Discovering Discrete Classical (Orthogonal) Polynomials: First Steps

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1 Why these polynomial sequences are called classical?

2 How to construct the families explicitly?
   - Classical Orthogonal Polynomials
   - The Favard’s theorem

3 The Schemes
   - The Classical Hypergeometric Orthogonal Polynomials
   - The Classical basic Hypergeometric Orth. Polyn.

4 Some Results
   - Characterization Theorem
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   - The Connection Problem
   - One example. Big $q$-Jacobi polynomials
THE CONSTRUCTION
They satisfy this Sturm-Liouville SODE

\((\rho_n)\) fulfill the Second Order Differential Equation

\[ \phi(x)y''(x) + \psi(x)y'(x) + \lambda y(x) = 0. \]

which, for the COP, is equivalent to

\[ \left( \phi(x)\rho(x)y'(x) \right)' + \lambda \rho(x)y(x) = 0. \]

All this is possible because there exist a weight function \(\rho(x)\) and an interval \((a, b) \subseteq \mathbb{R}\) so that

\[ \int_a^b \rho_n(x)\rho_m(x)\rho(x)dx = \kappa_n\delta_{n,m}. \]
The easiest way to discretize the SLE over a uniform lattice. In order to do that we divide the interval \((a,b)\) in subintervals of length \(h\), and we approximate

\[
\begin{align*}
y' &\sim \frac{1}{2} \left( \frac{y(x + h) - y(h)}{h} + \frac{y(x) - y(x - h)}{h} \right), \\
y'' &\sim \frac{1}{h} \left( \frac{y(x + h) - y(h)}{h} - \frac{y(x) - y(x - h)}{h} \right).
\end{align*}
\]

After some corrections we get

\[
\phi(x) \Delta \nabla y(x) + \psi(x) \Delta y(x) + \lambda y(x) = 0.
\]

here

\[
\Delta f(x) = f(x + 1) - f(x), \quad \nabla f(x) = f(x) - f(x - 1).
\]
From the continuous to the non uniform discrete world

In this case we discretize the SLE over a non uniform lattice \( \{x(s)\} \) with

\[
y'(x) \sim \frac{1}{2} \left( \frac{y(x(s + h)) - y(x(s))}{x(x + h) - x(s)} + \frac{y(x(s)) - (x(s - h))}{x(s) - x(s - h)} \right),
\]

and

\[
y''(x) \sim \frac{1}{x(s + h/2) - x(s - h/2)} \left( \frac{y(x(s + h)) - y(x(s))}{x(x + h) - x(s)} - \frac{y(x(s)) - y(x(s - h))}{x(s) - x(s - h)} \right).
\]

We are taking the points \( x(s \pm h) \), and \( x(s \pm h/2) \). Again, after some corrections, we get

\[
\phi(s) \frac{\Delta}{\Delta x(s - 1/2)} \nabla x(s) y(s) + \psi(s) \frac{\Delta}{\Delta x(s)} y(s) + \lambda y(s) = 0.
\]
THE BASICS
Let \((P_n)\) be a polynomial sequence and \(u\) be a functional.

**Property of orthogonality**

\[
\langle u, P_n P_m \rangle = d_n^2 \delta_{n,m}.
\]

**Distributional equation:**

\[
\mathcal{D}(\phi u) = \psi u, \quad \text{deg } \psi \geq 1, \text{ deg } \phi \leq 2.
\]

**Three-term recurrence relation:**

\[
xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).
\]

**The weight function**

\[
d\mu(z) = \omega(z) \, dz
\]

\[
\langle u, P \rangle = \int_\Gamma P(z) \, d\mu(z), \quad \Gamma \subset \mathbb{C}.
\]
1. Continuous classical orthogonal polynomials

\[ \frac{d}{dx}(\phi(x)\omega(x)) = \psi(x)\omega(x), \]

2. \(\Delta\)-classical orthogonal polynomials

\[ \nabla(\phi(x)\omega(x)) = \psi(x)\omega(x), \]

\[ \Delta f(x) = f(x + 1) - f(x), \quad \nabla f(x) = f(x) - f(x - 1), \]

