0 = R\frac{u_0}{u_2} = R\tan\phi\sinh\tau \to r\sinh\tau \equiv \tilde{t},$$

$$y_1 = R\frac{u_1}{u_2} = R\tan\phi\cosh\tau \to r\cosh\tau \equiv \tilde{x}.$$

$S_{EQ} = K_2^2 \to M^2 = X_S$ that corresponds to polar coordinates in the pseudo-Euclidean plane.



Equidistant system of Type I ($|u_2| \geq R$) and Type II ($|u_2| \leq R$).



Projective plane for equidistant system of Type I and II.

Some systems on H_1 are not symmetric ones with respect to axes u_1 and u_2. Let us analyze the projection on the plane $u_1 = R$. We can understand this kind of projection just like the projection on $u_2 = R$ of the rotated (on the $\pi/2$ trough the axes u_0) system coordinates: we have to make a change $u_1 \to -u_2$, $u_2 \to u_1$, so $K_1 \to K_2$, $K_2 \to -K_1$.

Equidistnat to Cartesian For rotated operator $\bar{S}_{EQ} = K_1^2$ in the contraction limit we have

$$\frac{\bar{S}_{EQ}}{R^2} = \pi_0^2 \to p_0^2 \simeq X_C.$$

For rotated equidistant system of Type II

$$u_0 = R \sin\phi \sinh\tau, \quad u_1 = R\cos\phi, \quad u_2 = -R\sin\phi\cosh\tau,$$

we obtain

$$\cot^2\phi = \frac{u_1^2}{u_2^2 - u_1^2} \simeq \frac{x^2}{R^2}, \quad \tan\tau = -\frac{u_0}{u_2} \simeq -\frac{t}{R}$$

and Beltrami coordinates contracts to Cartesian ones:

$$y_0 = -R\tanh\tau \to t, \quad y_1 = -R\cot\phi\frac{1}{\cosh\tau} \to x.$$

Some systems on H_1 are not symmetric ones with respect to axes u_1 and u_2. Let us analyze the projection on the plane $u_1 = R$. We can understand this kind of projection just like the projection on $u_2 = R$ of the rotated (on the $\pi/2$ trough the axes u_0) system coordinates: we have to make a change $u_1 \to -u_2$, $u_2 \to u_1$, so $K_1 \to K_2$, $K_2 \to -K_1$.

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$$u_0 = R \sin\phi \sinh\tau, \quad u_1 = R\cos\phi, \quad u_2 = -R\sin\phi\cosh\tau,$$

we obtain

$$\cot^2\phi = \frac{u_1^2}{u_2^2 - u_1^2} \simeq \frac{x^2}{R^2}, \quad \tan\tau = -\frac{u_0}{u_2} \simeq -\frac{t}{R}$$

and Beltrami coordinates contracts to Cartesian ones:

$$y_0 = -R\tanh\tau \to t, \quad y_1 = -R\cot\phi\frac{1}{\cosh\tau} \to x.$$

"State of the art": 2-dimensional hyperboloids

2-sheeted (9) → Euclidean (4)

System on H	System on E_2	Normalize constant	Solution contraction
Pseudo-Spher	Polar	V	V
Horicyclic	Cartesian	V	V
Equidistant	Cartesian	V	V
Semi-Circ-Parab	Cartesian	V	V
Hyper-Parab Discrete Continous	Cartesian	V in process	in process V
Ell-Parab	Cartesian Parabolic	V V	V V
Semi-Hyper	Cartesian Parabolic	contraction	of equations
Elliptic Elliptic$_{rot}$	Cartesian Elliptic Parabolic	contraction	of equations
Hyper	Cartesian	contraction	of equations

1-sheeted (9) → ps-Euclidean (9)

System on H	System on $E_{1,1}$	Normalize constant	Solution contraction
Pseudo-Spher	Cartesian	$V_{D,C}$	$V_{D,C}$
Horicyclic	Cartesian	$V_{D,C}$	$V_{D,C}$
Equidistant Eq$_{rot}$	Ps-Polar Cartesian	$V_{D,C}$ in	$V_{D,C}$ process
S-Circ-Parab$_{rot}$	Cartesian	in	process
Hyper-Parab HP$_{rot}$	Hyper III Parabolic I	Cartesian	in process
Ell-Parab	Hyper II Cartesian	in	process
Semi-Hyper SH$_{rot}$	Hyper I Parabolic I	Cartesian	in process
Elliptic	Cartesian Elliptic I	in	process
Hyper H$_{rot}$	Elliptic II Parabolic I	Elliptic III Ps-Polar	Cartesian

"State of the art": 2-dimensional hyperboloids

2-sheeted (9) $\to$ Euclidean (4)

System on H	System on E_2	Normalize constant	Solution contraction
Pseudo-Spher	Polar	V	V
Horicyclic	Cartesian	V	V
Equidistant	Cartesian	V	V
Semi-Circ-Parab	Cartesian	V	V
Hyper-Parab Discrete Continous	Cartesian	 V in process	 in process V
Ell-Parab	Cartesian Parabolic	V V	V V
Semi-Hyper	Cartesian Parabolic	contraction	of equations
Elliptic Elliptic$_{rot}$	Cartesian Elliptic Parabolic	contraction	of equations
Hyper	Cartesian	contraction	of equations

1-sheeted (9) $\to$ ps-Euclidean (9)

System on H	System on $E_{1,1}$	Normalize constant	Solution contraction
Pseudo-Spher	Cartesian	$V_{D,C}$	$V_{D,C}$
Horicyclic	Cartesian	$V_{D,C}$	$V_{D,C}$
Equidistant Eq$_{rot}$	Ps-Polar Cartesian	$V_{D,C}$ in	$V_{D,C}$ process
S-Circ-Parab$_{rot}$	Cartesian	in	process
Hyper-Parab HP$_{rot}$	Hyper III Parabolic I	Cartesian	in process
Ell-Parab	Hyper II Cartesian	in	process
Semi-Hyper SH$_{rot}$	Hyper I Parabolic I	Cartesian	in process
Elliptic	Cartesian Elliptic I	in	process
Hyper H$_{rot}$	Elliptic II Parabolic I	Elliptic III Ps-Polar	Cartesian

"State of the art": 2-dimensional hyperboloids

2-sheeted (9) $\to$ Euclidean (4)

System on H	System on E_2	Normalize constant	Solution contraction
Pseudo-Spher	Polar	V	V
Horicyclic	Cartesian	V	V
Equidistant	Cartesian	V	V
Semi-Circ-Parab	Cartesian	V	V
Hyper-Parab Discrete Continous	Cartesian	 V in process	 in process V
Ell-Parab	Cartesian Parabolic	V V	V V
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1-sheeted (9) $\to$ ps-Euclidean (9)

System on H	System on $E_{1,1}$	Normalize constant	Solution contraction
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Horicyclic	Cartesian	$V_{D,C}$	$V_{D,C}$
Equidistant Eq$_{rot}$	Ps-Polar Cartesian	$V_{D,C}$ in	$V_{D,C}$ process
S-Circ-Parab$_{rot}$	Cartesian	in	process
Hyper-Parab HP$_{rot}$	Hyper III Parabolic I	Cartesian	in process
Ell-Parab	Hyper II Cartesian	in	process
Semi-Hyper SH$_{rot}$	Hyper I Parabolic I	Cartesian	in process
Elliptic	Cartesian Elliptic I	in	process
Hyper H$_{rot}$	Elliptic II Parabolic I	Elliptic III Ps-Polar	Cartesian

$$\Psi^{I}_{\rho\alpha}(\xi^1,\xi^2) = \int W_{\rho\alpha\beta}\Psi^{II}_{\rho\beta}(\tilde{\xi}_1,\tilde{\xi}_2)d\beta + \sum_n W_{\rho\alpha\beta_n}\Psi^{II}_{\rho\beta_n}(\tilde{\xi}_1,\tilde{\xi}_2).$$

Orthogonal System	Coordinates	Solution
Spherical (S) $\tau > 0, 0 \leqslant \varphi < 2\pi$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\cos\varphi$ $u_2 = R\sinh\tau\sin\varphi$	$N_{\rho m}P^{\lvert m\rvert}_{i\rho-1/2}(\cosh\tau)e^{im\varphi}$ $m \in \mathbb{Z}$
Horicyclic (HO) $\tilde{x} \in \mathbb{R}, \tilde{y} > 0$	$u_0 = R\frac{\tilde{x}^2+\tilde{y}^2+1}{2\tilde{y}}$ $u_1 = R\frac{\tilde{x}^2+\tilde{y}^2-1}{2\tilde{y}}$ $u_2 = R\tilde{x}/\tilde{y}$	$N_{\rho s}\sqrt{\tilde{y}}\,K_{i\rho}(\lvert s\rvert\tilde{y})e^{is\tilde{x}}$ $s \in \mathbb{R} \setminus 0$
Equidistant (EQ) $\tau_1, \tau_2 \in \mathbb{R}$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$	$\Psi^{EQ(\pm)}_{\rho\nu} = N_{\rho\nu}\psi^{(\pm)}_{\rho\nu}(\tau_1)\,\frac{e^{i\nu\tau_2}}{\sqrt{2\pi}}$ $\nu \in \mathbb{R} \setminus 0$

$$\Psi_{\rho\alpha} = \int_{-\infty}^{\infty} W^{(+)}_{\rho\alpha\nu}\,\Psi^{EQ(+)}_{\rho\nu}(\tau_1,\tau_2)d\nu + \int_{-\infty}^{\infty} W^{(-)}_{\rho\alpha\nu}\,\Psi^{EQ(-)}_{\rho\nu}(\tau_1,\tau_2)d\nu$$

where $W_{\rho\alpha\nu}$ are the interbasis expansion coefficients.

