Sampling in Euclidean and Non-Euclidean Domains: A Unified Approach

NIST ACMD Seminar Series

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- Joint work with Jens Christensen (Colgate University).

Sampling Theory in $\mathbb{R}$

- $\mathbb{PW}_\Omega = \{f : f, \widehat{f} \in L^2, \mathrm{supp}(\widehat{f}) \subset [-\Omega, \Omega]\}$.

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Theorem (C-W-W-K-S-R-O-... Sampling Theorem)

Let $f \in \mathbb{PW}_\Omega$, $\delta_{nT}(t) = \delta(t - nT)$ *and* $\mathrm{sinc}_T(t) = \frac{\sin(\frac{2\pi}{T}t)}{\pi t}$.

a.) *If* $T \leq 1/2\Omega$*, then for all* $t \in \mathbb{R}$,

$$f(t) = T \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{2\pi}{T}(t - nT))}{\pi(t - nT)} = T\left(\left[\sum_{n\in\mathbb{Z}} \delta_{nT}\right] \cdot f\right) * \mathrm{sinc}_T$$

b.) *If* $T \leq 1/2\Omega$ *and* $f(nT) = 0$ *for all* $n \in \mathbb{Z}$*, then* $f \equiv 0$.

Formal Proof of W-K-S Sampling Theorem

Let $T > 0$ and let $f \in L^1([0, T))$. Assume that we can expand the T-periodization of f $(f_T)^\circ(t)$ in a Fourier series. This yields

$$\sum_{n\in\mathbb{Z}} f(t+nT) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \widehat{f}(n/T)e^{2\pi i(n/T)t} \quad (\mathrm{PSF})\,.$$

This extends to the class of Schwarz distributions as

$$\widehat{\sum_{n\in\mathbb{Z}} \delta_{nT}} = \frac{1}{T}\sum_{n\in\mathbb{Z}} \delta_{n/T} \quad (\mathrm{PSF2})\,.$$

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If $f \in \mathbb{PW}_\Omega$ and $T \leq 1/2\Omega$, then

$$\widehat{f}(\omega) = \left(\sum_{n\in\mathbb{Z}} \widehat{f}(\omega - \frac{n}{T})\right)\cdot\chi_{[-1/2T,1/2T)}(\omega) = \left(\sum_{n\in\mathbb{Z}} \left[\delta_{n/T}\right] * \widehat{f}\right)\cdot\chi_{[-1/2T,1/2T)},$$

which holds if and only if

$$f(t) = T\left(\left[\sum_{n\in\mathbb{Z}} \delta_{nT}\right]\cdot f\right) * \underset{T}{\operatorname{sinc}}(t)\,.$$

Beurling-Landau Densities

- A sequence Λ is *separated* or *uniformly discrete* if $q = \inf_k(\lambda_{k+l} - \lambda_k) > 0$. Distribution function

$$n_\Lambda(b) - n_\Lambda(a) = \mathrm{card}(\Lambda \cap (a, b]) \,,$$

normalized – $n_\Lambda(0) = 0$.

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- A discrete set Λ is a *set of sampling* for $\mathbb{PW}_\Omega$ if there exists a constant $C > 0$ such that $C\|f\|_2^2 \leq \sum_{\lambda_k \in \Lambda} |f(\lambda_k)|^2$ for every $f \in \mathbb{PW}_\Omega$.

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- Λ is a *set of uniqueness* for $\mathbb{PW}_\Omega$ if $f|_\Lambda = 0$ implies that $f = 0$.
- Λ is a *set of sampling and uniqueness* if there exists constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{\lambda_k \in \Lambda} |f(\lambda_k)|^2 \le B\|f\|_2^2 .$$

Beurling-Landau Densities, cont'd

Definition (Beurling-Landau Densities)

1.) The *Beurling-Landau lower density*

$$D^{-}(\Lambda) = \liminf_{r\to\infty} \inf_{t\in\mathbb{R}} \frac{(n_{\Lambda}(t+r)) - n_{\Lambda}(t)}{r}$$

2.) The *Beurling-Landau upper density*

$$D^{+}(\Lambda) = \limsup_{r\to\infty} \sup_{t\in\mathbb{R}} \frac{(n_{\Lambda}(t+r)) - n_{\Lambda}(t)}{r}$$

- For exact and stable reconstruction – $D^{-}(\Lambda) \geq 1$.
  Fails – $D^{-}(\Lambda) < 1$.
  (Note – There are sets of uniqueness with arbitrarily small density.)

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  Fails – $D^{-}(\Lambda) < 1$.
  (Note – There are sets of uniqueness with arbitrarily small density.)
- If $D^{+}(\Lambda) \leq 1$, then Λ is a *set of interpolation*.

