Realization Problems for Degree Sequences of Graphs and Hypergraphs

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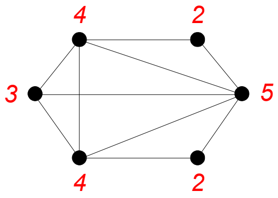
May 11, 2015

Research partially supported by Simons Foundation Collaboration Grant #206692 and NSF Grant DMS-08-38434

"EMSW21-MCTP: Research Experience for Graduate Students"

A nonnegative integer sequence $\pi = (d_1, \ldots, d_n)$ is graphic if π is the degree sequence of some graph G.



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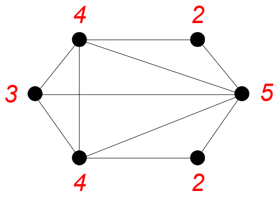

$\pi = (5,4,4,3,2,2)$

In this case, we say that G realizes or is a realization of π, and write

$$\pi = \pi(G) \text{ or } G = G(\pi).$$

Question

Given a sequence of nonnegative integers $\pi = (d_1, \ldots, d_n)$*, is* π *graphic?*

For instance, consider $\pi = (3, 3, 3, 3, 3, 3, 2, 2)$.

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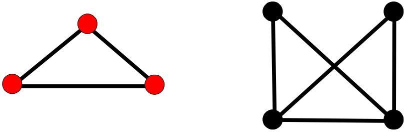

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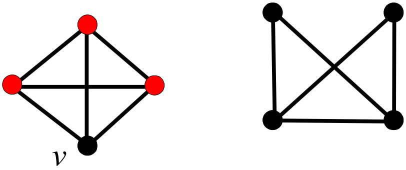

$$\pi = (3, 3, 3, 3, 3.3, 2, 2)$$

Theorem (Havel 1955, Hakimi 1962)

If $\pi = (d_1, \ldots, d_n)$ is a nonincreasing sequence of nonnegative integers, then π is graphic if and only if

$$\pi' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$$

is graphic.

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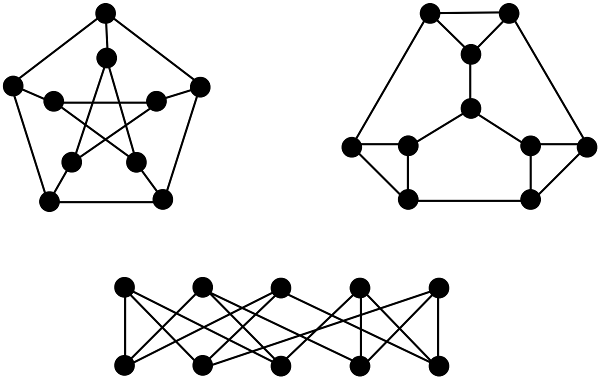
1. Havel (1955)/Hakimi (1962): Recursive (and constructive) characterization.
2. Erdős-Gallai Criteria (1960): Collection of n simple inequalities (constructive proof in 2009 due to Tripathi, Venugopalan and West).
3. G. Sierksma, and H. Hoogeveen (1991), *"Seven Criteria for Integer Sequences to be Graphic"*.



$$\pi = (3,3,3,3,3,3,3,3,3,3)$$

Question

What properties occur within the family of realizations of a graphic sequence π?

Question (Forcible)

Given a graph property $\mathcal{P}$ and a graphic sequence π, does every realization of π have $\mathcal{P}$?

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Question (Potential)

Given a graph property $\mathcal{P}$ and a graphic sequence π, does at least one realization of π have $\mathcal{P}$?

Many well-known theorems can be restated as forcible degree sequence results.

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Theorem (Dirac's Theorem 1952)

If G is a graph of order $n \geq 3$ and the minimum degree of G is at least $\frac{n}{2}$, then G has a hamiltonian (spanning) cycle.

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Theorem (Dirac's Theorem 1952)

If π is an n-term graphic sequence, and every entry of π is at least $\frac{n}{2}$, then π is forcibly hamiltonian.

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Theorem

If G is a planar graph on n vertices, then G has at most $3n - 6$ edges.

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Theorem

If π is an n-term graphic sequence with sum exceeding

$$6n-12=2(3n-6),$$

then π is forcibly nonplanar.

Example: F. Liljeros et al., The web of human sexual contacts *Nature* **411** (2001).

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3. Data and behavioral information place restrictions on network structure.