$$\Psi^{I}_{\rho\alpha}(\xi^1,\xi^2) = \int W_{\rho\alpha\beta}\Psi^{II}_{\rho\beta}(\tilde{\xi}_1,\tilde{\xi}_2)d\beta + \sum_n W_{\rho\alpha\beta_n}\Psi^{II}_{\rho\beta_n}(\tilde{\xi}_1,\tilde{\xi}_2).$$

Orthogonal System	Coordinates	Solution
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Horicyclic (HO) $\tilde{x}\in\mathbb{R}, \tilde{y} > 0$	$u_0 = R\frac{\tilde{x}^2+\tilde{y}^2+1}{2\tilde{y}}$ $u_1 = R\frac{\tilde{x}^2+\tilde{y}^2-1}{2\tilde{y}}$ $u_2 = R\tilde{x}/\tilde{y}$	$N_{\rho s}\sqrt{\tilde{y}}\,K_{i\rho}(\lvert s\rvert\tilde{y})e^{is\tilde{x}}$ $s\in\mathbb{R}\setminus 0$
Equidistant (EQ) $\tau_1,\tau_2\in\mathbb{R}$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$	$\Psi^{EQ(\pm)}_{\rho\nu} = N_{\rho\nu}\psi^{(\pm)}_{\rho\nu}(\tau_1)\frac{e^{i\nu\tau_2}}{\sqrt{2\pi}}$ $\nu\in\mathbb{R}\setminus 0$

$$\Psi_{\rho\alpha} = \int_{-\infty}^{\infty} W^{(+)}_{\rho\alpha\nu}\Psi^{EQ(+)}_{\rho\nu}(\tau_1,\tau_2)d\nu + \int_{-\infty}^{\infty} W^{(-)}_{\rho\alpha\nu}\Psi^{EQ(-)}_{\rho\nu}(\tau_1,\tau_2)d\nu$$

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Orthogonal System	Coordinates	Solution
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Horicyclic (HO) $\tilde{x} \in \mathbb{R}, \tilde{y} > 0$	$u_0 = R\frac{\tilde{x}^2+\tilde{y}^2+1}{2\tilde{y}}$ $u_1 = R\frac{\tilde{x}^2+\tilde{y}^2-1}{2\tilde{y}}$ $u_2 = R\tilde{x}/\tilde{y}$	$N_{\rho s}\sqrt{\tilde{y}}\,K_{i\rho}(\lvert s\rvert\tilde{y})e^{is\tilde{x}}$ $s \in \mathbb{R}\setminus 0$
Equidistant (EQ) $\tau_1,\tau_2 \in \mathbb{R}$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$	$\Psi^{EQ(\pm)}_{\rho\nu} = N_{\rho\nu}\psi^{(\pm)}_{\rho\nu}(\tau_1)\,\frac{e^{i\nu\tau_2}}{\sqrt{2\pi}}$ $\nu \in \mathbb{R}\setminus 0$

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Orthogonal System	Coordinates	Solution
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Horicyclic (HO) $\tilde{x}\in\mathbb{R}, \tilde{y} > 0$	$u_0 = R\frac{\tilde{x}^2+\tilde{y}^2+1}{2\tilde{y}}$ $u_1 = R\frac{\tilde{x}^2+\tilde{y}^2-1}{2\tilde{y}}$ $u_2 = R\tilde{x}/\tilde{y}$	$N_{\rho s}\sqrt{\tilde{y}}\,K_{i\rho}(\lvert s\rvert\tilde{y})e^{is\tilde{x}}$ $s\in\mathbb{R}\setminus 0$
Equidistant (EQ) $\tau_1,\tau_2\in\mathbb{R}$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$	$\Psi^{EQ(\pm)}_{\rho\nu} = N_{\rho\nu}\psi^{(\pm)}_{\rho\nu}(\tau_1)\,\frac{e^{i\nu\tau_2}}{\sqrt{2\pi}}$ $\nu\in\mathbb{R}\setminus 0$

$$\psi^{(+)}_{\rho\nu}(\tau_1) = (\cosh\tau_1)^{i\nu}\,{}_2F_1\left(\frac{1}{4}+i\frac{\nu-\rho}{2},\ \frac{1}{4}+i\frac{\nu+\rho}{2};\ \frac{1}{2};\ -\sinh^2\tau_1\right),$$

$$\psi^{(-)}_{\rho\nu}(\tau_1) = \sinh\tau_1(\cosh\tau_1)^{i\nu}\,{}_2F_1\left(\frac{3}{4}+i\frac{\nu-\rho}{2},\ \frac{3}{4}+i\frac{\nu+\rho}{2};\ \frac{3}{2};\ -\sinh^2\tau_1\right)$$

$$\Psi^{I}_{\rho\alpha}(\xi^1,\xi^2) = \int W_{\rho\alpha\beta}\Psi^{II}_{\rho\beta}(\tilde{\xi_1},\tilde{\xi_2})d\beta + \sum_n W_{\rho\alpha\beta_n}\Psi^{II}_{\rho\beta_n}(\tilde{\xi_1},\tilde{\xi_2}).$$

Orthogonal System	Coordinates	Solution
Spherical (S) $\tau > 0, 0 \leqslant \varphi < 2\pi$	$u_0 = R\cosh\tau$ $u_1 = R\sinh\tau\cos\varphi$ $u_2 = R\sinh\tau\sin\varphi$	$N_{\rho m}P^{\lvert m\rvert}_{i\rho-1/2}(\cosh\tau)e^{im\varphi}$ $m \in \mathbb{Z}$
Horicyclic (HO) $\tilde{x} \in \mathbb{R}, \tilde{y} > 0$	$u_0 = R\frac{\tilde{x}^2+\tilde{y}^2+1}{2\tilde{y}}$ $u_1 = R\frac{\tilde{x}^2+\tilde{y}^2-1}{2\tilde{y}}$ $u_2 = R\tilde{x}/\tilde{y}$	$N_{\rho s}\sqrt{\tilde{y}}\,K_{i\rho}(\lvert s\rvert\tilde{y})e^{is\tilde{x}}$ $s \in \mathbb{R}\setminus 0$
Equidistant (EQ) $\tau_1,\tau_2 \in \mathbb{R}$	$u_0 = R\cosh\tau_1\cosh\tau_2$ $u_1 = R\cosh\tau_1\sinh\tau_2$ $u_2 = R\sinh\tau_1$	$\Psi^{EQ(\pm)}_{\rho\nu} = N_{\rho\nu}\psi^{(\pm)}_{\rho\nu}(\tau_1)\,\frac{e^{i\nu\tau_2}}{\sqrt{2\pi}}$ $\nu \in \mathbb{R}\setminus 0$

$$\Psi_{\rho\alpha} = \int_{-\infty}^{\infty} W^{(+)}_{\rho\alpha\nu}\ \Psi^{EQ(+)}_{\rho\nu}(\tau_1,\tau_2)d\nu + \int_{-\infty}^{\infty} W^{(-)}_{\rho\alpha\nu}\ \Psi^{EQ(-)}_{\rho\nu}(\tau_1,\tau_2)d\nu$$

where $W_{\rho\alpha\nu}$ are the interbasis expansion coefficients.

Semi-circular parabolic coordinate system: solution

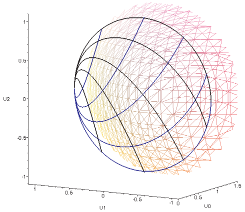
This system looks like

$$u_0 = R\frac{(\xi^2+\eta^2)^2+4}{8\xi\eta}, \quad u_1 = R\frac{(\xi^2+\eta^2)^2-4}{8\xi\eta}, \quad u_2 = R\frac{\eta^2-\xi^2}{2\xi\eta},$$

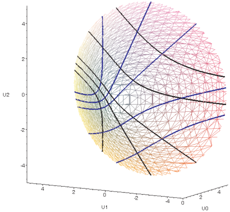
where $\xi, \eta > 0$. In contraction limit $R \to \infty$ we have:

$$\eta^2 = \frac{\sqrt{R^2+u_2^2}+u_2}{u_0-u_1} \to 1+\frac{x+y}{R}, \quad \xi^2 = \frac{\sqrt{R^2+u_2^2}-u_2}{u_0-u_1} \to 1+\frac{x-y}{R},$$

that is not suitable for contractions of solutions.