3. \(q\)-Hahn classical orthogonal polynomials

\[ \frac{\Delta[(\phi(s)\omega(s))]}{\Delta x(s - 1/2)} = \psi(s)\omega(s), \]

\[ x(s) = c_1 q^s + c_2 q^{-s} + c_3. \]
The Favard’s theorem

Let \((p_n)_{n \in \mathbb{N}_0}\) generated by the TTRR

\[ xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x). \]

Favard’s theorem

If \(\gamma_n \neq 0 \ \forall n \in \mathbb{N}\) then there exists a moments functional
\(\mathcal{L}_0 : \mathbb{P}[x] \rightarrow \mathbb{C}\) so that

\[ \mathcal{L}_0(p_n p_m) = r_n \delta_{n,m} \]

with \(r_n\) a non-vanishing normalization factor.
THE RELEVANT FAMILIES
The Classical Hypergeometric Orthogonal Polynomials

- Wilson
- Racah
- Cont. Dual Hahn
- Cont. Hahn
- Hahn
- Dual Hahn
- Meixner-Pollaczek
- Jacobi
- Meixner
- Krawchuk
- Laguerre
- Hermite
- Charlier

NIST, 2015
R. S. Costas-Santos: Discovering Discrete COP: First Steps
The scheme is too big to put it on here, let’s go outside to see it ;)}
SOME RESULTS
Characterization Theorems. The continuous version

Let \((P_n)\) be an OPS with respect to \(\omega\). The following statements are equivalent:

1. \(P_n\) is classical, i.e. \((\phi(x)\omega(x))' = \psi(x)\omega(x)\).
2. \((P'_{n+1})\) is a OPS.
3. \((P^{(k)}_{n+k})\) is a OPS for any integer \(k\).
4. (First structure relation)
   \[
   \phi(x)P'_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x).
   \]
5. (Second structure relation)
   \[
   P_n(x) = \alpha_n P'_{n+1}(x) + \beta_n P'_n(x) + \gamma_n P'_{n-1}(x).
   \]
6. (Eigenfunctions of SODE)
   \[
   \phi(x)P''(x) + \psi(x)P'(x) + \lambda P(x) = 0.
   \]
Let \((P_n)\) be an OPS with respect to \(\omega\). The following statements are equivalent:

1. \(P_n\) is classical, i.e. \((\phi(x)\omega(x))' = \psi(x)\omega(x)\).

2. The Rodrigues Formula for \(P_n\)

\[ P_n(x) = \frac{B_n}{\omega(x)} \frac{d^n}{dx^n} (\phi^n(x)\omega(x)), \quad B_n \neq 0. \]

3. \(\phi(x)(P_nP_{n-1})'(x) = \)
\[ g_nP_n^2(x) - (\psi(x) - \phi'(x))P_n(x)P_{n-1}(x) + h_nP_{n-1}^2(x) \]
The continuous and discrete COP can be written in terms of

\[ _rF_s \left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} \bigg| z \right) = \sum_{k \geq 0} \frac{(a_1)_k (a_2)_k \ldots (a_r)_k}{(b_1)_k (b_2)_k \ldots (b_s)_k} \frac{z^k}{k!}. \]

The \(q\)-discrete COP can be written in terms of

\[ _r\varphi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \bigg| z \right) = \sum_{k \geq 0} \frac{(a_1; q)_k \ldots (a_r; q)_k}{(b_1; q)_k \ldots (b_s; q)_k} \left((-1)^k q^{k(1+s-r)}\right) \frac{z^k}{(q; q)_k}. \]

\((a)_k = a(a + 1) \cdots (a + k - 1)\)

\((a; q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1})\)
The connection problem is the problem of finding the coefficients \( c_{k;n} \) in the expansion of \( P_n \) in terms of another sequence of polynomials \( R_k \), i.e.

\[
P_n(x) = \sum_{k=0}^{n} c_{k;n} R_k(x).
\]

We are interested into obtaining such coefficients for Classical orthogonal polynomials in a enough ‘general’ context.
The example. Big $q$-Jacobi polynomials

Again let’s go to File 2 :D
Some References

FINALLY....

THANK YOU
FOR YOUR ATTENTION !!