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4. Is it possible to efficiently construct a *typical* member of $\mathcal{N}$?

   (Sample $\mathcal{N}$ uniformly at random)

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Further, suppose that no pair of members can have both a red and a blue relationship.

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Further, suppose that no pair of members can have both a red and a blue relationship.

Question

Is it possible to construct a network in which each member has the prescribed number of red and blue interactions?

$$\pi_{blue} = (2, 2, 3, 2, 3, 2, 2) \quad \text{and} \quad \pi_{red} = (2, 3, 0, 4, 2, 2, 3).$$

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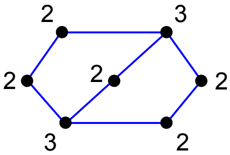

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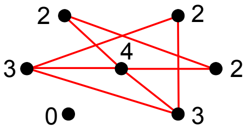

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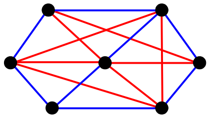

The Asymptotics of the Potential Function

Joint with:

Timothy LeSaulnier
NSA

Casey Moffatt
CU Denver

Paul Wenger
Rochester Institute of Technology

A graphic sequence π is potentially H-graphic if there is a realization of π that contains H as a subgraph.

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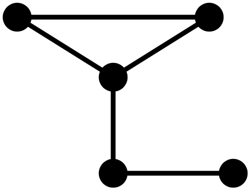
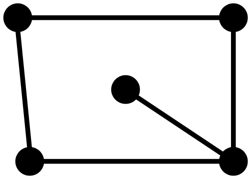
Example: $\pi = (3, 2, 2, 2, 1)$ is potentially K_3-graphic.





We are interested in studying subgraph inclusion in the framework, inspired by the classical extremal literature.

Problem (The Turán Problem)

Determine

$$\mathrm{ex}(n, H),$$

the maximum number of edges in an n-vertex graph that does not contain H as a subgraph.

Given $\pi = (d_1, \ldots, d_n)$, let

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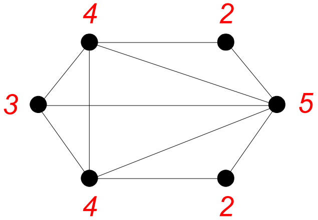
Recall: $\sigma(\pi(G)) = 2|E(G)|$.



$\pi = (5,4,4,3,2,2)$

Problem (Erdős-Jacobson-Lehel 1991)

Determine $\sigma(H,n)$, the minimum even integer such that any n-term graphic sequence with

$$\sigma(\pi) \geq \sigma(H,n)$$

is potentially H-graphic.

We refer to $\sigma(H,n)$ as the potential number of H.

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$$\sigma(\pi) \geq \sigma(H,n)$$

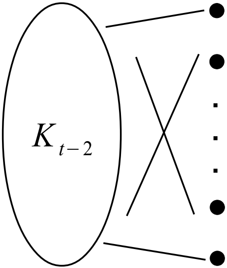
is potentially H-graphic.

Problem (The Turán Problem (Restated))

Determine the minimum integer $ex(n,H)$ such that every n-term graphic π with $\sigma(\pi) > 2ex(H,n)$ is forcibly H-graphic.

Conjecture (Erdős-Jacobson-Lehel 1991)

$$\sigma(K_t, n) = (t-2)(2n-t+1)+2.$$



$$\pi = \left((n-1)^{t-2}, (t-2)^{n-t+2}\right)$$

$\sigma(H, n)$ has been determined for several families, and various specific graphs.

- K_t (E-J-L 1991; Gould, Jacobson, Lehel 1999; Li, Song 1998; **Li, Song, Luo 1999**)

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We determine $\sigma(H, n)$ asymptotically for all H.

If F is an induced subgraph of H, we will write $F \leq H$.

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Let $|H| = k$. For

$$\alpha(H) + 1 \leq i \leq k,$$

let

$$\nabla_i(H) = \min\{\Delta(F) \mid F \leq H,\ |F| = i\}.$$

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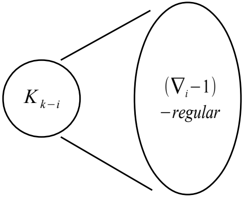
In other words, every i-vertex induced subgraph F of H has maximum degree at least ∇_i.