Semi-circular parabolic system.



Projective plane.

The Laplace-Beltrami operator is invariant under the rotation, so we

Semi-circular parabolic coordinate system: solution

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$$u_0 = R\frac{(\xi^2+\eta^2)^2+4}{8\xi\eta}, \quad u_1 = R\frac{(\xi^2+\eta^2)^2-4}{8\xi\eta}, \quad u_2 = R\frac{\eta^2-\xi^2}{2\xi\eta},$$

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that is not suitable for contractions of solutions.
The Laplace-Beltrami operator is invariant under the rotation, so we can introduce the equivalent semi-circular parabolic system of coordinate connected with the above one by the rotation about axis u_0 through the angle $\pi/4$, then as $R \to \infty$:

$$\eta^2 \to 1+\sqrt{2}\frac{x}{R}, \qquad \xi^2 \to 1+\sqrt{2}\frac{y}{R}.$$

The Laplace-Beltrami equation takes the following form

$$\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\Psi(\xi,\eta)=-(\rho^2+1/4)\Psi(\xi,\eta).$$

The separation of variable leads to two differential equations:

$$\frac{d^2L_1}{d\xi^2}+\left(A+\frac{\rho^2+1/4}{\xi^2}\right)L_1=0,\ \frac{d^2L_2}{d\eta^2}+\left(-A+\frac{\rho^2+1/4}{\eta^2}\right)L_2=0,$$

where $\Psi(\xi,\eta)=L_1(\xi)L_2(\eta)$ and A is the separation constant. These equations are related by the change $\xi\to i\eta$ and coincide with the Bessel equations. The wave function $\Psi(\xi,\eta)$, depending of the sign of separation constant A, have the form:

$$\Psi^{(1)}(\xi,\eta)=N_{\rho A}\sqrt{\xi\eta}\left[J_{i\rho}\left(\sqrt{|A|}\xi\right)+J_{-i\rho}\left(\sqrt{|A|}\xi\right)\right]K_{i\rho}\left(\sqrt{|A|}\eta\right)$$

for $A>0$ and

$$\Psi^{(2)}(\xi,\eta)=N_{\rho A}\sqrt{\xi\eta}\,K_{i\rho}\left(\sqrt{|A|}\xi\right)\left[J_{i\rho}\left(\sqrt{|A|}\eta\right)+J_{-i\rho}\left(\sqrt{|A|}\eta\right)\right]$$

for $A<0$, $N_{\rho A}$ is a normalization constant. Let us note, that

$$\Psi^{(2)}_{\rho A}(\xi,\eta)=\Psi^{(1)}_{\rho A}(\eta,\xi).$$

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Interbases expansions through equidistant basis

$$\Psi^{(1)}_{\rho A}(\xi,\eta) = \int\limits_{-\infty}^{\infty} T^{(+)}_{\rho A\nu}\, \Psi^{(+)}_{\rho\nu}(\tau_1,\tau_2) d\nu + \int\limits_{-\infty}^{\infty} T^{(-)}_{\rho A\nu}\, \Psi^{(-)}_{\rho\nu}(\tau_1,\tau_2) d\nu.$$

After the long but not complicated algebraic calculations we obtain that

$$T^{(+)}_{\rho A\nu} = \frac{N_{\rho A}}{N^{(+)}_{\rho\nu}} \frac{(|A|)^{-\frac{1}{2}+i\nu}}{2\pi\, 2^{1/2+2i\nu}}\, \Gamma\left(\frac{1}{2} - i\nu\right) \left[\frac{\Gamma\left(\frac{1}{4} - i\frac{\nu-\rho}{2}\right)}{\Gamma\left(\frac{3}{4} + i\frac{\nu+\rho}{2}\right)} + \frac{\Gamma\left(\frac{1}{4} - i\frac{\nu+\rho}{2}\right)}{\Gamma\left(\frac{3}{4} + i\frac{\nu-\rho}{2}\right)}\right]$$

$$T^{(-)}_{\rho A\nu} = \frac{N_{\rho A}}{N^{(-)}_{\rho\nu}} \frac{(|A|)^{-\frac{1}{2}+i\nu}}{\pi\, 2^{1/2+2i\nu}}\, \Gamma\left(\frac{1}{2} - i\nu\right) \left[\frac{\Gamma\left(\frac{3}{4} - i\frac{\nu-\rho}{2}\right)}{\Gamma\left(\frac{1}{4} + i\frac{\nu+\rho}{2}\right)} + \frac{\Gamma\left(\frac{3}{4} - i\frac{\nu+\rho}{2}\right)}{\Gamma\left(\frac{1}{4} + i\frac{\nu-\rho}{2}\right)}\right]$$

For normalization constants of SCP basis $N^{(1)}_{\rho A}$ we take into account:

$$T^{(+)}_{\rho A\nu} {T^{(+)}_{\rho A'\nu}}^{*} + T^{(-)}_{\rho A\nu} {T^{(-)}_{\rho A'\nu}}^{*} = \frac{\pi R^2}{\rho \tanh \pi\rho/2} N_{\rho A} {N_{\rho A'}}^{*} \, (|A|)^{-\frac{1}{2}+i\nu} (|A'|)^{-\frac{1}{2}+i\nu}$$

and obtain

$$\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \Psi^{(1,2)}_{\rho A}(\xi,\eta) {\Psi^{(1,2)}_{\rho' A'}(\xi,\eta)}^{*} \frac{\xi^2+\eta^2}{4\xi^2\eta^2} d\xi d\eta = \frac{4\pi^2 |N_{\rho A}|^2}{\rho \tanh \frac{\pi\rho}{2}} \delta(|A| - |A'|)\delta(\rho - \rho').$$

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and obtain

$$\int\limits_0^{\infty}\int\limits_0^{\infty}\Psi^{(1,2)}_{\rho A}(\xi,\eta){\Psi^{(1,2)}_{\rho' A'}(\xi,\eta)}^{*}\,\frac{\xi^2+\eta^2}{4\xi^2\eta^2}d\xi d\eta = \frac{4\pi^2\,|N_{\rho A}|^2}{\rho\tanh\frac{\pi\rho}{2}}\,\delta(|A|-|A'|)\delta(\rho-\rho').$$

Therefore the SCP basis will be normalized on delta functions if

$$N_{\rho A} = \frac{\sqrt{\rho \tanh \frac{\pi\rho}{2}}}{2\pi R}.$$

It is easy to prove that

$$\int\limits_{-\infty}^{\infty} T^{(+)}_{\rho A\nu} T^{(+)\,*}_{\rho A\nu'}\, dA = \int\limits_{-\infty}^{\infty} T^{(-)}_{\rho A\nu} T^{(-)\,*}_{\rho A\nu'}\, dA = \delta(\nu - \nu')$$

and correspondingly to construct the inverse expansions:

$$\Psi^{(\pm)}_{\rho\nu}(\tau_1, \tau_2) = \int\limits_{-\infty}^{\infty} T^{(\pm)*}_{\rho A\nu} \left\{ \theta(A)\, \Psi^{(1)}_{\rho A}(\xi, \eta) \pm \theta(-A)\, \Psi^{(2)}_{\rho A}(\xi, \eta) \right\} dA,$$

where $\theta(x)$ is a step function: $\theta(x) = 0$ for $x < 0$, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 1/2$ for $x = 0$.

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Orthogonal System	Coordinates	Solution
Elliptic-Parabolic (EP) $a \geq 0, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \gamma > 0$	$u_0 = \frac{R}{\sqrt{\gamma}} \frac{\cosh^2 a - \sin^2 \theta + \gamma}{2\cos\theta\cosh a}$ $u_1 = \frac{R}{\sqrt{\gamma}} \frac{\cosh^2 a - \sin^2 \theta - \gamma}{2\cos\theta\cosh a}$ $u_2 = R\tan\theta\tanh a$	$\Psi^{(\pm)}_{\rho\mu}(a, \theta)$

Separation of variables $\Rightarrow$Poeschl-Teller and Rosen-Morse potentials:

$$\frac{d^2\Psi_1}{d\theta^2} + \left(-\mu^2 + \frac{\rho^2 + 1/4}{\cos^2\theta}\right)\Psi_1 = 0, \qquad \frac{d^2\Psi_2}{da^2} + \left(\mu^2 - \frac{\rho^2 + 1/4}{\cosh^2 a}\right)\Psi_2 = 0.$$

