Computational Combinatorics and the search for Uniquely K_r-Saturated Graphs

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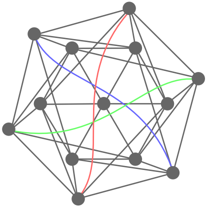
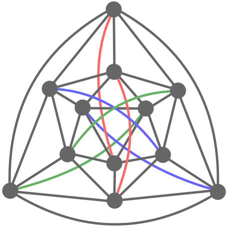
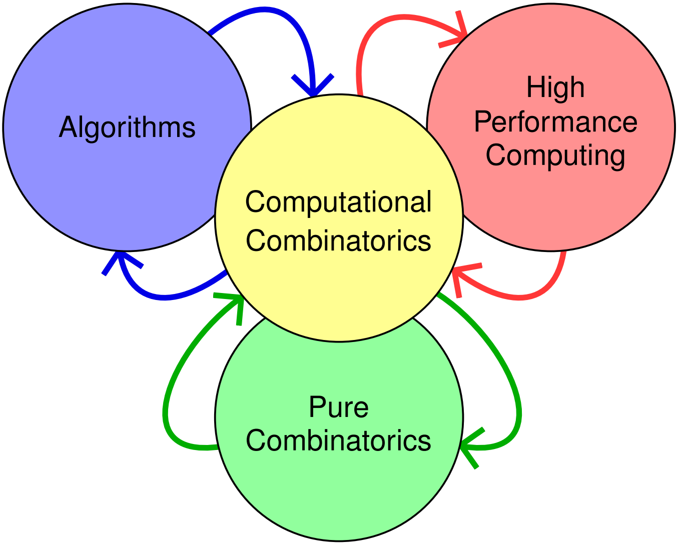




Joint work with Derrick Stolee, Iowa State University.

What is Computational Combinatorics?



What is Computational Combinatorics?

Using a combination of

- **pure mathematics**,
- **algorithms**, and
- **computational resources**

to solve problems in pure combinatorics by

- **providing evidence** for conjectures,
- finding **examples** and **counterexamples**, and
- **discovering and proving theorems**.

The Goal

Determine if certain **combinatorial objects** exist with given **structural** or **extremal** properties.

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Examples:

1. Is there a **projective plane** of order 10?
2. When do **strongly regular graphs** exist?
3. How many **Steiner triple systems** are there of order 19?

The Goal

Determine if certain **combinatorial objects** exist with given **structural** or **extremal** properties.

Examples:

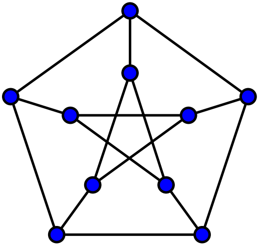
1. Is there a **projective plane** of order 10?
   (Lam, Thiel, Swiercz, 1989)

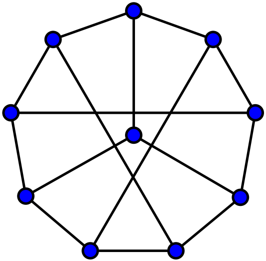
2. When do **strongly regular graphs** exist?
   (Spence 2000, Coolsaet, Degraer, Spence 2006, many others)

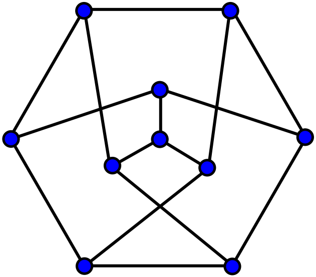
3. How many **Steiner triple systems** are there of order 19?
   (Kaski, Östergård, 2004)

Combinatorial Object: Graphs

A **graph** G of **order** n is composed of a set $V(G)$ of n vertices and a set $E(G)$ of edges, where the edges are unordered pairs of vertices.

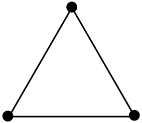




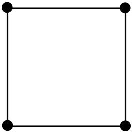


Combinatorial Object: Graphs

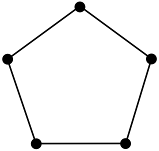
Cycles C_k



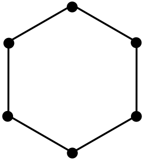
C_3



C_4



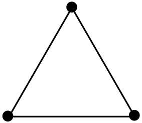
C_5



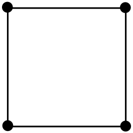
C_6

Combinatorial Object: Graphs

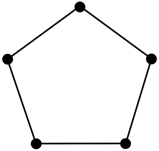
Cycles C_k



C_3



C_4



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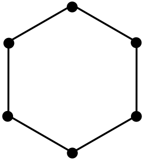
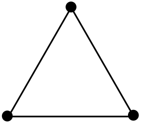
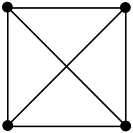


C_6

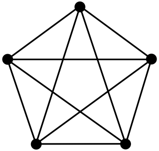
Complete Graphs K_r (cliques)



K_3



K_4



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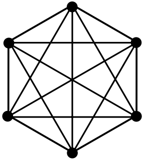

K_6

Main Technique: Combinatorial Search

Goal: Determine if certain combinatorial objects exist with given structural or extremal properties.

Idea: Build objects *piece-by-piece* from *base examples* to enumerate all desired examples of a given order.

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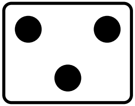
The computer performs a long, detailed case analysis.

Our job is to efficiently design the case analysis, using **algorithms**:

- Combinatorial Generation
- Combinatorial Optimization
- Graph Algorithms

Example: Generating Graphs by Edges

We can build graphs starting at $\overline{K_n}$ by adding edges.