Sampling Theory in $\mathbb{R}^d$

- Let $T > 0$ and let $g(t)$ be a function such that $\mathrm{supp}\, f \subseteq [0, T]^d$. Assume that we can expand the T-periodization of f $(f_T)^\circ(t)$ in a Fourier series, we get

$$\sum_{n \in \mathbb{Z}^d} f(t + nT) = \frac{1}{T^d} \sum_{n \in \mathbb{Z}^d} \widehat{f}(n/T) e^{2\pi i n \cdot t/T} \quad \text{(PSF)}\,.$$

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- As before, we get

$$\sum_{n\in\mathbb{Z}^d} f(nT) = \frac{1}{T^d} \sum_{n\in\mathbb{Z}^d} \widehat{f}(n/T) \ \text{(PSF1)}\,,$$

$$\widehat{\sum_{n\in\mathbb{Z}^d} \delta_{nT}} = \frac{1}{T^d} \sum_{n\in\mathbb{Z}^d} \delta_{n/T} \ \text{(PSF2)}\,.$$

Sampling Theory in $\mathbb{R}^d$, Cont'd

- We can write Poisson summation for an arbitrary lattice by a change of coordinates. Let $\mathbf{A}$ be an invertible $d \times d$ matrix, $\Lambda = \mathbf{A}\,\mathbb{Z}^d$, and $\Lambda^{\perp} = (\mathbf{A}^T)^{-1}\mathbb{Z}^d$ be the dual lattice. Then

$$\begin{aligned}\sum_{\lambda\in\Lambda} f(t+\lambda) &= \sum_{n\in\mathbb{Z}^d}(f\circ\mathbf{A})(\mathbf{A}^{-1}t+n) = \sum_{n\in\mathbb{Z}^d}\widehat{(f\circ\mathbf{A})}(n)e^{2\pi i n\cdot\mathbf{A}^{-1}(t)}\\ &= \frac{1}{|\det\mathbf{A}|}\sum_{n\in\mathbb{Z}^d}\widehat{f}((\mathbf{A}^T)^{-1}(n))e^{2\pi i(\mathbf{A}^T)^{-1}(n)\cdot t}.\end{aligned}$$

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$$\begin{aligned}\sum_{\lambda\in\Lambda} f(t+\lambda) &= \sum_{n\in\mathbb{Z}^d} (f\circ\mathbf{A})(\mathbf{A}^{-1}t+n) = \sum_{n\in\mathbb{Z}^d} \widehat{(f\circ\mathbf{A})}(n)e^{2\pi i n\cdot\mathbf{A}^{-1}(t)} \\ &= \frac{1}{|\det\mathbf{A}|}\sum_{n\in\mathbb{Z}^d} \widehat{f}((\mathbf{A}^T)^{-1}(n))e^{2\pi i(\mathbf{A}^T)^{-1}(n)\cdot t}.\end{aligned}$$

- Since $|\det\mathbf{A}| = \mathrm{vol}(\Lambda)$, we can write this as

$$\sum_{\lambda\in\Lambda} f(t+\lambda) = \frac{1}{\mathrm{vol}(\Lambda)}\sum_{\beta\in\Lambda^{\perp}} \widehat{f}(\beta)e^{2\pi i\beta\cdot t} \ \ (\mathrm{PSF}).$$

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- This extends again to the Schwartz class of distributions as

$$\widehat{\sum_{\lambda\in\Lambda}\delta_\lambda} = \frac{1}{\mathrm{vol}(\Lambda)}\sum_{\beta\in\Lambda^\perp}\delta_\beta \ \ (\mathrm{PSF2}).$$

Sampling Theory in $\mathbb{R}^d$, Cont'd

- The dual sampling lattice can be written as $\Lambda^{\perp} = \{\lambda^{\perp} : \lambda^{\perp} = z_1\omega_1 + z_2\omega_2 + \ldots + z_d\omega_d\}$. This creates a fundamental sampling parallelpiped $\mathbf{\Omega}_{\mathcal{P}}$ in $\widehat{\mathbb{R}^d}$. If the region $\mathbf{\Omega}_{\mathcal{P}}$ is a hyper-rectangle, we get

$$f(t) = \frac{1}{\mathrm{vol}(\Lambda)} \sum_{n \in \mathbb{Z}^d} f(n_1\omega_1, \ldots) \frac{\sin(\frac{\pi}{\omega_1}(t - n_1\omega_1))}{\pi(t - n_1\omega_1)} \cdot \ldots \cdot \frac{\sin(\frac{\pi}{\omega_d}(t - n_d\omega_d))}{\pi(t - n_d\omega_d)} .$$

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- If, however, $\mathbf{\Omega}_{\mathcal{P}}$ is a general parallelepiped, we first have to compute the inverse Fourier transform of $\chi_{\mathbf{\Omega}_{\mathcal{P}}}$. Let $\mathcal{S}$ be the generalized sinc function

$$\mathcal{S} = \frac{1}{\mathrm{vol}(\Lambda)} (\chi_{\mathbf{\Omega}_{\mathcal{P}}})^{\vee} .$$

  Then, the sampling formula becomes

$$f(t) = \sum_{\lambda \in \Lambda} f(\lambda) \mathcal{S}(t - \lambda) .$$

Sampling Theory in $\mathbb{R}^d$, Cont'd

Definition (Nyquist Tiles for $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$)

Let

$$\mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}} = \{f \text{ continuous} : f \in L^2(\mathbb{R}^d), \widehat{f} \in L^2(\widehat{\mathbb{R}^d}), \mathrm{supp}(\widehat{f}) \subset \mathbf{\Omega}_{\mathcal{P}}\}.$$

Let $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$. The *Nyquist Tile* $\mathbb{NT}(f)$ for f is the parallelepiped of minimal area in $\widehat{\mathbb{R}^d}$ such that $\mathrm{supp}(\widehat{f}) \subseteq \mathbb{NT}(f)$. A *Nyquist Tiling* is the set of translates $\{\mathbb{NT}(f)_k\}_{k \in \mathbb{Z}^d}$ of Nyquist tiles which tile $\widehat{\mathbb{R}^d}$.