$\pi_i(H,n)$ is the degree sequence of the following graph.



$$\sigma(\pi_i) \approx (2(k-i) + \nabla_i - 1)n$$

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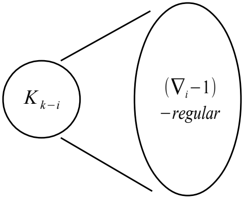

$$\sigma(\pi_i) \approx (2(k-i) + \nabla_i - 1)n$$

Claim

$$\sigma(H, n) \geq \sigma(\pi_i).$$

Let

$$\widetilde{\sigma}(H) = \max_{\alpha(H)+1 \le i \le |H|} \left\{ 2(k-i) + \nabla_i - 1 \right\}.$$

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Theorem (F, Moffatt, LeSaulnier, Wenger 2015+)

If H is a graph and n is a positive integer, then

$$\sigma(H,n) = \widetilde{\sigma}(H)n + o(n).$$

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Theorem (The Erdős-Stone-Simonovits Theorem)

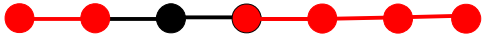
If H is a graph with chromatic number $\chi(H) \geq 2$, then

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

$\alpha(P_7)=4$

$\nabla_5=1$

$\tilde{\sigma}_4=(2(7-5)+1-1)=4$



$\nabla_6=2$

$\tilde{\sigma}_6=(2(7-6)+2-1)=3$

$\nabla_7=2$

$\tilde{\sigma}_7=(2(7-7)+2-1)=1$

$\alpha(P_7)=4$

$\nabla_5=1$

$\tilde{\sigma}_4=(2(7-5)+1-1)=4$

$\nabla_6=2$

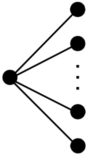
$\tilde{\sigma}_6=(2(7-6)+2-1)=3$

$\nabla_7=2$

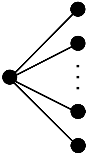
$\tilde{\sigma}_7=(2(7-7)+2-1)=1$

Claim

$$\sigma(P_7, n) \approx 4n$$



$$\alpha(K_{1,k-1}) + 1 = |K_{1,k-1}| = k$$



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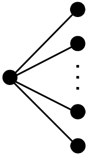

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$$\nabla_k = \Delta(K_{1,k-1}) = k - 1$$

$$\widetilde{\sigma}(K_{1,k-1}) = (2(k-k) + \nabla_k - 1) = k - 2$$

Claim

$$\sigma(K_{1,k-1}, n) \approx (k-2)n$$

Let K_k be the complete graph of order k.

$$\alpha(K_k) = 1$$

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$$\widetilde{\sigma}_i(K_k) = (2(k-i) + (i-1) - 1) = 2k - i - 2$$

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Claim

$$\sigma(K_k, n) \approx (2k-4)n$$

E-J-L: $\sigma(K_k, n) = (k-2)(2n-k+1) + 2$

Stability - what does a graphic sequence π that

(a) is not potentially H-graphic, but

(b) has $\sigma(\pi)$ close to $\sigma(H, n)$

look like?

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(a) is not potentially H-graphic, but

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Joint w/ C. Erbes, R. Martin and P. Wenger.

1. Editing Results (akin to Erdős 1970; Pikhurko & Taraz 2005)
2. Stable & non-stable families.

Degree Sequences of Uniform Hypergraphs

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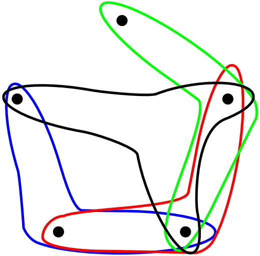
University of Nebraska-Lincoln

Catherine Erbes
Hiram College

Stephen Hartke
University of Colorado Denver

Ben Reiniger, Hannah Spinoza
University of Illinois at Urbana-Champaign

A hypergraph G is k-uniform, or is a k-graph if every edge of G contains exactly k vertices.



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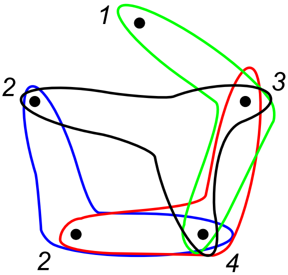

$$\pi(H) = (4, 3, 2, 2, 1)$$

The degree sequence of a k-uniform hypergraph H is the list of the degrees of vertices in H.

There are many complex networks modeled using hypergraphs:

1. Social Networks: family dynamics, group conversation, overlapping communities.
2. Biological Networks: protein interactions, chemical reactions.
3. Education Networks: classroom collaborations, group discussions.