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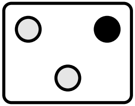

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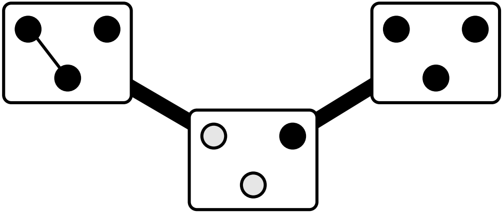

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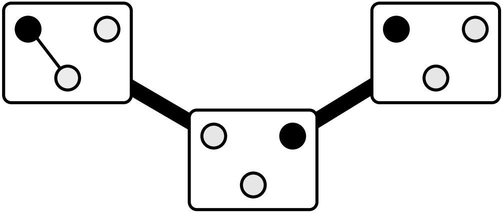

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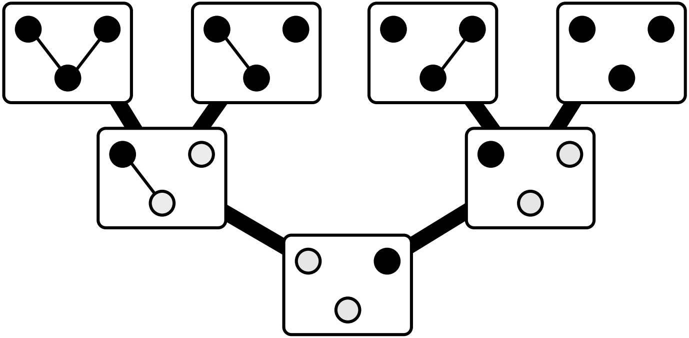

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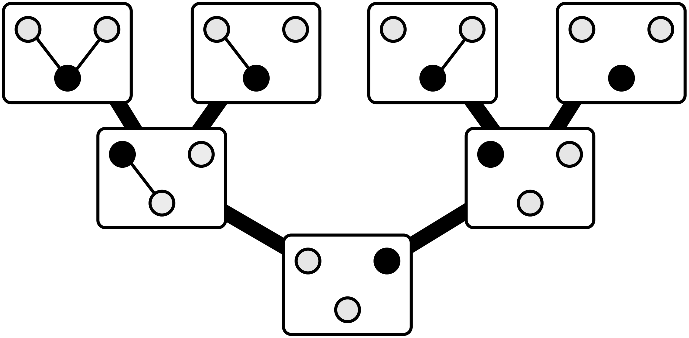

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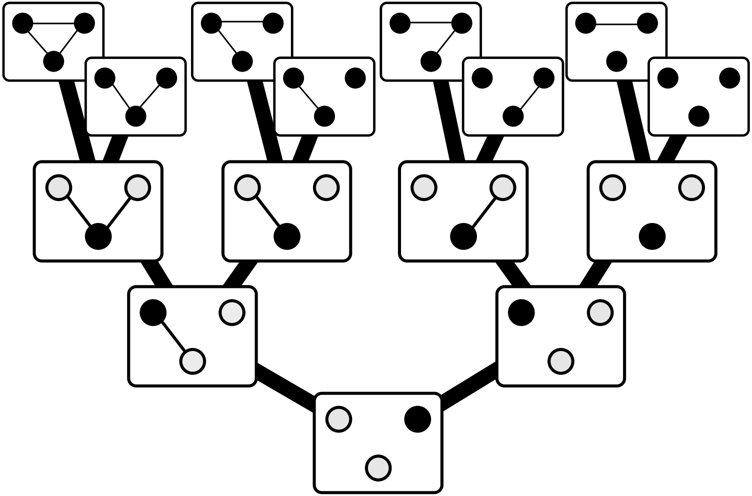

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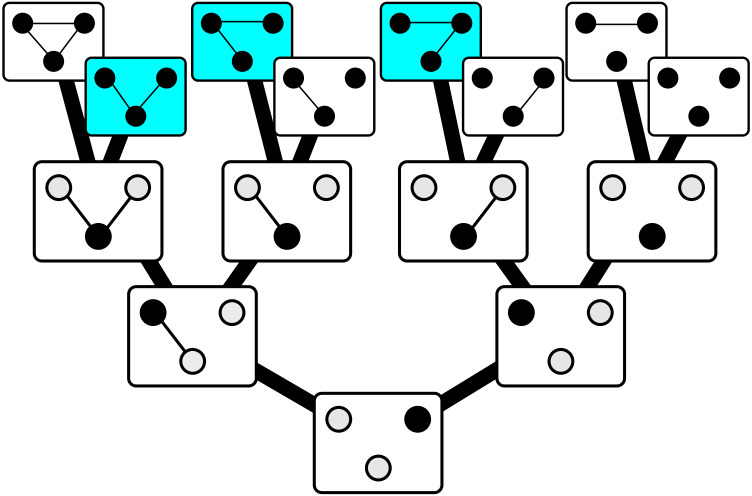

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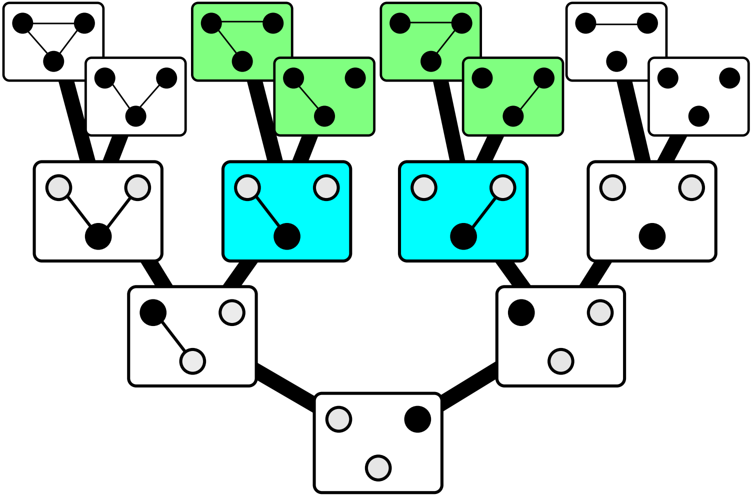