Sampling Theory in $\mathbb{R}^d$, Cont'd

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Definition (Sampling Group for $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$)

Let $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$ with *Nyquist Tile* $\mathbb{NT}(f)$. The *Sampling Group* $\mathbb{G}$ is a symmetry group of translations such that $\mathbb{NT}(f)$ tiles $\widehat{\mathbb{R}^d}$.

Sampling Theory in $\mathbb{R}^d$, Cont'd

We use our sampling lattices to develop *Voronoi cells* corresponding to the sampling lattice. These cells will be, in the Euclidean case, our Nyquist tiles.

Definition (Voronoi Cells in $\widehat{\mathbb{R}^d}$)

Let $\Lambda = \{\lambda_k \in \mathbb{R}^d\}$ be a sampling set for $f \in \mathbb{PW}_\Omega$. Let $\Lambda^\perp$ be the dual lattice in frequency space. Then, the Voronoi cells $\{\Phi_k\}$, the Voronoi partition $\mathcal{VP}(\Lambda^\perp)$, and partition norm $\|\mathcal{VP}(\Lambda^\perp)\|$ corresponding to the sampling lattice are defined as follows.

1.) The *Voronoi cells* $\Phi_k = \{\omega \in \widehat{\mathbb{R}^d} : \mathrm{dist}(\omega, \lambda_k^\perp) \leq \inf_{j \neq k} \mathrm{dist}(\omega, \lambda_j^\perp)\}$,

2.) The *Voronoi partition* $\mathcal{VP}(\Lambda^\perp) = \{\Phi_k \in \widehat{\mathbb{R}^d}\}_{k \in \mathbb{Z}^d}$,

3.) The *partition norm* $\|\mathcal{VP}(\Lambda^\perp)\| = \sup_{k \in \mathbb{Z}^d} \sup_{\omega, \nu \in \Phi_k} \mathrm{dist}(\omega, \nu)$.

Sampling Theory in $\mathbb{R}^d$, Cont'd

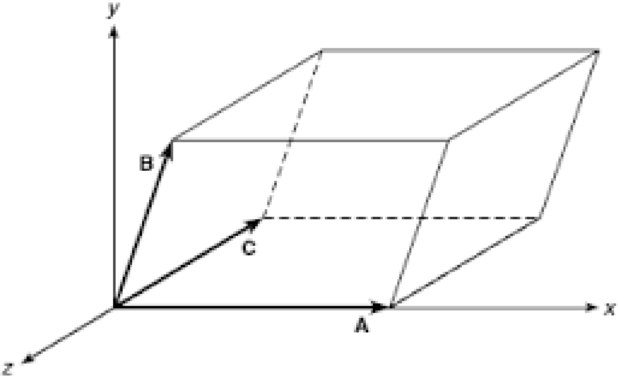

Figure: 3D Nyquist Cell

Sampling Theory in $\mathbb{R}^d$, Cont'd

Theorem (Nyquist Tiling for Euclidean Space (C, (3)(2016)))

Let $f \in \mathbb{PW}_{\mathbf{\Omega}_{\mathcal{P}}}$, and let $\Lambda = \{\lambda_k \in \mathbb{R}^d\}_{k \in \mathbb{Z}^d}$ be the sampling grid which samples f exactly at Nyquist. Let $\Lambda^{\perp}$ be the dual lattice in frequency space. Then the Voronoi partition $\mathcal{VP}(\Lambda^{\perp}) = \{\Phi_k \in \widehat{\mathbb{R}^d}\}_{k \in \mathbb{Z}^d}$ equals the Nyquist Tiling, i.e.,

$$\{\Phi_k \in \widehat{\mathbb{R}^d}\}_{k \in \mathbb{Z}^d} = \{\mathbb{NT}(f)_k\}_{k \in \mathbb{Z}^d} .$$

Moreover, the partition norm equals the volume of $\Lambda^{\perp}$, i.e.,

$$\|\mathcal{VP}(\Lambda^{\perp})\| = \sup_{k \in \mathbb{Z}^d} \sup_{\omega, \nu \in \Phi_k} \mathrm{dist}(\omega, \nu) = \mathrm{vol}(\Lambda^{\perp}) ,$$

and the sampling group $\mathbb{G}$ is exactly the group of motions that preserve $\Lambda^{\perp}$.

Why Use Voronoi Cells?

- Allows us to create a unified construction of Sampling in all geometries.

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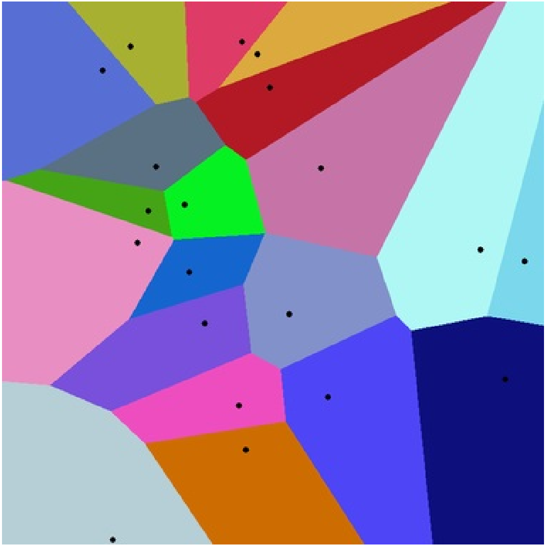
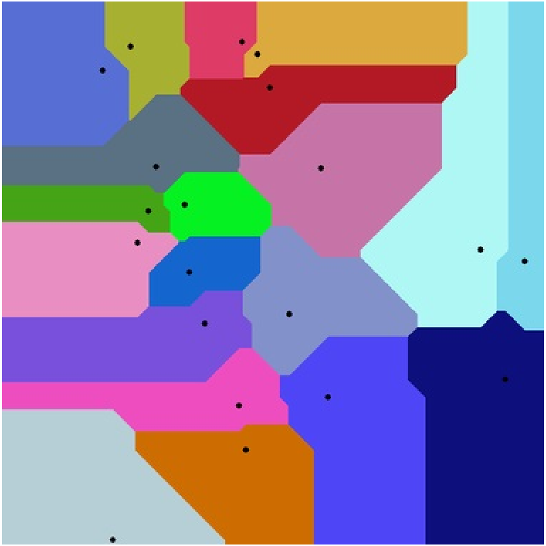
- Allows us to create a unified construction of Sampling in all geometries.
- For a fixed grid, if the geometry changes, the Voronoi Cells give the correct Nyquist Tiles for that geometry.
- Reduces the question of Sampling purely to the optimal sampling grid.