The complete set of EP functions is two mutually orthogonal bases ($\mathbf{N}^{(\pm)}_{\rho\mu} = ?$):

$$\begin{aligned}
\Psi^{(+)}_{\rho\mu}(a,\theta) &= N^{(+)}_{\rho\mu} (\cos\theta)^{i\mu} \, {}_2F_1\left(\frac{1}{4} - i\frac{\rho+\mu}{2}, \frac{1}{4} + i\frac{\rho-\mu}{2}; \frac{1}{2}; -\tan^2\theta\right) \\
&\times (\cosh a)^{i\mu} \, {}_2F_1\left(\frac{1}{4} - i\frac{\rho+\mu}{2}, \frac{1}{4} + i\frac{\rho-\mu}{2}; \frac{1}{2}; \tanh^2 a\right) \\
\Psi^{(-)}_{\rho\mu}(a,\theta) &= N^{(-)}_{\rho\mu} (\cos\theta)^{i\mu} \tanh\theta \, {}_2F_1\left(\frac{3}{4} - \frac{i(\rho+\mu)}{2}, \frac{3}{4} + \frac{i(\rho-\mu)}{2}; \frac{3}{2}; -\tan^2\theta\right) \\
&\times (\cosh a)^{i\mu} \tanh a \, {}_2F_1\left(\frac{3}{4} - \frac{i(\rho+\mu)}{2}, \frac{3}{4} + \frac{i(\rho-\mu)}{2}; \frac{3}{2}; \tanh^2 a\right)
\end{aligned}$$

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&\times (\cosh a)^{i\mu} \tanh a \, {}_2F_1\left(\frac{3}{4} - \frac{i(\rho+\mu)}{2}, \frac{3}{4} + \frac{i(\rho-\mu)}{2}; \frac{3}{2}; \tanh^2 a\right)
\end{aligned}$$

EP↔S

For the fixed ρ are connected :

$$\Psi^{EQ(\pm)}_{\rho\mu}(a,\theta) = \sum_{m=-\infty}^{\infty} E^{(\pm)}_{\rho\mu m}\Psi^{S}_{\rho m}(\tau,\varphi).$$

Coordinates are expressed:

$$\cos^2\theta = \frac{e^{-\tau}}{\cosh\tau - \sinh\tau\cos\varphi},\ \cosh^2 a = \frac{e^{\tau}}{\cosh\tau - \sinh\tau\cos\varphi}.$$

In limit $\tau \to 0$, after a long calculation we obtain (of Saalschütz type)

$$E^{(+)}_{\rho\mu m} = N^{(+)}_{\rho\mu}(-1)^{|m|}\,|\Gamma(1/2 + i\rho - |m|)|\,\frac{\left(\frac{1}{4} + \frac{i(\rho+\mu)}{2}\right)_{|m|}\left(\frac{1}{4} - \frac{i(\rho-\mu)}{2}\right)_{|m|}}{(1/2)_{|m|}}$$

$$\times\frac{R\sqrt{2}\cosh\pi\rho}{\sqrt{\rho\sinh\pi\rho}}\,{}_4F_3\left(\begin{matrix} -|m|, & \frac{1}{4} - \frac{i(\rho+\mu)}{2}, & \frac{1}{4} + \frac{i(\rho-\mu)}{2}, & 1/2 - |m| \\ \frac{1}{2}, & \frac{3}{4} - \frac{i(\rho+\mu)}{2} - |m|, & \frac{3}{4} + \frac{i(\rho-\mu)}{2} - |m| & \end{matrix}\middle|\, 1\right).$$

For the fixed ρ are connected :

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$$\times \frac{R\sqrt{2}\cosh\pi\rho}{\sqrt{\rho\sinh\pi\rho}} \, {}_4F_3\left(\begin{array}{cccc} -|m|, & \frac{1}{4} - \frac{i(\rho+\mu)}{2}, & \frac{1}{4} + \frac{i(\rho-\mu)}{2}, & 1/2 - |m| \\ \frac{1}{2}, & \frac{3}{4} - \frac{i(\rho+\mu)}{2} - |m|, & \frac{3}{4} + \frac{i(\rho-\mu)}{2} - |m| & \end{array} \middle| 1 \right).$$

Wilson-Racah polynomials

$$p_n(t^2) \equiv p_n(t^2,\alpha,\beta,\gamma,\delta) = (\alpha+\beta)_n\,(\alpha+\gamma)_n\,(\alpha+\delta)_n$$
$$\times \quad {}_4F_3\left(\begin{array}{cccc} -n, & \alpha+\beta+\gamma+\delta+n-1, & \alpha-t, & \alpha+t; \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta, & \end{array}\middle|\, 1\right)$$

and are orthogonal with respect to the inner product:

$$\frac{1}{2\pi}\int_0^\infty p_n(-t^2)p_{n'}(-t^2)\left|\frac{\Gamma(\alpha+it)\Gamma(\beta+it)\Gamma(\gamma+it)\Gamma(\delta+it)}{\Gamma(2it)}\right|^2 dt = n!(\alpha+\beta+\gamma+\delta+n-1)_n$$
$$\times\Gamma(\alpha+\beta+n)\Gamma(\alpha+\gamma+n)\frac{\Gamma(\alpha+\delta+n)\Gamma(\beta+\gamma+n)\Gamma(\beta+\delta+n)\Gamma(\gamma+\delta+n)}{\Gamma(\alpha+\beta+\gamma+\delta+2n)}\delta_{nn'}\,.$$

So

$$E^{(+)}_{\rho\mu m} = \frac{R\pi^2\sqrt{2}}{\sqrt{\rho\sinh\pi\rho}}\,\frac{(-1)^{|m|}\,N^{(+)}_{\rho\mu}}{[\Gamma(1/2+|m|)]^2}\,\frac{|\Gamma(1/2+i\rho-|m|)|}{|\Gamma(1/2+i\rho)|^2}\;p_{|m|}\left(-\frac{\mu^2}{4}\right).$$

Taking account the orthogonality condition for Wilson-Racah polynomials we can prove

$$\int_{-\infty}^{\infty} E^{(+)}_{\rho\mu m}\,{E^{(+)}_{\rho\mu m'}}^{*}\,d\mu = \frac{1}{2}\,[\delta_{m,m'}+\delta_{m,-m'}],$$

if we choose normalized constant $N^{(+)}_{\rho\mu}$ in form

$$N^{(+)}_{\rho\mu} = \frac{1}{4\pi^2 R}\,\frac{\left|\Gamma\left(\frac{1}{4}+\frac{i(\rho+\mu)}{2}\right)\Gamma\left(\frac{1}{4}+\frac{i(\rho-\mu)}{2}\right)\right|^2}{|\Gamma(i\rho)\,\Gamma(i\mu)|}.$$

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$$E^{(+)}_{\rho\mu m} = \frac{R\pi^2\sqrt{2}}{\sqrt{\rho\sinh\pi\rho}}\,\frac{(-1)^{|m|}\,N^{(+)}_{\rho\mu}}{[\Gamma(1/2+|m|)]^2}\,\frac{|\Gamma(1/2+i\rho-|m|)|}{|\Gamma(1/2+i\rho)|^2}\;p_{|m|}\left(-\frac{\mu^2}{4}\right).$$

Taking account the orthogonality condition for Wilson-Racah polynomials we can prove

$$\int_{-\infty}^{\infty} E^{(+)}_{\rho\mu m}\,{E^{(+)}_{\rho\mu m'}}^*\,d\mu = \frac{1}{2}\,[\delta_{m,m'}+\delta_{m,-m'}],$$

if we choose normalized constant $N^{(+)}_{\rho\mu}$ in form

$$N^{(+)}_{\rho\mu} = \frac{1}{4\pi^2 R}\,\frac{\left|\Gamma\left(\frac{1}{4}+\frac{i(\rho+\mu)}{2}\right)\Gamma\left(\frac{1}{4}+\frac{i(\rho-\mu)}{2}\right)\right|^2}{|\Gamma(i\rho)\,\Gamma(i\mu)|}.$$

Contraction of EP basis to Parabolic one

Let $\gamma = 1$. In the contraction limit $R \to \infty$:

$$\cos^2\theta \to 1 - \frac{v^2}{R}, \qquad \cosh^2 a \to 1 + \frac{u^2}{R},$$

where (u, v) are the parabolic coordinates $x = (u^2 - v^2)/2$, $y = uv$ on E_2. Taking $\mu \sim kR + \frac{\lambda}{2k}$, $\rho \sim kR$ in (2) $\Rightarrow$ Eqs. for parabolic-cylinder functions:

$$\left(\frac{d^2}{du^2} + k^2u^2 + \lambda\right)\Phi^{EP}(u) = 0, \qquad \left(\frac{d^2}{dv^2} + k^2v^2 - \lambda\right)\Phi^{EP}(v) = 0.$$

$$N^{(+)}_{\rho\mu} \sim \frac{1}{4\pi^2}\sqrt{\frac{k}{R}}\left|\Gamma\left(\frac{1}{4} + \frac{i\lambda}{4k}\right)\right|^2$$

and

$$\Psi^{(+)}_{\rho\mu} \sim \sqrt{\frac{k}{R}}\left|\Gamma\left(\frac{1}{4} + \frac{i\lambda}{4k}\right)\right|^2 \frac{e^{\frac{ik}{2}(u^2+v^2)}}{4\pi^2}\, {}_1F_1\left(\frac{1}{4} - \frac{i\lambda}{4k}; \frac{1}{2}; -iku^2\right) {}_1F_1\left(\frac{1}{4} + \frac{i\lambda}{4k}; \frac{1}{2}; -ikv^2\right)$$

$$= D_{-\frac{1}{2}+i\frac{\lambda}{2k}}(u\sqrt{-2ik})\, D_{-\frac{1}{2}+i\frac{\lambda}{2k}}(v\sqrt{-2ik}),$$

where $D_\nu(z)$ is a parabolic-cylinder function

$$D_\nu(z) = 2^{\nu/2}\sqrt{\pi}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)}\, {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) - \frac{z\sqrt{2}}{\Gamma\left(-\frac{\nu}{2}\right)}\, {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right)\right].$$