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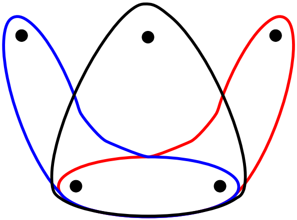
Example: $\pi = (3, 3, 1, 1, 1)$



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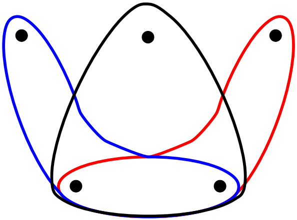

π is 3-graphic (but not 2-graphic).

The k-graph G k-realizes or is a k-realization of π.

Question

Given a sequence of nonnegative integers $\pi = (d_1, \ldots, d_n)$*, is* π *graphic?*

Bhave, Bam and Deshpande (2009) give an Erdős-Gallai-type characterization of loopless linear hypergraphs.

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Achuthan, Achuthan and Simanihuruk: None of these necessary conditions are sufficient.

Theorem (Dewdney 1975)

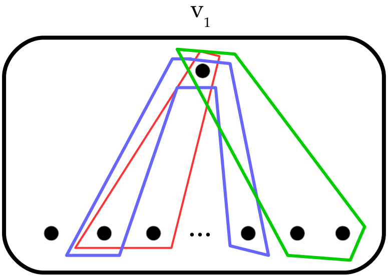
Let $\pi = (d_1, \dots, d_n)$ be a nonincreasing sequence of nonnegative integers. π is k-graphic if and only if there exists a nonincreasing sequence

$$\pi' = (d'_2, \dots, d'_n)$$

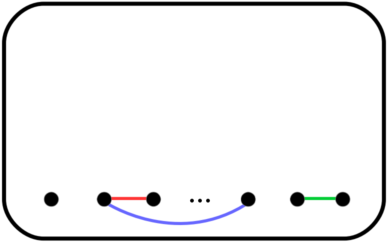
of nonnegative integers such that

1. *π' is $(k-1)$-graphic,*
2. *$\sum_{i=2}^{n} d'_i = (k-1)d_1$, and*
3. *$\pi'' = (d_2 - d'_2, d_3 - d'_3, \dots, d_n - d'_n)$ is k-graphic.*

The $(k-1)$ graph obtained by deleting v_1 from its incident edges is the link of v_1.



The $(k-1)$ graph obtained by deleting v_1 from its incident edges is the link of v_1.



Theorem (Dewdney 1975)

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. π is k-graphic if and only if there exists a nonincreasing sequence

$$\pi' = (d'_2, \ldots, d'_n) \text{ the "link sequence"}$$

of nonnegative integers such that

1. *π' is $(k-1)$-graphic, - link seq. is $(k-1)$-graphic*
2. *$\sum_{i=2}^{n} d'_i = (k-1)d_1$ - link seq. can be "expanded" to include v_1 with the right degree*
3. *$\pi'' = (d_2 - d'_2, d_3 - d'_3, \ldots, d_n - d'_n)$ is k-graphic.*
   completed link + v_1 can be added to the rest of the graph to realize π.

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Havel-Hakimi requires that we only check one residual sequence:

$$(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n).$$

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Problem

Efficiently characterize k-graphic sequences, or show that the associated decision problem is NP-complete.

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Efficiently characterize k-graphic sequences, or show that the associated decision problem is NP-complete.

In lieu of an efficient characterization, we obtain several sharp sufficient conditions.

Theorem (BEFHRST 2013)

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence with

$$d_1 = \Delta \qquad \textit{and} \qquad d_t \geq \Delta - 1.$$

If k divides $\Sigma(d_i)$ and

$$\binom{t-1}{k-1} \geq \Delta,$$

then π is k-graphic. This result is sharp.

Corollary (BEFHRST 2013)

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence with $d_1 = \Delta$, and let p be the minimum integer such that

$$\Delta \leq \binom{p-1}{k-1}.$$

If k divides Σd_i and

$$\sigma(\pi) \geq (\Delta - 1)p + 1,$$

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Proof uses poset methods akin to Aigner-Triesch (graphs - 1994), Duval-Reiner (2002 - hypergraphs) and others.

Theorem (Barrus, Hartke, Jao and West 2012)

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing 2-graphic sequence with

$$d_1 = \Delta \qquad \text{and} \qquad d_n = \delta.$$

If

$$n \geq \frac{(\Delta + \delta - 1)^2 - \ell}{4\delta},$$

where

$$\ell = \Delta + \delta + 1 \pmod 2,$$

then π is 2-graphic.