Why Use Voronoi Cells, Cont'd?



Why Use Voronoi Cells, Cont'd?



Geometry of Surfaces

- Understand the geometry by understanding the group of motions that preserve the geometry – Klein's *Erlangen Program*.

Euclidean Geometry

- Euclidean Geometry – Rotations and Translations.

$$\varphi_{\theta,\alpha} = e^{i\theta} z - \alpha \, .$$

- Length –

$$\mathcal{L}_E(\Gamma) = \int_\Gamma |dz| \, .$$

- $$\mathcal{L}_E(\varphi_{\theta,\alpha}(\Gamma)) = \mathcal{L}_E(\Gamma) \, .$$

Spherical Geometry

- Spherical Geometry – Möbius Transformations

$$\varphi_{\alpha,\beta} = \frac{\alpha z - \beta}{-\overline{\beta} z - \overline{\alpha}} \, ,$$

where $|\alpha|^2 + |\beta|^2 = 1$.

- Length –

$$\mathcal{L}_S(\Gamma) = \int_\Gamma \frac{2\,|dz|}{1 + |z|^2} \, .$$

- $$\mathcal{L}_S(\varphi_{\alpha,\beta}(\Gamma)) = \mathcal{L}_S(\Gamma) \, .$$

Spherical Geometry, Cont'd

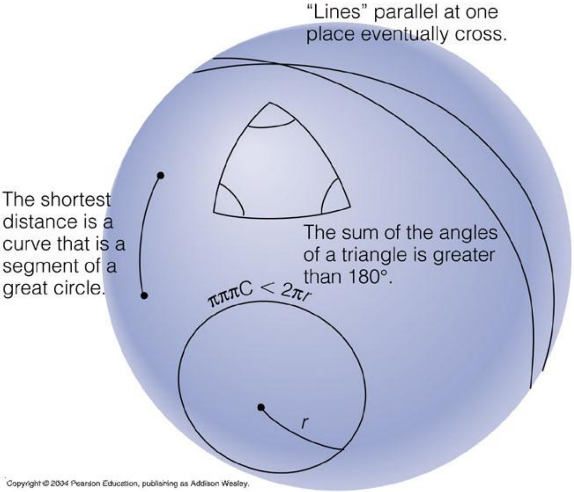


Figure: Spherical Geometry

Hyperbolic Geometry

- Hyperbolic Geometry – Möbius-Blaschke Transformations

$$\varphi_{\theta,\alpha} = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z}, \; \alpha \in \mathbb{D},$$

  where $|\alpha| < 1$.
- Length –

$$\mathcal{L}_H(\Gamma) = \int_\Gamma \frac{2\,|dz|}{1 - |z|^2}.$$

- $$\mathcal{L}_H(\varphi_{\theta,\alpha}(\Gamma)) = \mathcal{L}_H(\Gamma).$$

Hyperbolic Geometry, Cont'd

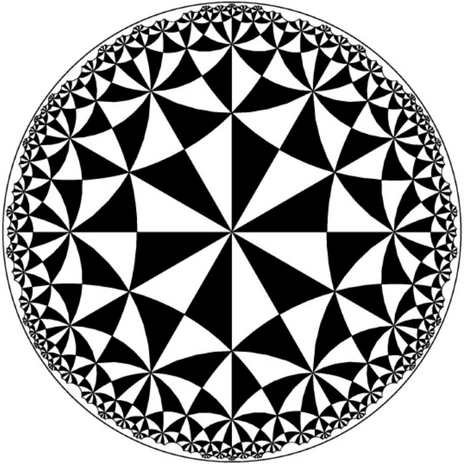


Figure: Hyperbolic Tesselation – $\mathrm{SU}(1,1)$

Hyperbolic Geometry, Cont'd

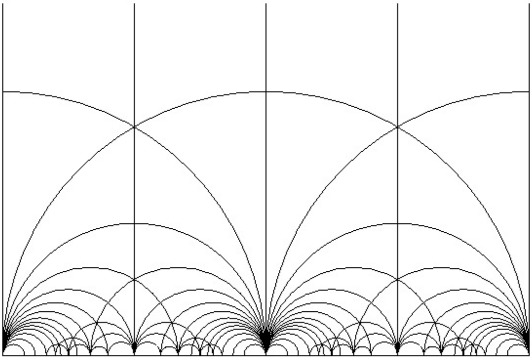
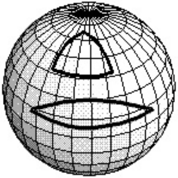


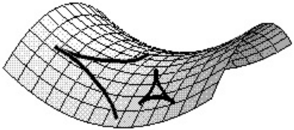
Figure: Hyperbolic Upper Half Plane $\mathbb{H}$ – $\mathrm{SL}(2,\mathbb{R})$

Hyperbolic Geometry, Cont'd

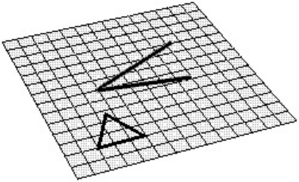
Curvature “in a Nutshell”



Universe with *positive* curvature. Diverging line converge at great distances. Triangle angles add to more than 180°.