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$$N^{(+)}_{\rho\mu} \sim \frac{1}{4\pi^2}\sqrt{\frac{k}{R}}\left|\Gamma\left(\frac{1}{4} + \frac{i\lambda}{4k}\right)\right|^2$$

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and

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Orthogonal System	Coordinates	Solution
Hyperbolic-Parabolic (HP) $b>0, \theta\in(0,\pi), \gamma>0$	$u_0 = \frac{R}{\sqrt{\gamma}} \frac{\cosh^2 b - \sin^2\theta + \gamma}{2\sin\theta\sinh b}$ $u_1 = \frac{R}{\sqrt{\gamma}} \frac{\cosh^2 b - \sin^2\theta - \gamma}{2\sin\theta\sinh b}$ $u_2 = R\cot\theta\coth b$	Discrete for $s^2>0$ Continuous for $s^2<0$

Separation of variables $\Psi(b,\theta)=\psi(b)\psi(\theta)$:

$$\frac{d^2\psi}{db^2} + \left(-s^2 + \frac{1/4+\rho^2}{\sinh^2 b}\right)\psi = 0, \qquad \frac{d^2\psi}{d\theta^2} + \left(s^2 + \frac{1/4+\rho^2}{\sin^2\theta}\right)\psi = 0$$

Eigenvalue problem is singular: $b=0$; at the both ends of the interval $\theta\in(0,\pi)$. There exist two spectrums of separation constant s.

Continuous spectrum:

$$\psi_{1,2}(b) = (\sinh b)^{1/2\pm i\rho}\, {}_2F_1\left(\frac{1}{2}\pm i(\rho+s), \frac{1}{2}\pm i(\rho-s); 1\pm i\rho; -\sinh^2\frac{b}{2}\right),$$

$$\psi_{\rho s}^{(+)}(\theta) = (\sin\theta)^{i\rho+1/2}\, {}_2F_1\left(\frac{1}{4}+i\frac{\rho+s}{2}, \frac{1}{4}+i\frac{\rho-s}{2}; \frac{1}{2}; \cos^2\theta\right),$$

$$\psi_{\rho s}^{(-)}(\theta) = (\sin\theta)^{i\rho+1/2}\cos\theta\, {}_2F_1\left(\frac{3}{4}+i\frac{\rho+s}{2}, \frac{3}{4}+i\frac{\rho-s}{2}; \frac{3}{2}; \cos^2\theta\right).$$

Orthogonal System	Coordinates	Solution
Hyperbolic-Parabolic (HP) $b > 0, \theta \in (0, \pi), \gamma > 0$	$u_0 = \frac{R}{\sqrt{\gamma}} \frac{\cosh^2 b - \sin^2 \theta + \gamma}{2 \sin\theta \sinh b}$ $u_1 = \frac{R}{\sqrt{\gamma}} \frac{\cosh^2 b - \sin^2 \theta - \gamma}{2 \sin\theta \sinh b}$ $u_2 = R \cot\theta \coth b$	Discrete for $s^2 > 0$; Continuous for $s^2 < 0$

Separation of variables $\Psi(b, \theta) = \psi(b)\psi(\theta)$:

$$\frac{d^2\psi}{db^2} + \left(-s^2 + \frac{1/4 + \rho^2}{\sinh^2 b}\right)\psi = 0, \qquad \frac{d^2\psi}{d\theta^2} + \left(s^2 + \frac{1/4 + \rho^2}{\sin^2 \theta}\right)\psi = 0$$

Eigenvalue problem is singular: $b = 0$; at the both ends of the interval $\theta \in (0, \pi)$. There exist two spectrums of separation constant s.

Continuous spectrum:

$$\psi_{1,2}(b) = (\sinh b)^{1/2 \pm i\rho} \, {}_2F_1\left(\frac{1}{2} \pm i(\rho + s), \frac{1}{2} \pm i(\rho - s); 1 \pm i\rho; -\sinh^2 \frac{b}{2}\right),$$

$$\psi^{(+)}_{\rho s}(\theta) = (\sin\theta)^{i\rho + 1/2} \, {}_2F_1\left(\frac{1}{4} + i\frac{\rho + s}{2}, \frac{1}{4} + i\frac{\rho - s}{2}; \frac{1}{2}; \cos^2\theta\right),$$

$$\psi^{(-)}_{\rho s}(\theta) = (\sin\theta)^{i\rho + 1/2} \cos\theta \, {}_2F_1\left(\frac{3}{4} + i\frac{\rho + s}{2}, \frac{3}{4} + i\frac{\rho - s}{2}; \frac{3}{2}; \cos^2\theta\right).$$

Orthogonal System	Coordinates	Solution
Hyperbolic-Parabolic (HP) $b>0, \theta\in(0,\pi), \gamma>0$	$u_0 = \frac{R}{\sqrt{\gamma}}\frac{\cosh^2 b - \sin^2\theta + \gamma}{2\sin\theta\sinh b}$ $u_1 = \frac{R}{\sqrt{\gamma}}\frac{\cosh^2 b - \sin^2\theta - \gamma}{2\sin\theta\sinh b}$ $u_2 = R\cot\theta\coth b$	Discrete for $s^2>0$
		Continuous for $s^2<0$

Separation of variables $\Psi(b,\theta) = \psi(b)\psi(\theta)$:

$$\frac{d^2\psi}{db^2} + \left(-s^2 + \frac{1/4+\rho^2}{\sinh^2 b}\right)\psi = 0, \qquad \frac{d^2\psi}{d\theta^2} + \left(s^2 + \frac{1/4+\rho^2}{\sin^2\theta}\right)\psi = 0$$

Eigenvalue problem is singular: $b=0$; at the both ends of the interval $\theta\in(0,\pi)$.
There exist two spectrums of separation constant s.

Discrete spectrum:

$$\begin{aligned}
\psi_{\rho s}(b) &= \sqrt{\sinh b}\, Q^{-i\rho}_{-1/2+s}(\cosh b);\\
\psi^{(+)}_{\rho s}(\theta) &= (\sin\theta)^{i\rho+1/2}\, {}_2F_1\left(\frac{1}{4}+\frac{i\rho+s}{2}, \frac{1}{4}+\frac{i\rho-s}{2}; \frac{1}{2}; \cos^2\theta\right),\\
\psi^{(-)}_{\rho s}(\theta) &= (\sin\theta)^{i\rho+1/2}\cos\theta\, {}_2F_1\left(\frac{3}{4}+\frac{i\rho+s}{2}, \frac{3}{4}+\frac{i\rho-s}{2}; \frac{3}{2}; \cos^2\theta\right).
\end{aligned}$$

HP through EQ

Interbasis expansion we take in the form:

$$\Psi^{HP(\pm)}_{\rho s}(\theta, b) = \int\limits_{-\infty}^{\infty} A^{(\pm)}_{\rho s \nu} \Psi^{EQ(\pm)}_{\rho\nu}(\tau_1, \tau_2) d\nu, \quad \Psi^{EQ(\pm)}_{\rho\nu}(\tau_1, \tau_2) = N^{(\pm)}_{\rho\nu}\, \psi^{(\pm)}_{\rho\nu}(\tau_1)\ e^{i\nu\tau_2},$$

$$\begin{aligned} \psi^{(+)}_{\rho\nu}(\tau_1) &= (\cosh\tau_1)^{-1/2-i\rho}\, {}_2F_1\left(\frac{1}{4} + i\frac{\rho-\nu}{2}, \frac{1}{4} + i\frac{\rho+\nu}{2}; \frac{1}{2}; \tanh^2\tau_1\right), \\ \psi^{(-)}_{\rho\nu}(\tau_1) &= \tanh\tau_1 (\cosh\tau_1)^{-1/2-i\rho}\, {}_2F_1\left(\frac{3}{4} + i\frac{\rho-\nu}{2}, \frac{3}{4} + i\frac{\rho+\nu}{2}; \frac{3}{2}; \tanh^2\tau_1\right) \end{aligned}$$

and the normalized constants have the form

$$N^{(+)}_{\rho\nu} = \frac{\left|\Gamma\left(\frac{1}{4} + i\frac{\rho+\nu}{2}\right)\Gamma\left(\frac{1}{4} + i\frac{\rho-\nu}{2}\right)\right|}{R\sqrt{8\pi^3}\,|\Gamma(i\rho)|}, \quad N^{(-)}_{\rho\nu} = \frac{\left|\Gamma\left(\frac{3}{4} + i\frac{\rho+\nu}{2}\right)\Gamma\left(\frac{3}{4} + i\frac{\rho-\nu}{2}\right)\right|}{R\sqrt{2\pi^3}\,|\Gamma(i\rho)|}.$$