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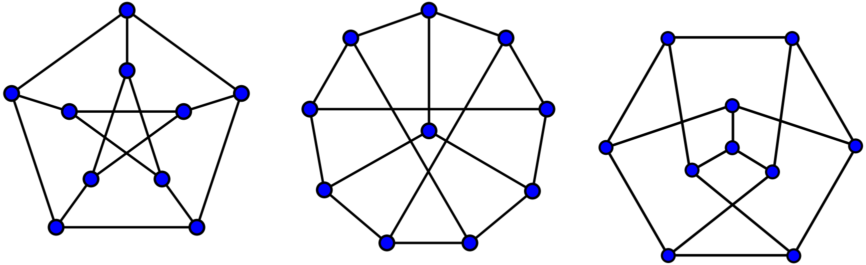

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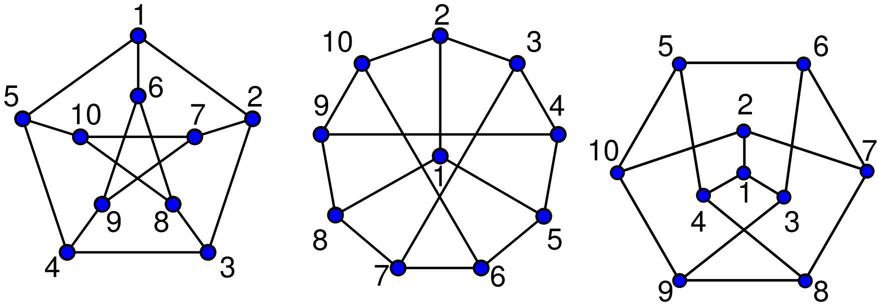
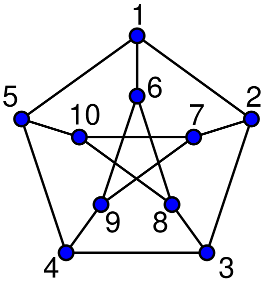
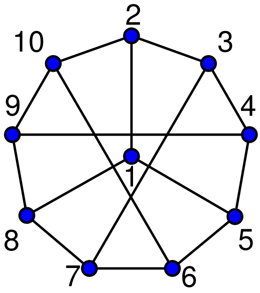
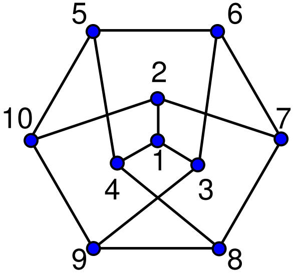


Example: Generating Graphs by Edges

An **isomorphism** between G_1 and G_2 is a bijection from $V(G_1)$ to $V(G_2)$ that induces a bijection from $E(G_1)$ to $E(G_2)$.







Labeled Versus Unlabeled Objects

A **labeled** graph has a linear ordering on the vertices.

An **unlabeled** graph represents an isomorphism class of graphs.

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An **unlabeled** graph represents an isomorphism class of graphs.

Most interesting graph properties are **invariant under isomorphism**.

n	Labeled graphs of order n
6	32,768
7	2,097,152
8	268,435,456
9	68,719,476,736
10	35,184,372,088,832
11	36,028,797,018,963,968
12	73,786,976,294,838,206,464
13	302,231,454,903,657,293,676,544
14	2,475,880,078,570,760,549,798,248,448
15	40,564,819,207,303,340,847,894,502,572,032

$$2^{\binom{n}{2}} \approx 2^{\theta(n^2)}$$

n	Unlabeled connected graphs of order n
6	85
7	509
8	4,060
9	41,301
10	510,489
11	7,319,447
12	117,940,535
13	2,094,480,864
14	40,497,138,011
15	845,480,228,069

OEIS Sequence A002851 Grows $2^{\Omega(n^2)}$.

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Requires about **1 day** of CPU Time.

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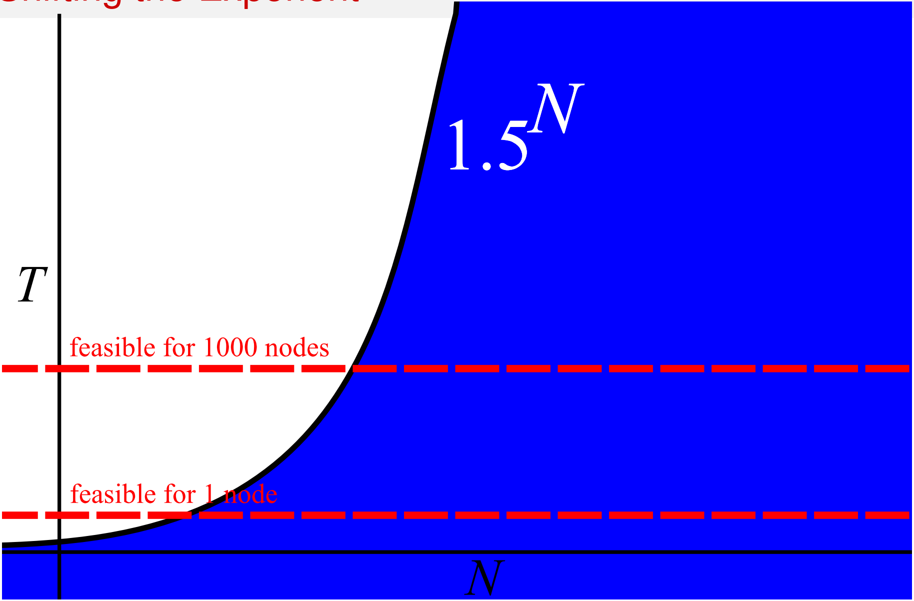
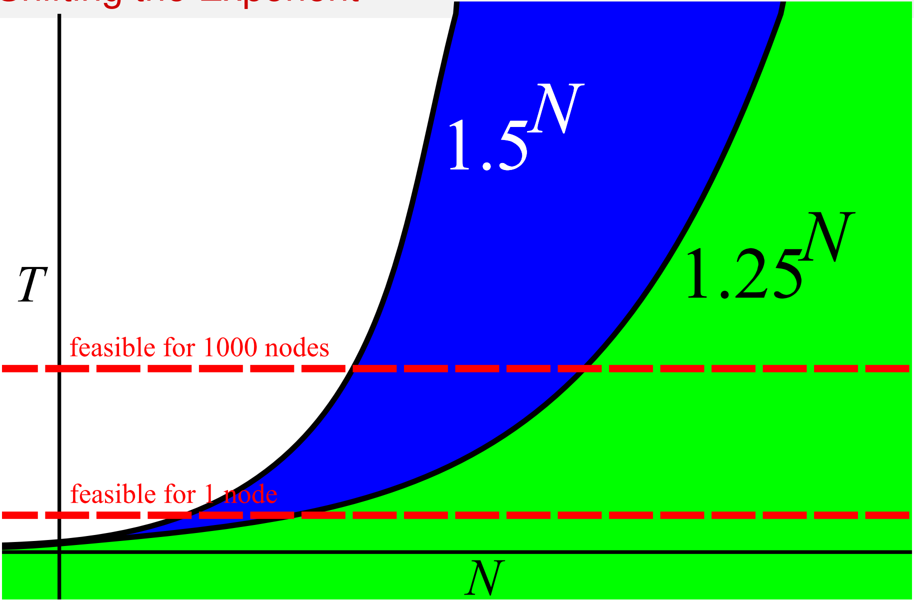
Requires over **1 year** of CPU Time.