Universe with *negative* curvature. Lines diverge at ever increasing angles. Triangle angles add to less than 180°.



Universe with no curvature. Lines diverge at constant angle. Triangle angles add to 180°.

Theorem

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- *Every surface admits a Riemannian metric of constant Gaussian curvature* κ.
- *Every simply connected Riemann surface (universal covering space* $\widetilde{\mathcal{S}}$*) is conformally equivalent to one of the following:*
    - *The Plane* $\mathbb{C}$ *– Euclidean Geometry –* $\kappa = 0$ *–*

$$\left\langle \left\{ e^{i\theta} z + \alpha \right\}, \circ \right\rangle$$

    - *The Riemann Sphere* $\widetilde{\mathbb{C}}$ *– Spherical Geometry –* $\kappa = 1$ *–*

$$\left\langle \left\{ \frac{\alpha z - \beta}{-\overline{\beta} z - \overline{\alpha}} \right\}, \circ \right\rangle, \ \textit{where } |\alpha|^2 + |\beta|^2 = 1 \,.$$

    - *The Poincaré Disk* $\mathbb{D}$ *– Hyperbolic Geometry –* $\kappa = -1$ *–*

$$\left\langle \left\{ e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z} \right\}, \circ \right\rangle, \ \textit{where } |\alpha| < 1 \,.$$

General Surface

- Given connected Riemann surface $\mathcal{S}$ and its universal covering space $\widetilde{\mathcal{S}}$, $\mathcal{S}$ is isomorphic to $\widetilde{\mathcal{S}}/\Gamma$, where the group Γ is isomorphic to the fundamental group of $\mathcal{S}$, $\pi_1(\mathcal{S})$.

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- The corresponding covering is simply the quotient map which sends every point of $\widetilde{\mathcal{S}}$ to its orbit under Γ.
- A *fundamental domain* is a subset of $\widetilde{\mathcal{S}}$ which contains exactly one point from each of these orbits.

Sampling in Spherical Geometry

- The sphere is compact, and its study requires different tools.
- Sampling on the sphere is how to sample a band-limited function, an Nth degree spherical polynomial, at a finite number of locations, such that all of the information content of the continuous function is captured.

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- The open question is the establishment of the optimal Beurling-Landau densities. This leads to questions about sphere tiling.

Fourier Analysis in Hyperbolic Geometry

- Let dz denote the area measure on the unit disc $\mathbb{D} = \{z \mid |z| < 1,$ and let the measure dv be given by the $\mathrm{SU}(1,1)$-invariant measure on $\mathbb{D}$, given by $dv(z) = dz/(1 - |z|^2)^2$.

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- For $f \in L^1(\mathbb{D}, dv)$ the *Fourier-Helgason transform* (FHT) –

$$\widehat{f}(\lambda, b) = \int_{\mathbb{D}} f(z) e^{(-i\lambda+1)\langle z,b\rangle}\, dv(z)$$

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- FHT Inversion

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{T}} \widehat{f}(\lambda, b) e^{(i\lambda+1)\langle z,b\rangle} \lambda \tanh(\lambda\pi/2)\, d\lambda\, db\,.$$

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  Then, since we don't know Nyquist, we cover the bands with disks.
- There is a natural number N such that for any sufficiently small r there are points $x_j \in \mathbb{D}$ for which $B(x_j, r/4)$ are disjoint, $B(x_j, r/2)$ cover $\mathbb{D}$ and $1 \leq \sum_j \chi_{B(x_j,r)} \leq N$. Such a collection of $\{x_j\}$ will be called an (r, N)-lattice. Let ϕ_j be smooth non-negative functions which are supported in $B(x_j, r/2)$ and satisfy that $\sum_j \phi_j = 1_{\mathbb{D}}$

Sampling in Hyperbolic Geometry, Cont'd

- Let ϕ_j be smooth non-negative functions which are supported in $B(x_j, r/2)$ and satisfy that $\sum_j \phi_j = 1_{\mathbb{D}}$. Define the operator

$$Tf(x) = P_\Omega \left(\sum_j f(x_j)\phi_j(x) \right),$$

where P_Ω is the orthogonal projection from $L^2(\mathbb{D}, dv)$ onto $\mathbb{PW}_\Omega(\mathbb{D})$.

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  where P_Ω is the orthogonal projection from $L^2(\mathbb{D}, dv)$ onto $\mathbb{PW}_\Omega(\mathbb{D})$.
- By decreasing r one can obtain the inequality $\|I - T\| < 1$, in which case T can be inverted by

$$T^{-1}f = \sum_{k=0}^{\infty} (I - T)^k f.$$

Sampling in Hyperbolic Geometry, Cont'd

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- For given samples, we can calculate Tf and the Neumann series, which provides the recursion formula

$$f_{n+1} = f_n + Tf - Tf_n\,.$$

Then $f_{n+1} \to f$ as $n \to \infty$ in norm. The rate of convergence – $\|f_n - f\| \leq \|I - T\|^{n+1}\|f\|$.