HP basis for discrete spectrum:

$$\Psi^{HP(+)}_{\rho s}(\theta, b) = C^{(+)}_{\rho s}\psi_{\rho s}(b)\,(\sin\theta)^{i\rho+\frac{1}{2}}\, {}_2F_1\left(\frac{1}{4} + \frac{i\rho+s}{2}, \frac{1}{4} + \frac{i\rho-s}{2}; \frac{1}{2}; \cos^2\theta\right),$$

$$\Psi^{HP(-)}_{\rho s}(\theta, b) = C^{(-)}_{\rho s}\psi_{\rho s}(b)\,\cos\theta\,(\sin\theta)^{i\rho+\frac{1}{2}}\, {}_2F_1\left(\frac{3}{4} + \frac{i\rho+s}{2}, \frac{3}{4} + \frac{i\rho-s}{2}; \frac{3}{2}; \cos^2\theta\right),$$

where $\mathbf{C}^{(\pm)}_{\rho\mathbf{s}} = ???$, $\psi_{\rho s}(b) = \sqrt{\sinh b}\, Q^{-i\rho}_{-1/2+s}(\cosh b)$.

HP through EQ

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$$\Psi^{HP(\pm)}_{\rho s}(\theta, b) = \int\limits_{-\infty}^{\infty} A^{(\pm)}_{\rho s \nu} \Psi^{EQ(\pm)}_{\rho\nu}(\tau_1, \tau_2) d\nu, \quad \Psi^{EQ(\pm)}_{\rho\nu}(\tau_1, \tau_2) = N^{(\pm)}_{\rho\nu}\, \psi^{(\pm)}_{\rho\nu}(\tau_1)\; e^{i\nu\tau_2},$$

$$\psi^{(+)}_{\rho\nu}(\tau_1) = (\cosh\tau_1)^{-1/2-i\rho}\, {}_2F_1\left(\frac{1}{4} + i\frac{\rho-\nu}{2}, \frac{1}{4} + i\frac{\rho+\nu}{2}; \frac{1}{2}; \tanh^2\tau_1\right),$$

$$\psi^{(-)}_{\rho\nu}(\tau_1) = \tanh\tau_1 (\cosh\tau_1)^{-1/2-i\rho}\, {}_2F_1\left(\frac{3}{4} + i\frac{\rho-\nu}{2}, \frac{3}{4} + i\frac{\rho+\nu}{2}; \frac{3}{2}; \tanh^2\tau_1\right)$$

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$$N^{(+)}_{\rho\nu} = \frac{\left|\Gamma\left(\frac{1}{4} + i\frac{\rho+\nu}{2}\right)\Gamma\left(\frac{1}{4} + i\frac{\rho-\nu}{2}\right)\right|}{R\sqrt{8\pi^3}\,|\Gamma(i\rho)|}, \quad N^{(-)}_{\rho\nu} = \frac{\left|\Gamma\left(\frac{3}{4} + i\frac{\rho+\nu}{2}\right)\Gamma\left(\frac{3}{4} + i\frac{\rho-\nu}{2}\right)\right|}{R\sqrt{2\pi^3}\,|\Gamma(i\rho)|}.$$

HP basis for discrete spectrum:

$$\Psi^{HP(+)}_{\rho s}(\theta, b) = C^{(+)}_{\rho s}\psi_{\rho s}(b)\,(\sin\theta)^{i\rho+\frac{1}{2}}\, {}_2F_1\left(\frac{1}{4} + \frac{i\rho+s}{2}, \frac{1}{4} + \frac{i\rho-s}{2}; \frac{1}{2}; \cos^2\theta\right),$$

$$\Psi^{HP(-)}_{\rho s}(\theta, b) = C^{(-)}_{\rho s}\psi_{\rho s}(b)\,\cos\theta\,(\sin\theta)^{i\rho+\frac{1}{2}}\, {}_2F_1\left(\frac{3}{4} + \frac{i\rho+s}{2}, \frac{3}{4} + \frac{i\rho-s}{2}; \frac{3}{2}; \cos^2\theta\right),$$

where $\mathbf{C}^{(\pm)}_{\rho\mathbf{s}} =???$, $\psi_{\rho s}(b) = \sqrt{\sinh b}\, Q^{-i\rho}_{-1/2+s}(\cosh b)$.

$HP \leftrightarrow EQ$:

$$\cos^2\theta = \frac{u_0 - \sqrt{u_1^2 + R^2}}{u_0 - u_1} \quad = \quad \frac{\cosh\tau_1\cosh\tau_2 - \sqrt{\cosh^2\tau_1\sinh^2\tau_2 + 1}}{e^{-\tau_2}\cosh\tau_1},$$

$$\cosh^2 b = \frac{u_0 + \sqrt{u_1^2 + R^2}}{u_0 - u_1} \quad = \quad \frac{\cosh\tau_1\cosh\tau_2 + \sqrt{\cosh^2\tau_1\sinh^2\tau_2 + 1}}{e^{-\tau_2}\cosh\tau_1}.$$

As $\tau_1 \sim 0$ we have $\cos^2\theta \sim \tau_1^2 e^{\tau_2}(2\cosh\tau_2)^{-1}$, $\cosh^2 b \sim 1 + e^{2\tau_2} = 2e^{\tau_2}\cosh\tau_2$.

$$A_{\rho s\nu}^{(\pm)}\psi_{\rho\nu}^{(\pm)}(\tau_1) = \frac{1}{2\pi N_{\rho\nu}^{(\pm)}}\int_{-\infty}^{+\infty}\Psi_{\rho s}^{HP(\pm)}(\cos\theta, b)e^{-i\nu\tau_2}d\tau_2,$$

$$\psi_{\rho s}(b) = \sqrt{\sinh b}\, Q^{-i\rho}_{-1/2+s}(\cosh b) \sim$$

$$e^{\frac{\tau_2}{2}} q\left(1 + e^{2\tau_2}\right)^{-\frac{1}{4}+\frac{i\rho - s}{2}} e^{-i\rho\tau_2}\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4} + \frac{s-i\rho}{2}\right)_k\left(\frac{1}{4} + \frac{s-i\rho}{2}\right)_k}{(1+s)_k k!}(2e^{\tau_2}\cosh\tau_2)^{-k}$$

$\psi_{\rho\nu}^{(+)}(\tau_1) \sim 1$, $\psi_{\rho\nu}^{(-)}(\tau_1) \sim \tau_1$.

Collecting all the terms we comes to integral for even solution

$$\int_0^{+\infty}\frac{\cosh\left(\frac{1}{4} - \frac{i\rho+s}{2} - i\nu - k\right)\tau_2}{(\cosh\tau_2)^{\frac{1}{4}-\frac{i\rho-s}{2}+k}}d\tau_2 = 2^{-\frac{7}{4}+\frac{s-i\rho}{2}+k}\,\mathrm{B}\left(\frac{i\nu+s}{2} + k, \frac{1}{4} - \frac{i(\rho+\nu)}{2}\right)$$

$HP \leftrightarrow EQ$:

$$\cos^2\theta = \frac{u_0 - \sqrt{u_1^2 + R^2}}{u_0 - u_1} \quad = \quad \frac{\cosh\tau_1\cosh\tau_2 - \sqrt{\cosh^2\tau_1\sinh^2\tau_2 + 1}}{e^{-\tau_2}\cosh\tau_1},$$

$$\cosh^2 b = \frac{u_0 + \sqrt{u_1^2 + R^2}}{u_0 - u_1} \quad = \quad \frac{\cosh\tau_1\cosh\tau_2 + \sqrt{\cosh^2\tau_1\sinh^2\tau_2 + 1}}{e^{-\tau_2}\cosh\tau_1}.$$

As $\tau_1 \sim 0$ we have $\cos^2\theta \sim \tau_1^2 e^{\tau_2}(2\cosh\tau_2)^{-1}$, $\cosh^2 b \sim 1 + e^{2\tau_2} = 2e^{\tau_2}\cosh\tau_2$.

$$A^{(\pm)}_{\rho s\nu}\psi^{(\pm)}_{\rho\nu}(\tau_1) = \frac{1}{2\pi N^{(\pm)}_{\rho\nu}} \int\limits_{-\infty}^{+\infty} \Psi^{HP(\pm)}_{\rho s}(\cos\theta, b) e^{-i\nu\tau_2} d\tau_2,$$

$$\psi_{\rho s}(b) = \sqrt{\sinh b}\, Q^{-i\rho}_{-1/2+s}(\cosh b) \sim$$

$$e^{\frac{\tau_2}{2}} q \left(1 + e^{2\tau_2}\right)^{-\frac{1}{4}+\frac{i\rho - s}{2}} e^{-i\rho\tau_2} \sum_{k=0}^{\infty} \frac{\left(\frac{3}{4} + \frac{s-i\rho}{2}\right)_k \left(\frac{1}{4} + \frac{s-i\rho}{2}\right)_k}{(1+s)_k k!} (2e^{\tau_2}\cosh\tau_2)^{-k}$$

$\psi^{(+)}_{\rho\nu}(\tau_1) \sim 1$, $\psi^{(-)}_{\rho\nu}(\tau_1) \sim \tau_1$.