Shifting the Exponent



Shifting the Exponent



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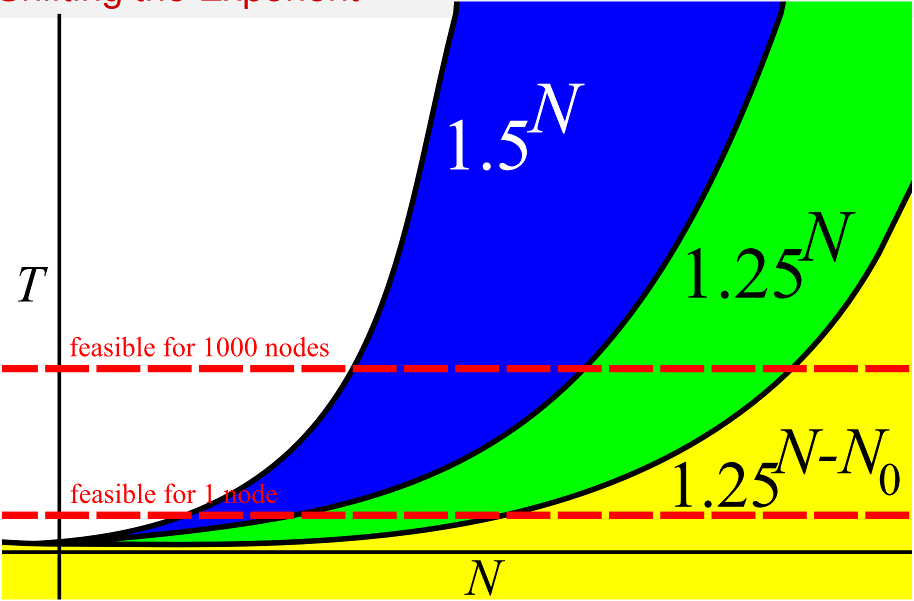

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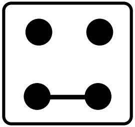
Unlabeled Graphs



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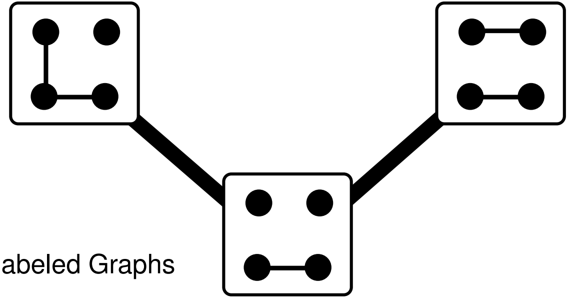

Unlabeled Graphs

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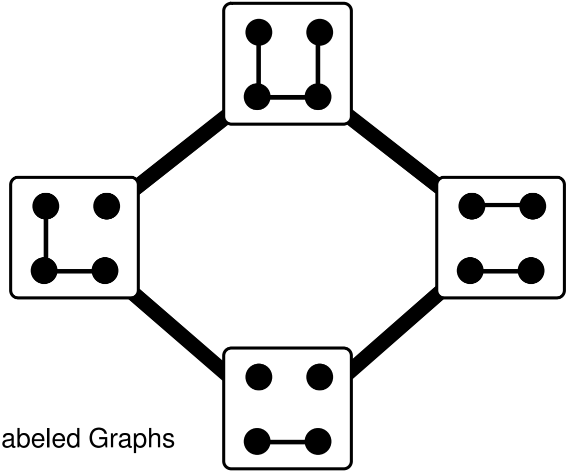

Unlabeled Graphs

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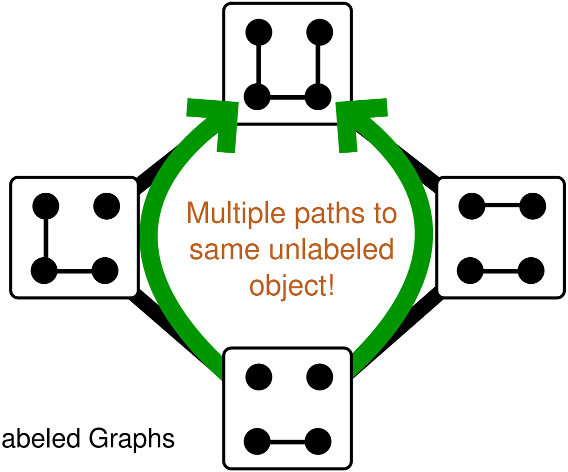


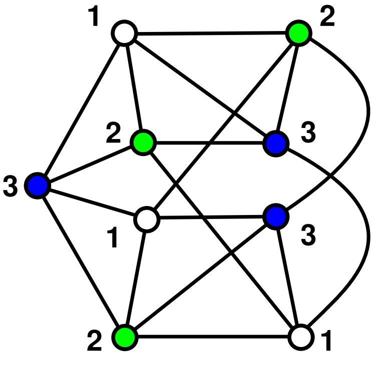
Multiple paths to same unlabeled object!