Improves upon a result of Zverovich and Zverovich (1992) when $\Delta + \delta$ is even.

Corollary (BEFHRST 2013)

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence with

$$d_1 = \Delta \qquad \text{and} \qquad d_n = \delta,$$

and let p be the minimum integer such that

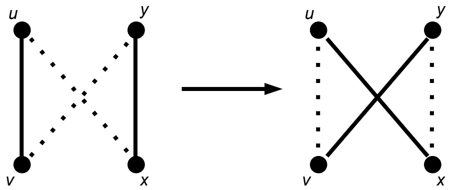
$$\Delta \leq \binom{p-1}{k-1}.$$

If k divides $\sigma(\pi)$ and

$$n \geq \frac{(\Delta - 1)p - \Delta + \delta + 1}{\delta},$$

then π is k-graphic.

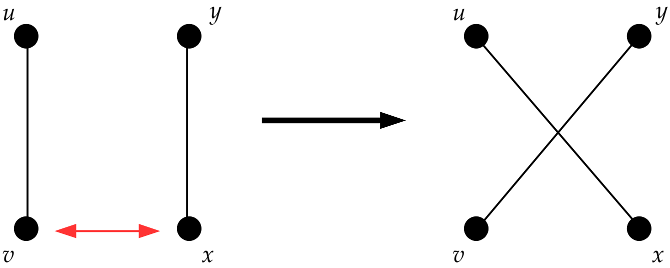
The 2-switch (or edge-exchange or rewiring or infusion) operation:



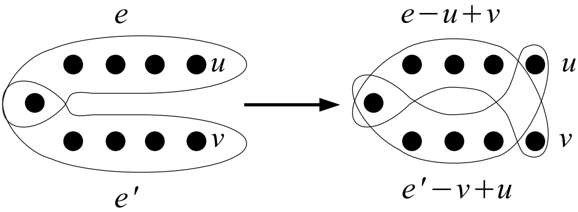
Theorem (Petersen 1891)

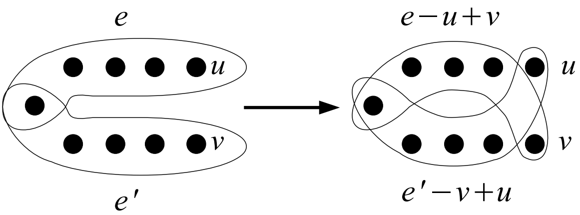
If G_1 and G_2 are realizations of a 2-graphic sequence π, then G_1 can be transformed into G_2 by a finite sequence of 2-switches.

What about k-graphic sequences?



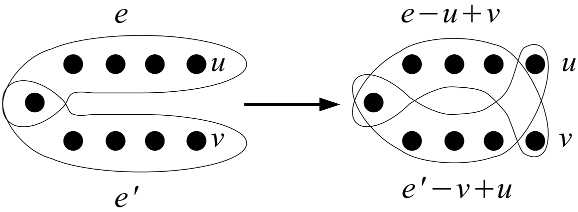
uv	$\longrightarrow$	ux
yx		yv


e
u
v
e'
$e-u+v$
u
v
$e'-v+u$



Theorem (BEFHRST 2013)

If G_1 and G_2 are realizations of a k-graphic sequence, then G_1 can be transformed into G_2 by a finite sequence of 2-switches.



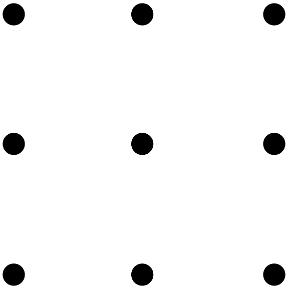
This extends a result of Kocay and Li (2007) for $k = 3$.

There is an issue, however - the intermediate graphs between G_1 and G_2 may have multiple edges.

An i-switch is exchanges i edges for i non-edges in a graph, while maintaining the degree of each vertex.

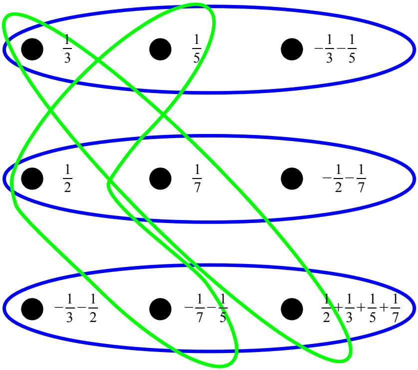
Theorem (Gabelman 1961; BEFHRST 2013)

For every $k \geq 3$, there is a k-graphic sequence π with distinct simple realizations such that for any $i < k$, there is no i-switch that can be performed to change one realization into another.