Collecting all the terms we comes to integral for even solution

$$\int\limits_0^{+\infty} \frac{\cosh\left(\frac{1}{4} - \frac{i\rho+s}{2} - i\nu - k\right)\tau_2}{(\cosh\tau_2)^{\frac{1}{4} - \frac{i\rho - s}{2} + k}} d\tau_2 = 2^{-\frac{7}{4} + \frac{s-i\rho}{2} + k}\, \mathrm{B}\left(\frac{i\nu + s}{2} + k, \frac{1}{4} - \frac{i(\rho+\nu)}{2}\right)$$

HP $\leftrightarrow$ *EQ*:

$$\cos^2\theta = \frac{u_0 - \sqrt{u_1^2 + R^2}}{u_0 - u_1} \quad = \quad \frac{\cosh\tau_1\cosh\tau_2 - \sqrt{\cosh^2\tau_1\sinh^2\tau_2 + 1}}{e^{-\tau_2}\cosh\tau_1},$$

$$\cosh^2 b = \frac{u_0 + \sqrt{u_1^2 + R^2}}{u_0 - u_1} \quad = \quad \frac{\cosh\tau_1\cosh\tau_2 + \sqrt{\cosh^2\tau_1\sinh^2\tau_2 + 1}}{e^{-\tau_2}\cosh\tau_1}.$$

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$$A^{(\pm)}_{\rho s\nu}\psi^{(\pm)}_{\rho\nu}(\tau_1) = \frac{1}{2\pi N^{(\pm)}_{\rho\nu}}\int\limits_{-\infty}^{+\infty}\Psi^{HP(\pm)}_{\rho s}(\cos\theta, b)e^{-i\nu\tau_2}d\tau_2,$$

$$\psi_{\rho s}(b) = \sqrt{\sinh b}\, Q^{-i\rho}_{-1/2+s}(\cosh b) \sim$$

$$e^{\frac{\tau_2}{2}}q\left(1 + e^{2\tau_2}\right)^{-\frac{1}{4}+\frac{i\rho-s}{2}} e^{-i\rho\tau_2}\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4}+\frac{s-i\rho}{2}\right)_k\left(\frac{1}{4}+\frac{s-i\rho}{2}\right)_k}{(1+s)_k k!}(2e^{\tau_2}\cosh\tau_2)^{-k}$$

$\psi^{(+)}_{\rho\nu}(\tau_1) \sim 1$, $\psi^{(-)}_{\rho\nu}(\tau_1) \sim \tau_1$.

Collecting all the terms we comes to integral for even solution

$$\int\limits_0^{+\infty}\frac{\cosh\left(\frac{1}{4}-\frac{i\rho+s}{2}-i\nu-k\right)\tau_2}{(\cosh\tau_2)^{\frac{1}{4}-\frac{i\rho-s}{2}+k}}d\tau_2 = 2^{-\frac{7}{4}+\frac{s-i\rho}{2}+k}\,\mathrm{B}\left(\frac{i\nu+s}{2}+k, \frac{1}{4}-\frac{i(\rho+\nu)}{2}\right)$$

Finally

$$A_{\rho s\nu}^{(+)} = \frac{e^{\pi\rho}\pi R}{2^{s+1}}|\Gamma(i\rho)|C_{\rho s}^{(+)} \frac{\Gamma\left(\frac{1}{2}+s-i\rho\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{s-i\rho}{2}\right)\right|^2} \frac{\Gamma\left(\frac{s+i\nu}{2}\right)}{\Gamma\left(1+\frac{s-i\nu}{2}\right)} \sqrt{\frac{\Gamma\left(\frac{1}{4}-i\frac{\rho+\nu}{2}\right)}{\Gamma\left(\frac{1}{4}+i\frac{\rho+\nu}{2}\right)} \frac{\Gamma\left(\frac{1}{4}+i\frac{\rho-\nu}{2}\right)}{\Gamma\left(\frac{1}{4}-i\frac{\rho-\nu}{2}\right)}}.$$

To define the normalization constant $C_{\rho s}^{(+)}$: $\int\limits_{-\infty}^{\infty} A_{\rho s\nu}^{(+)} A_{\rho s'\nu}^{(+)*} d\nu = \delta_{ss'}$

$$\int\limits_{-\infty}^{\infty} A_{\rho s\nu}^{(+)} A_{\rho s'\nu}^{(+)*} d\nu = C_{\rho s}^{(+)} C_{\rho s'}^{(+)*} \frac{e^{2\pi\rho}\pi^2 R^2}{2^{2+s+s'}} \frac{|\Gamma(i\rho)|^2 \Gamma\left(\frac{1}{2}+s-i\rho\right)\Gamma\left(\frac{1}{2}+s'+i\rho\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{s-i\rho}{2}\right)\right|^2 \left|\Gamma\left(\frac{1}{4}+\frac{s'-i\rho}{2}\right)\right|^2} \int\limits_{-\infty}^{\infty} f(\nu) d\nu,$$

where

$$f(\nu) = \frac{\Gamma\left(\frac{s+i\nu}{2}\right)\Gamma\left(\frac{s'-i\nu}{2}\right)}{\Gamma\left(1+\frac{s-i\nu}{2}\right)\Gamma\left(1+\frac{s'+i\nu}{2}\right)}.$$

Integral is absolutely convergent (is Melling-Barnes of third type). Residue theorem:

$$\int\limits_{-\infty}^{\infty} f(\nu) d\nu = 2\pi i \sum_{k=0}^{N} Res[f(\nu), \nu_k],$$

where ν_k are poles of $f(\nu)$ in upper complex semi plane: $\nu_k = i(s+2k),\ k = 0, 1, \ldots$

Finally

$$A^{(+)}_{\rho s\nu} = \frac{e^{\pi\rho}\pi R}{2^{s+1}}|\Gamma(i\rho)|C^{(+)}_{\rho s}\frac{\Gamma\left(\frac{1}{2}+s-i\rho\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{s-i\rho}{2}\right)\right|^2}\frac{\Gamma\left(\frac{s+i\nu}{2}\right)}{\Gamma\left(1+\frac{s-i\nu}{2}\right)}\sqrt{\frac{\Gamma\left(\frac{1}{4}-i\frac{\rho+\nu}{2}\right)}{\Gamma\left(\frac{1}{4}+i\frac{\rho+\nu}{2}\right)}\frac{\Gamma\left(\frac{1}{4}+i\frac{\rho-\nu}{2}\right)}{\Gamma\left(\frac{1}{4}-i\frac{\rho-\nu}{2}\right)}}.$$

To define the normalization constant $C^{(+)}_{\rho s}$: $\int\limits_{-\infty}^{\infty} A^{(+)}_{\rho s\nu}A^{(+)*}_{\rho s'\nu}d\nu = \delta_{ss'}$

$$\int\limits_{-\infty}^{\infty} A^{(+)}_{\rho s\nu}A^{(+)*}_{\rho s'\nu}d\nu = C^{(+)}_{\rho s}C^{(+)*}_{\rho s'}\frac{e^{2\pi\rho}\pi^2R^2}{2^{2+s+s'}}\frac{|\Gamma(i\rho)|^2\Gamma\left(\frac{1}{2}+s-i\rho\right)\Gamma\left(\frac{1}{2}+s'+i\rho\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{s-i\rho}{2}\right)\right|^2\left|\Gamma\left(\frac{1}{4}+\frac{s'-i\rho}{2}\right)\right|^2}\int\limits_{-\infty}^{\infty} f(\nu)d\nu,$$

where

$$f(\nu) = \frac{\Gamma\left(\frac{s+i\nu}{2}\right)\Gamma\left(\frac{s'-i\nu}{2}\right)}{\Gamma\left(1+\frac{s-i\nu}{2}\right)\Gamma\left(1+\frac{s'+i\nu}{2}\right)}.$$

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$$Res[f(\nu), \nu_k] = \frac{2}{i(-1)^k k!} \frac{\Gamma(\frac{s+s'}{2} + k)}{\Gamma(1+s+k)\,\Gamma\left(1 + \frac{s'-s}{2} - k\right)}.$$

$$\int\limits_{-\infty}^{\infty} f(\nu) d\nu = 4\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(\frac{s+s'}{2} + k\right)}{\Gamma(1+s+k)\,\Gamma\left(1 + \frac{s'-s}{2} - k\right)} =$$

$$= 4 \sin\left(\pi \frac{s-s'}{2}\right) \frac{\Gamma\left(\frac{s+s'}{2}\right)\Gamma\left(\frac{s-s'}{2}\right)}{\Gamma(1+s)} {}_2F_1\left(\frac{s+s'}{2}, \frac{s-s'}{2}; 1+s; 1\right) =$$

$$= \frac{16}{s^2 - s'^2} \sin\left(\pi \frac{s-s'}{2}\right).$$

Let us note, that $\int\limits_{-\infty}^{\infty} f(\nu) d\nu = 0$ if $s = s' + 2k$, $k \in \mathbb{N}$ and is equal to $4\pi/s$ if $s' \sim s$.