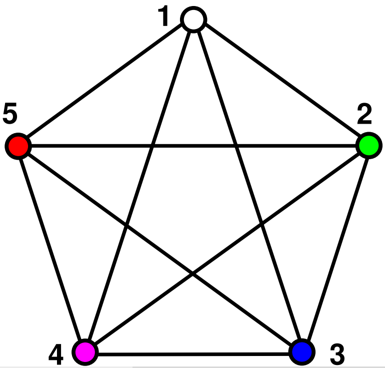
Unlabeled Graphs

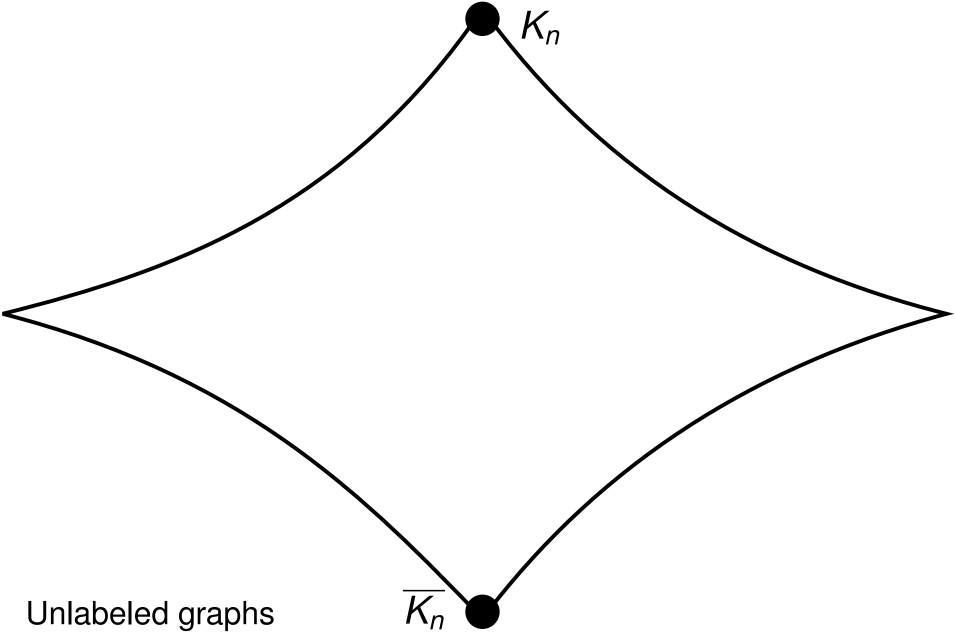
Toy Example

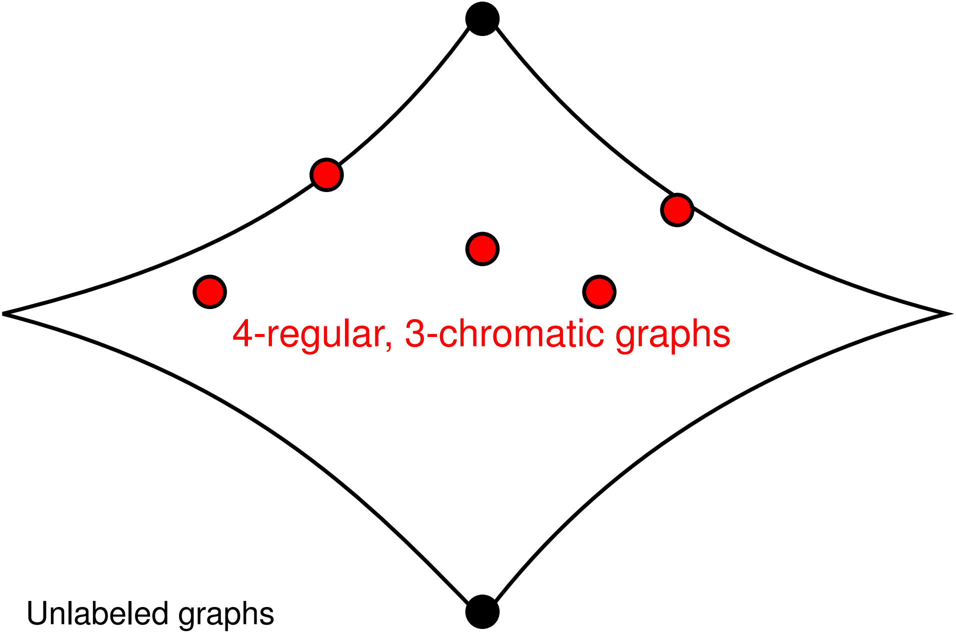
Suppose we are searching for graphs which are:

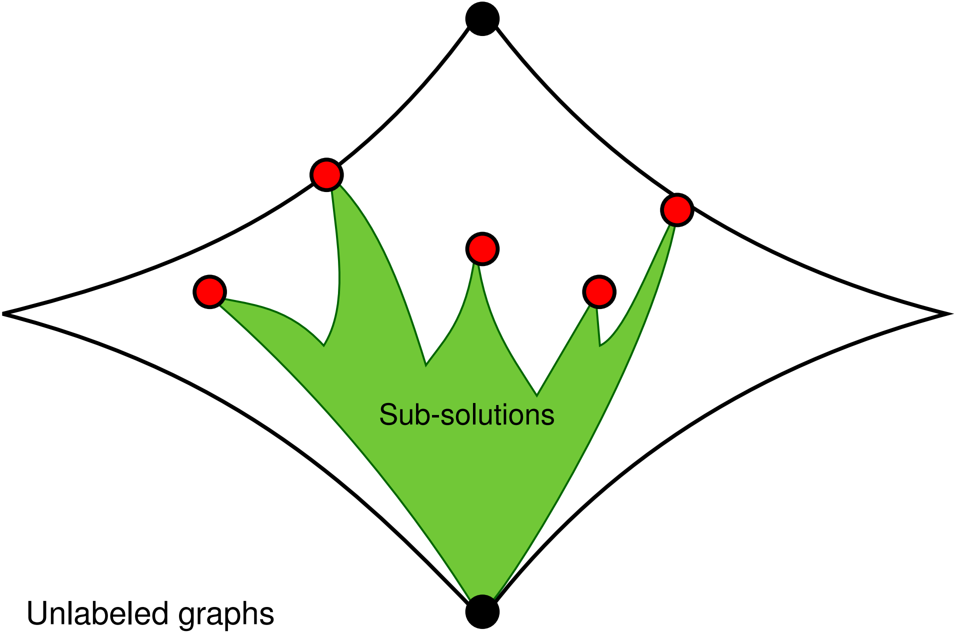
1. **4-regular**: All vertices have 4 incident edges.
2. **3-colorable**: The vertices can be colored with three colors so that no edge is monochromatic.