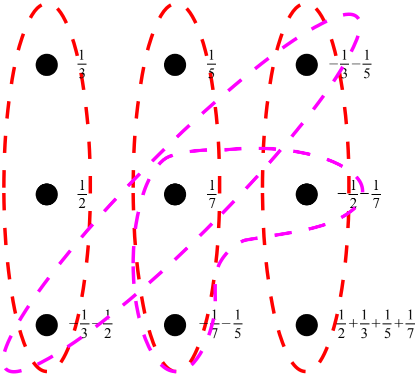


$\bullet\ \frac{1}{3}$	$\bullet\ \frac{1}{5}$	$\bullet\ -\frac{1}{3}-\frac{1}{5}$
$\bullet\ \frac{1}{2}$	$\bullet\ \frac{1}{7}$	$\bullet\ -\frac{1}{2}-\frac{1}{7}$
$\bullet\ -\frac{1}{3}-\frac{1}{2}$	$\bullet\ -\frac{1}{7}-\frac{1}{5}$	$\bullet\ \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$

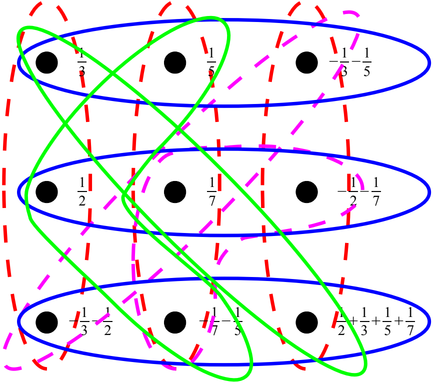
- Weight vertices so that the only 3-sets with zero sum are rows or columns.



- Weight vertices so that the only 3-sets with zero sum are rows or columns.
- Edges are rows and 3-sets with positive sum.

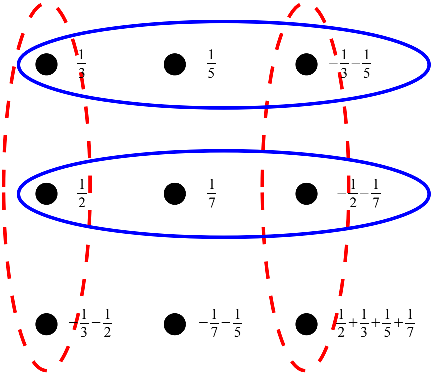


- Weight vertices so that the only 3-sets with zero sum are rows or columns.
- Edges are rows and 3-sets with positive sum.
- Nonedges are columns and 3-sets with negative sum.



- To exchange edge set F_1 and nonedge set F_2, we need

$$\sum_{e\in F_1}\sum_{v\in e} wt(v) = \sum_{e\in F_2}\sum_{v\in e} wt(v).$$



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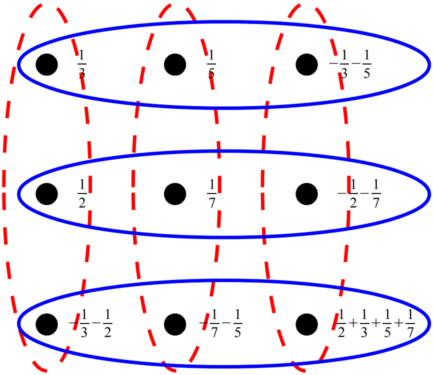

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- This means $F_1 \subseteq$ Rows and $F_2 \subseteq$ Columns.
- Maintaining vertex degrees requires $F_1 =$ Rows and $F_2 =$ Columns.

Problem

Determine a minimal family of edge exchanges such that, given realizations H_1 and H_2 of a k-graphic sequence π, H_1 can be transformed into H_2 by a sequence of edge exchanges such that each intermediate k-graph is simple.

The realm of k-graphic sequences is wide open - go forth and explore!

Theorem (Gu and Lai 2013)

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing k-graphic sequence. Then π has a t-edge-connected realization iff

- $d_n \geq t$
- $d_1 \geq \frac{k(n-1)}{k-1}$ *if* $t = 1$.

This extends results of Boonyasombat (1984) for $t = 1$ and Edmonds (1964) for graphic sequences.