The normalization constants for HP wave functions:

$$C_{\rho s}^{(+)} = \frac{\sqrt{2s}}{\pi R e^{\pi\rho} |\Gamma(i\rho)|} \left| \frac{\Gamma\left(\frac{1}{4} + \frac{s+i\rho}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{s+i\rho}{2}\right)} \right|, \quad C_{\rho s}^{(-)} = \frac{2\sqrt{2s}}{\pi R e^{\pi\rho} |\Gamma(i\rho)|} \left| \frac{\Gamma\left(\frac{3}{4} + \frac{s+i\rho}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{s+i\rho}{2}\right)} \right|.$$

$$Res[f(\nu),\nu_k] = \frac{2}{i(-1)^k k!}\frac{\Gamma(\frac{s+s'}{2}+k)}{\Gamma\left(1+s+k\right)\Gamma\left(1+\frac{s'-s}{2}-k\right)}.$$

$$\int\limits_{-\infty}^{\infty} f(\nu)d\nu = 4\pi\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma\left(\frac{s+s'}{2}+k\right)}{\Gamma\left(1+s+k\right)\Gamma\left(1+\frac{s'-s}{2}-k\right)} =$$

$$= 4\sin\left(\pi\frac{s-s'}{2}\right)\frac{\Gamma\left(\frac{s+s'}{2}\right)\Gamma\left(\frac{s-s'}{2}\right)}{\Gamma\left(1+s\right)}{}_2F_1\left(\frac{s+s'}{2},\frac{s-s'}{2};1+s;1\right) =$$

$$= \frac{16}{s^2-s'^2}\sin\left(\pi\frac{s-s'}{2}\right).$$

Let us note, that $\int\limits_{-\infty}^{\infty} f(\nu)d\nu = 0$ if $s = s'+2k$, $k\in\mathbb{N}$ and is equal to $4\pi/s$ if $s'\sim s$.

The normalization constants for HP wave functions:

$$C^{(+)}_{\rho s} = \frac{\sqrt{2s}}{\pi Re^{\pi\rho}|\Gamma(i\rho)|}\left|\frac{\Gamma\left(\frac{1}{4}+\frac{s+i\rho}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{s+i\rho}{2}\right)}\right|,\quad C^{(-)}_{\rho s} = \frac{2\sqrt{2s}}{\pi Re^{\pi\rho}|\Gamma(i\rho)|}\left|\frac{\Gamma\left(\frac{3}{4}+\frac{s+i\rho}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{s+i\rho}{2}\right)}\right|.$$

Hyperbolic parabolic to a Cartesian basis

Hyperbolic parabolic basis contracting to a Cartesian basis on E_2. In these coordinates the points on the hyperbola are given by $\theta \in (0, \pi)$, $b > 0$]:

$$u_0 = R\frac{\cosh^2 b + \cos^2\theta}{2\sinh a \sin\theta}, \quad u_1 = R\frac{\sinh^2 b - \sin^2\theta}{2\sinh a\sin\theta}, \quad u_3 = R\cot\theta\coth b.$$

From these relations we see that

$$\cos^2\theta = \frac{u_0 - \sqrt{u_1^2 + R^2}}{u_0 - u_1}, \qquad \cosh^2 b = \frac{u_0 + \sqrt{u_1^2 + R^2}}{u_0 - u_1}.$$

In the limit as $R \to \infty$ we can choose

$$\cos^2\theta \to \frac{y^2}{2R^2}, \qquad \cosh^2 b \to 2\left(1 + \frac{x}{R}\right).$$

The hyperbolic parabolic basis function on hyperbolid can be chosen in the form

$$\Psi^{HP}(b,\theta) = (\sinh b\,\sin\theta)^{1/2}\; P^{i\rho}_{is-1/2}(\cosh b)\; P^{i\rho}_{is-1/2}(\cos\theta). \tag{4}$$

To proceed further with this limit we take $\rho^2 \sim k^2 R^2$ and $s^2 \sim (k_1^2 - k_2^2) R^2$ (the case of $s^2 < 0$, or $k_1^2 < k_2^2$, corresponds to the discrete spectrum of constant s, and we do not consider this case here) where $k_1^2 + k_2^2 = k^2$

$$\sqrt{\sin\theta}\ P^{i\rho}_{is-1/2}(\cos\theta) \sim P^{ikR}_{i\sqrt{k_1^2-k_2^2}R-1/2}\left(\frac{y}{\sqrt{2}R}\right)$$

$$\sim \frac{2^{ikR}\sqrt{\pi}\exp(ik_2 y)}{\Gamma\left(\frac{3}{4}-\frac{iR}{2}\left(k+\sqrt{k_1^2-k_2^2}\right)\right)\Gamma\left(\frac{3}{4}-\frac{iR}{2}\left(k-\sqrt{k_1^2-k_2^2}\right)\right)}.$$

For the limit of the b dependent part of the eigenfunctions we must proceed differently. In fact we need to calculate the limit of

$$P^{ikR}_{i\sqrt{k_1^2-k_2^2}R-1/2}\left(\sqrt{2\left(1+\frac{x}{R}\right)}\right)$$

as $R \to \infty$.

We know that the leading terms of this expansion have the form

$$A \exp(ik_1 x) + B \exp(-ik_1 x),$$

and we now make use of this fact. By this we mean that

$$\lim_{R\to\infty} P^{ikR}_{i\sqrt{k_1^2-k_2^2}R-1/2}\left(\sqrt{2\left(1+\frac{x}{R}\right)}\right) = A \exp(ik_1 x) + B \exp(-ik_1 x),$$

where the constants A and B depend on R. It remains to determine A and B. To do this let us consider $x = 0$. We then need to determine the following limit

$$\lim_{R\to\infty} P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}\left(\sqrt{2}\right) = A + B. \tag{5}$$

This can be done using the method of stationary phase.

From the integral representation formula

$$\frac{\Gamma(-\nu-\mu)\Gamma(1+\nu-\mu)}{\Gamma(1/2-\mu)}\sqrt{\frac{\pi}{2}}\;P_\nu^\mu(z)$$
$$=(z^2-1)^{-\mu/2}\int_0^\infty (z+\cosh t)^{\mu-1/2}\cosh\left([\nu+1/2]\,t\right)\,dt,$$

the above limit requires us to calculate as $R\to\infty$

$$\int_0^\infty \left(\sqrt{2}+\cosh t\right)^{ikR-1/2}\cos\left(\left[R\sqrt{k_1^2-k_2^2}\right]t\right)\,dt.$$

We obtain

$$P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}\left(\sqrt{2}\right)\sim\frac{2^{-\frac{5}{4}+\frac{iR}{2}\left(\sqrt{k_1^2-k_2^2}-k\right)}\Gamma\left(\frac{1}{2}-ikR\right)}{\Gamma\left[\frac{1}{2}-iR\left(\sqrt{k_1^2-k_2^2}+k\right)\right]\Gamma\left[\frac{1}{2}+iR\left(\sqrt{k_1^2-k_2^2}-k\right)\right]}$$
$$\left(\frac{i}{Rk_1}\right)^{1/2}\left(\frac{k_1-\sqrt{k_1^2-k_2^2}}{k+\sqrt{k_1^2-k_2^2}}\right)^{iR\sqrt{k_1^2-k_2^2}}\left(\frac{k}{k-k_1}\right)^{ikR}.$$

By considering the expression for the derivatives of the Legendre function (6) at $x = 0$, we derive the expression

$$\frac{d}{dx} P_\nu^\mu(z)\bigg|_{x=0} \sim -ik_1\, P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}\left(\sqrt{2}\right) \sim ik_1(A-B),$$

then

$$P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}\left(\sqrt{2}\right) \sim -A+B.$$

Comparing the above relation with (5), we obtain that $A = 0$ and B is equal to (6), that is

$$P^{i\rho}_{is-\frac{1}{2}}(\cosh b) \to Be^{-ixk_1}.$$

Finally, solution (4) contracts as follows:

$$\Psi_{\rho s}(b,\theta) \sim \frac{2^{ikR}\sqrt{\pi}B}{\Gamma\left(\frac{3}{4} - iR\frac{k+\sqrt{k_1^2-k_2^2}}{2}\right)\Gamma\left(\frac{3}{4} - iR\frac{k-\sqrt{k_1^2-k_2^2}}{2}\right)}\ \exp(ik_2y - ik_1x).$$

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