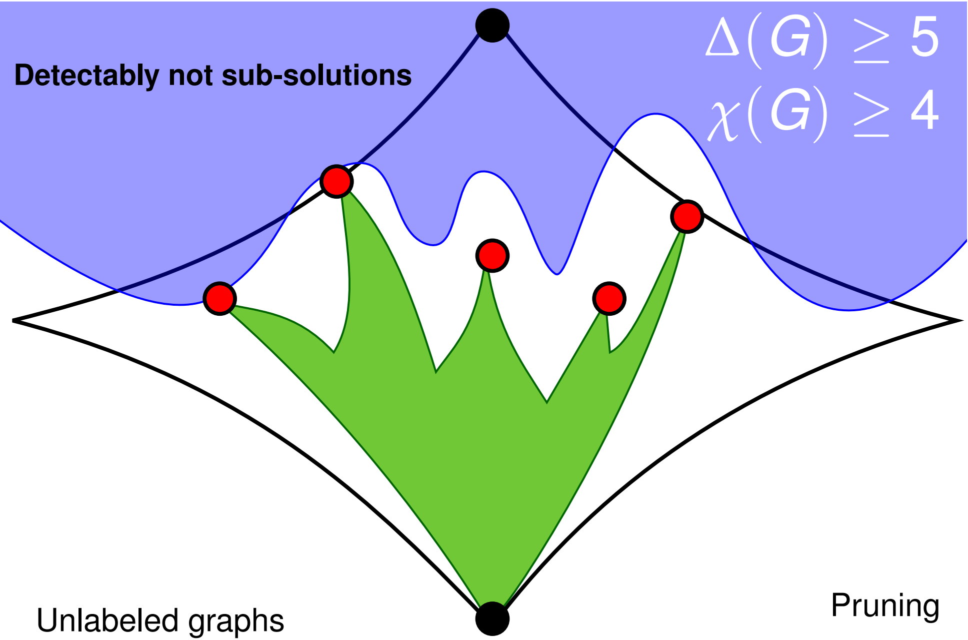


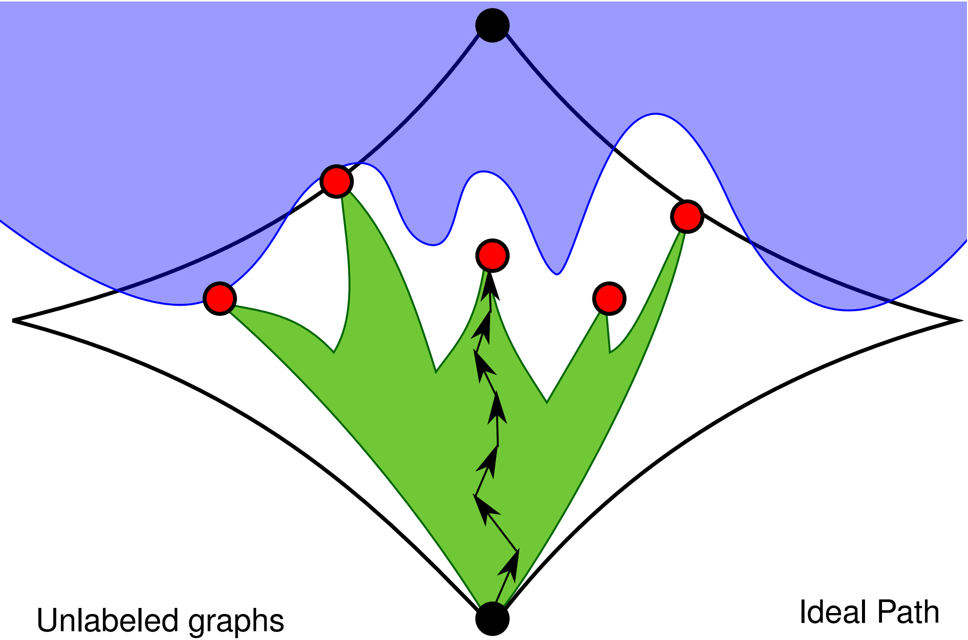


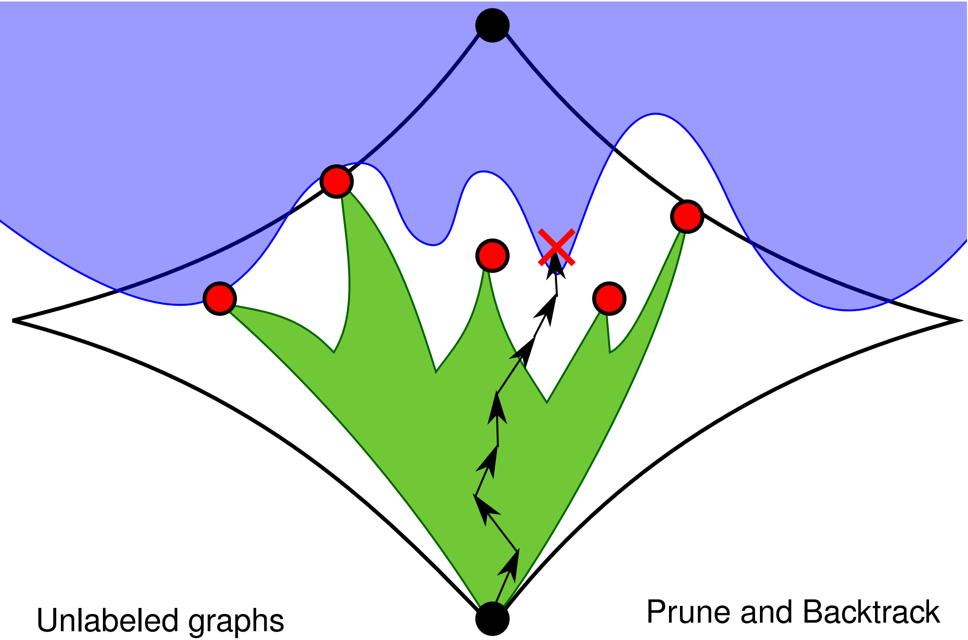

K_n
$\overline{K_n}$
Unlabeled graphs


4-regular, 3-chromatic graphs
Unlabeled graphs

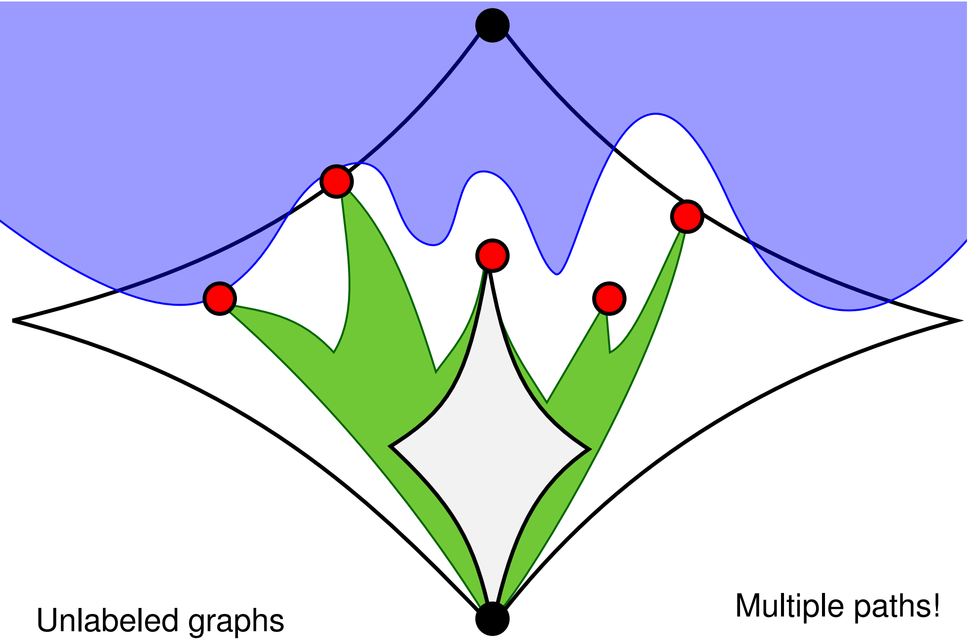

Sub-solutions
Unlabeled graphs


Detectably not sub-solutions
$\Delta(G) \geq 5$
$\chi(G) \geq 4$
Unlabeled graphs
Pruning


Unlabeled graphs
Ideal Path

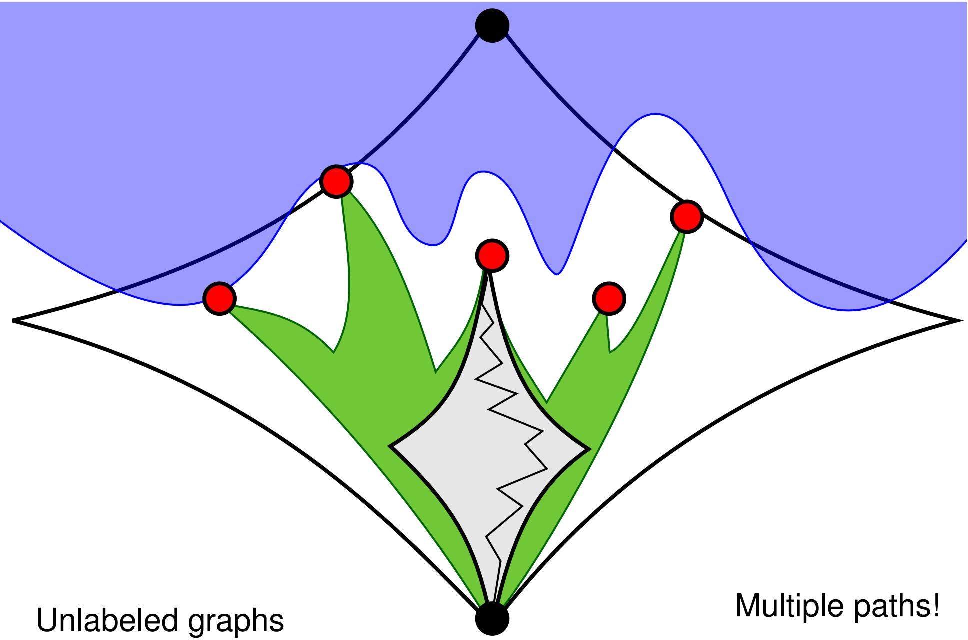


Unlabeled graphs
Prune and Backtrack



Unlabeled graphs

Multiple paths!