Sampling in Hyperbolic Geometry, Cont'd

Theorem (Irregular Sampling by Iteration (C, (2016))

Let $\mathcal{S}$ be a Riemann surface whose universal covering space $\widetilde{\mathcal{S}}$ is hyperbolic. Then there exists an (r, N)-lattice on $\mathcal{S}$ such that given $f \in \mathbb{PW}_\Omega = \mathbb{PW}_\Omega(\mathcal{S})$, f can be reconstructed from its samples on the lattice via the recursion formula

$$f_{n+1} = f_n + Tf - Tf_n \,.$$

We have $f_{n+1} \to f$ as $n \to \infty$ in norm. The rate of convergence is $\|f_n - f\| \le \|I - T\|^{n+1}\|f\|$.

This, however, leaves open questions about densities. We address this in the next few frames.

Sampling in Hyperbolic Geometry, Cont'd

Equip the unit disc $\mathbb{D}$ with normalized area measure $d\sigma(z)$, let $\mathcal{O}$ be the set of holomorphic functions, and let $1 \leq p < \infty$ be given.

Definition (Bergman Space)

$$A^p(\mathbb{D}) = L^p(\mathbb{D}, d\sigma) \cap \mathcal{O}(\mathbb{D}) .$$

This is a reproducing kernel Banach space with reproducing kernel

$$K(z, w) = \frac{1}{(1 - \overline{w}z)^2} .$$

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- Define
$$\rho(z,\zeta) = \left| \frac{z-\zeta}{1-\bar{z}\zeta} \right|$$
- Let $\Gamma_k = \{z_k\}$ be a set of uniformly discrete points, that is
$$\inf_{j \neq k} \rho(z_j, z_k) > 0$$

Sampling in Hyperbolic Geometry, Cont'd

- For each z let $n_z(r)$ be the number of points from Γ_k in the disk $|\zeta| < r$, and define

$$N_z(r) = \int_0^r n_z(\tau) d\tau$$

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$$A(r) = \int_0^r a(\rho)d\rho$$

Sampling in Hyperbolic Geometry, Cont'd

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Sampling in Hyperbolic Geometry, Cont'd

Theorem (Seip and Schuster)

Let Λ be a set of distinct points in $\mathbb{D}$.

1.) *A sequence Λ is a set of sampling for A^p if and only if it is a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence Λ' for which $D^-(\Lambda') > 1/p$.*

2.) *A sequence Λ is a set of interpolation for A^p if and only if it is uniformly discrete and $D^+(\Lambda) < 1/p$.*

Sampling in Hyperbolic Geometry, Cont'd

- Recall: (FHT) –

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- Because of $\langle z, b\rangle$, $\widehat{f}(\lambda, b)$ is not analytic.
- Seip and Schuster results do not apply.

Sampling in Hyperbolic Geometry, Cont'd

Let dist denote the weighted distance in $\mathbb{R}^+ \times \mathbb{T}$, weighted by $\frac{1}{2\pi}\lambda \tanh(\lambda\pi/2)$. Using this distance, we can define the following.

Definition (Voronoi Cells in $\widehat{\mathbb{D}}$)

Let $\Lambda = \{\lambda_k \in \mathbb{D} : k \in \mathbb{N}\}$ be a sampling set on $\mathbb{D}$. Let $\Lambda^\perp \subset \mathbb{R}^+ \times \mathbb{T}$ be the dual lattice in frequency space. Then, the Voronoi cells $\{\Phi_k\}$, the Voronoi partition $\mathcal{VP}(\Lambda^\perp)$, and partition norm $\|\mathcal{VP}(\Lambda^\perp)\|$ corresponding to the sampling lattice are defined as follows.

1.) The *Voronoi cells* $\Phi_k = \{\omega \in \widehat{\mathbb{D}} : \mathrm{dist}(\omega, \lambda_k^\perp) \leq \inf_{j \neq k} \mathrm{dist}(\omega, \lambda_j^\perp)\}$,

2.) The *Voronoi partition* $\mathcal{VP}(\Lambda^\perp) = \{\Phi_k \in \widehat{\mathbb{D}}\}_{k \in \mathbb{Z}^d}$,

3.) The *partition norm* $\|\mathcal{VP}(\Lambda^\perp)\| = \sup_{k \in \mathbb{Z}^d} \sup_{\omega, \nu \in \Phi_k} \mathrm{dist}(\omega, \nu)$.

Sampling on a General Surface

- Recall the following.
- Given connected Riemann surface $\mathcal{S}$ and its universal covering space $\widetilde{\mathcal{S}}$, $\mathcal{S}$ is isomorphic to $\widetilde{\mathcal{S}}/\Gamma$, where the group Γ is isomorphic to the fundamental group of $\mathcal{S}$, $\pi_1(\mathcal{S})$.

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- A *fundamental domain* is a subset of $\widetilde{\mathcal{S}}$ which contains exactly one point from each of these orbits.

Sampling on a General Surface

- By the Uniformization Theorem – The only covering surface of Riemann sphere $\widetilde{\mathbb{C}}$ is itself, with the covering map being the identity. The plane $\mathbb{C}$ is the universal covering space of itself, the once punctured plane $\mathbb{C} \setminus \{z_0\}$ (with covering map $\exp(z - z_0)$), and all tori $\mathbb{C}/\Gamma$, where Γ is a parallelogram generated by $z \longmapsto z + n\gamma_1 + m\gamma_2$, $n, m \in \mathbb{Z}$ and γ_1, γ_2 are two fixed complex numbers linearly independent over $\mathbb{R}$.

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- The universal covering space of every other Riemann surface is the hyperbolic disk $\mathbb{D}$.
- Therefore, the establishment of exact the Beurling-Landau densities for functions in Paley-Wiener spaces in spherical and especially hyperbolic geometries will allow the development of sampling schemes on arbitrary Riemann surfaces.

Sampling on a General Surface, cont'd

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Sampling on a General Surface, cont'd

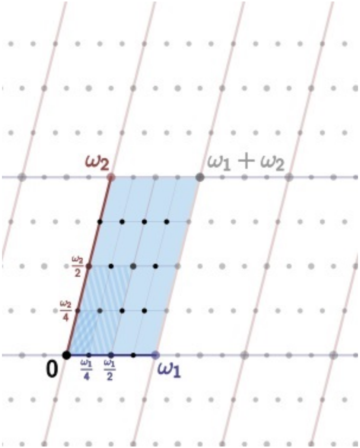

Figure: Torus Fundamental Domain

Sampling on a General Surface, cont'd

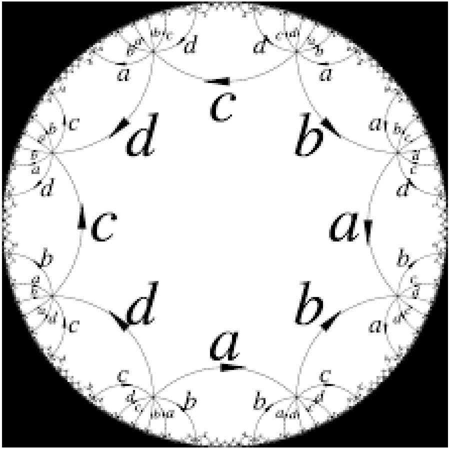

Figure: Two Torus Fundamental Domain

Sampling on a General Surface, cont'd

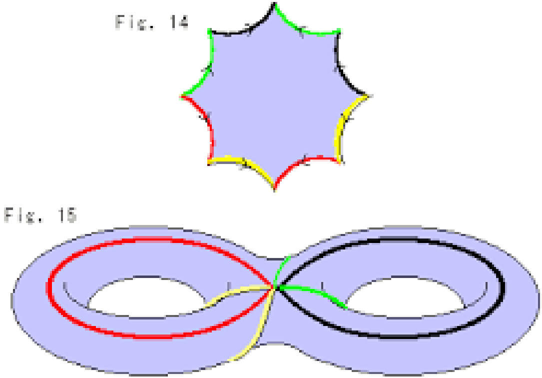

Figure: Two Torus Fundamental Domain – A Second Look

Sampling on a General Surface, cont'd

- Compact vs. Non-Compact, cont'd

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- Sampling on a **non-compact** surface is how to sample a band-limited function at an infinite number of locations, such that all of the information content of the continuous function is captured.
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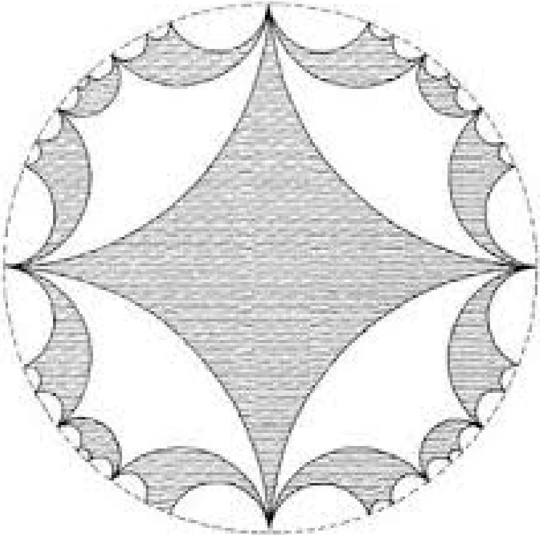

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- These are challenging problems.
- Maybe the following will help ... .

Sampling on a General Surface, cont'd

Applications

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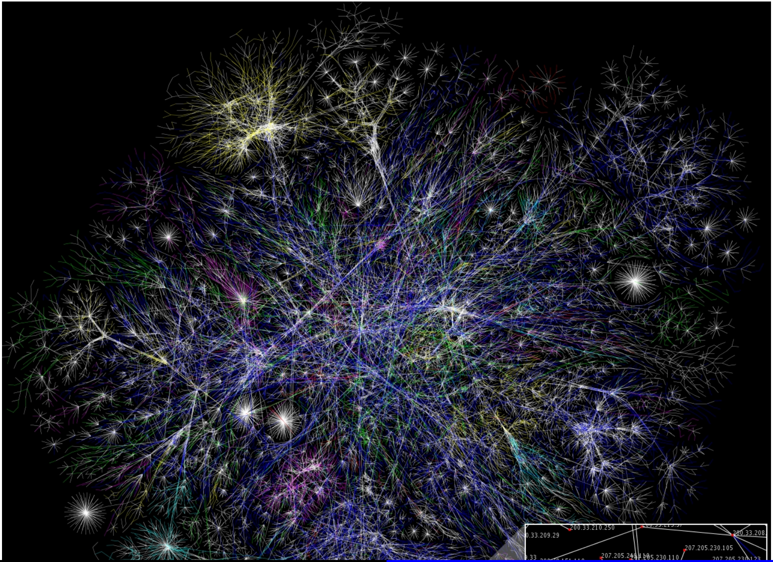
Network Tomography



The Radon Transform

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- *The Radon Transform* in two dimensional space is the mapping defined by the projection or line integral of $f \in L^1$ along all possible lines L. In higher dimensions, given a function f, the Radon Transform of f, designated by $\mathcal{R}(f) = \check{f}$, is determined by integrating over each hyperplane in the space.

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- The n dimensional Radon Transform $\mathcal{R}_n$ is related to the n dimensional Fourier Transform by $\mathcal{R}_n(f) = \mathcal{F}_1^{-1}\mathcal{F}_n(f)$. This allows us to use Fourier methods in computations, and get relations of shifting, scaling, convolution, differentiation, and integration.

The Radon Transform, cont'd

- The inversion formula is necessary to recover desired information about internal structure. The formula can be derived in an even and odd part, then unified analogously to the Fourier Series. The unified inversion formula is

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The Radon Transform, cont'd

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  where Υ_0 is the Helgason Operator.
- In hyperbolic space, the Radon Transform is a $1-1$ mapping on the space of continuous functions with exponential decrease. This makes it the natural tool to use.

Network Tomography

- Munzner has proven that the internet has a hyperbolic structure. Therefore, a good tool to deal with the network problems we are interested in is the discrete Radon transform on trees and its inversion formula.

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- A tree T is a finite or countable collection V of vertices $v_j, j = 0, 1, \ldots$ and a collection E of edges $e_j k = (v_j, v_k)$, i.e., pairs of vertices. For every edge, we can associate a non-negative number ω corresponding to the traffic along that edge. The values of ω will increase or decrease depending on traffic.

Network Tomography, Cont'd

- We wish to determine the weight ω for the case of general weighted graphs. We begin by considering relatively simple regions of interest in a graph and suitable choices for the data of the ω-Neumann boundary value problem to produce a linear system of equations for the values of ω, computing the actual weight from the knowledge of the Dirichlet data (output) for convenient choices of the Neumann data (input).

Network Tomography, Cont'd

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- We can then compute the discrete Laplacian of a weighted subgraph, getting the boundary value data (Dirichlet data).

Network Tomography, Cont'd

- A theorem by Berenstein and Chung give us uniqueness. We can solve for the information via the Neumann matrix N. We then use the Neumann-to-Dirichlet map to get the information as boundary values. Uniqueness carries through. Thus, each subnetwork is distinct and can be solved individually. This allows us to piece together the whole network as a collection of subnetworks, which it turn, can be solved uniquely as a set of linear equations.

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- The key equation to solve is the following in the end. Set S be a network with boundary ∂S, let ω_1, ω_2 be weights on two paths in the network, and let f_1, f_2 be the amount of information on those paths, modeled as real valued functions. Then we wish to solve, for $j = 1, 2$

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0 & x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \psi(z) & z \in \partial S \\ \int_S f_j \, d\omega_j = K & \end{cases}$$

Network Tomography, Cont'd

- The importance of the uniqueness theorem must be discussed to understand its importance in the problem. Looking at the internet as modeled as a hyperbolic graph allows for the natural use of the Neumann-to-Dirichlet map, and thus the hyperbolic Radon Transform. Obviously, the inverse of the Radon Transform completes the problem with its result giving the interior data.

Network Tomography, Cont'd

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- The discrete Radon transform is injective in this setting, and therefore invertible. If increased traffic is detected, we can use the inverse Radon transform to focus in on particular signals. Given that these computations are just matrix multiplications, the computations can be done in real time on suitable subnetworks.

References

Berenstein, C.A.: Local tomography and related problems. Radon transforms and tomography, Contemp. Math., **278**, Amer. Math. Soc., Providence, RI, 3-14 (2001)

Casey, S.D: "Harmonic Analysis in Hyperbolic Space: Theory and Application," to appear in *Novel Methods in Harmonic Analysis*, Birkhäuser/Springer, New York (2016)

Casey, S.D and Christensen, J.G.: "Sampling in Euclidean and Non-Euclidean Domains: A Unified Approach," Chapter 9 in *Sampling Theory, a Renaissance*, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, – 331-359 (2015)

Christensen, J.G., and Ólafsson, G.: Sampling in spaces of bandlimited functions on commutative spaces. Excursions in harmonic analysis (Volume 1). Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 35–69 (2013)

Farkas, H.M, and Kra, I.: Riemann Surfaces. Springer-Verlag, New York (1980)

Feichtinger, H., and Pesenson, I.: A reconstruction method for band-limited signals in the hyperbolic plane. Sampling Theory in Signal and Image Processing, **4** (3), 107–119 (2005)

Forster, O.: Lectures on Riemann Surfaces. Springer-Verlag, New York (1981)

Gröchenig, K.: Reconstruction algorithms in irregular sampling. Math. Comp., **59** 199, 181–194, (1992)

Helgason, S.: Groups and Geometric Analysis. American Mathematical Society, Providence, RI (2000)

Schuster, A.P.: Sets of sampling and interpolation in Bergman spaces. Proc. Amer. Math. Soc., **125** (6), 1717–1725, (1997)

Seip, K.: Beurling type density theorems in the unit disk. Inventiones Mathematicae, **113**, 21–39, (1993)

Seip, K.: Regular sets of sampling and interpolation for weighted Bergman spaces. Proc. Amer. Math. Soc., **117** (1), 213–220, (1993)