Unlabeled graphs
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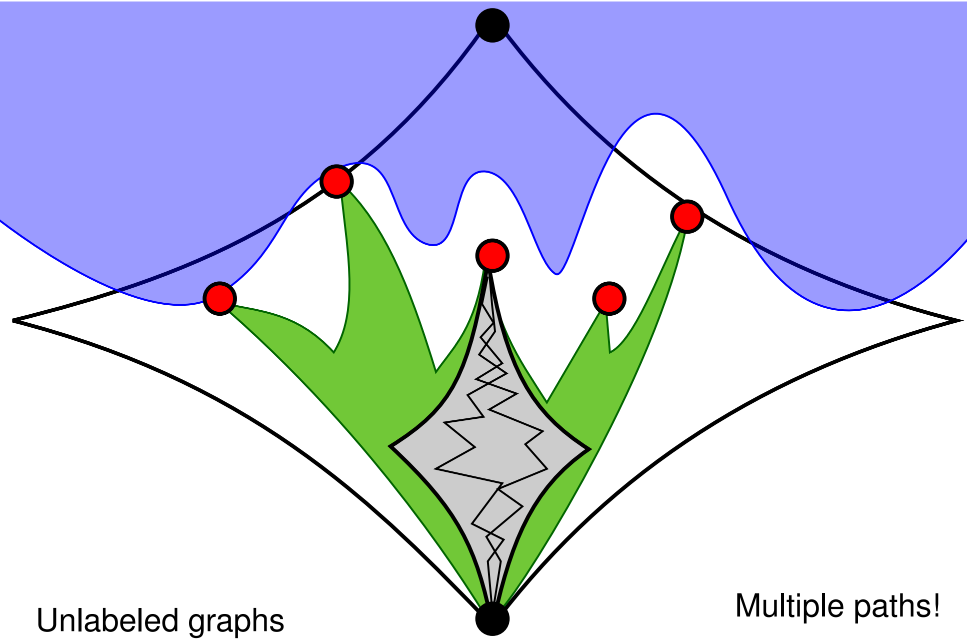

Unlabeled graphs

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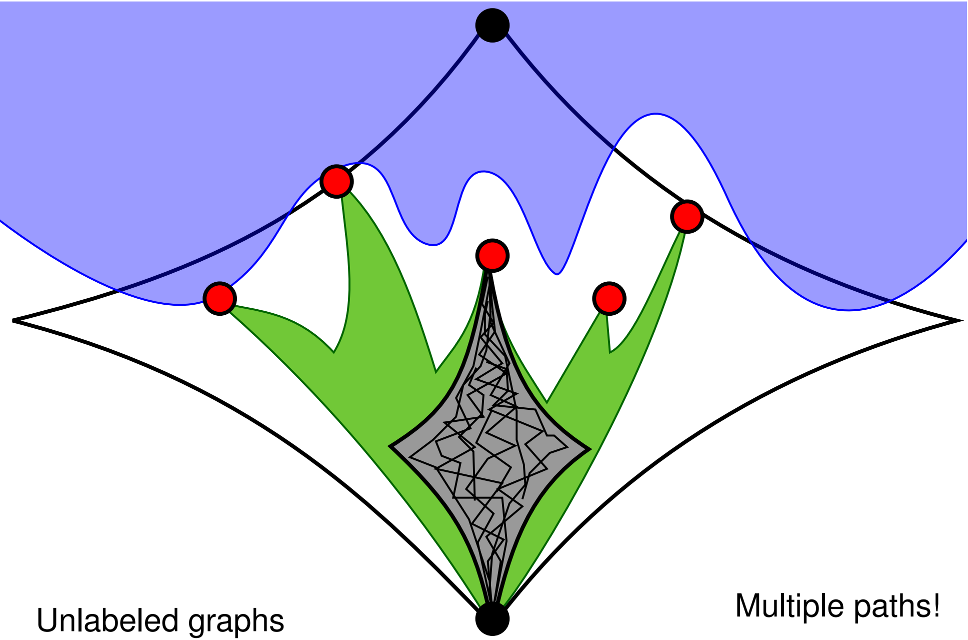

Unlabeled graphs

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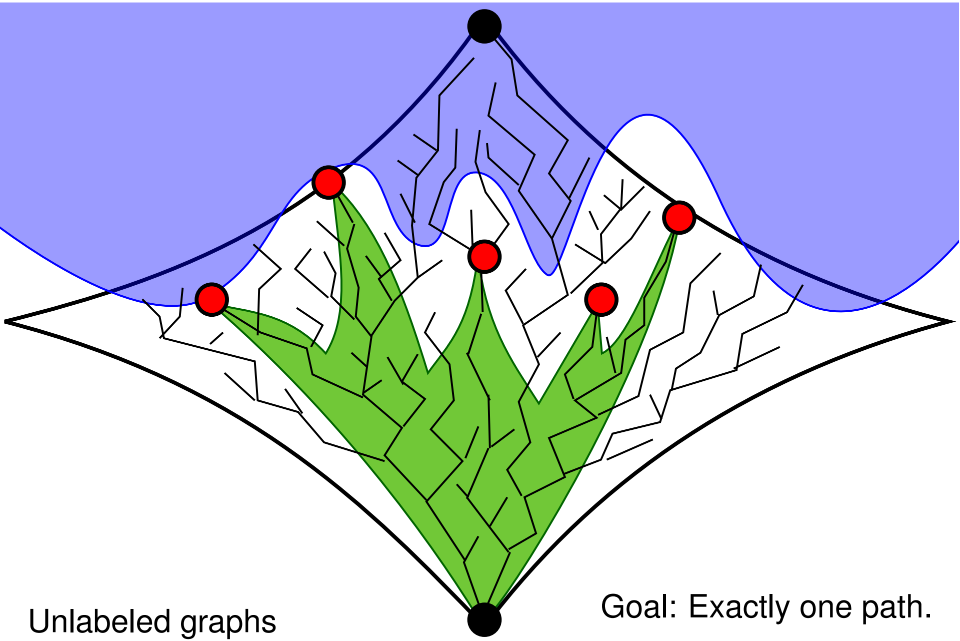
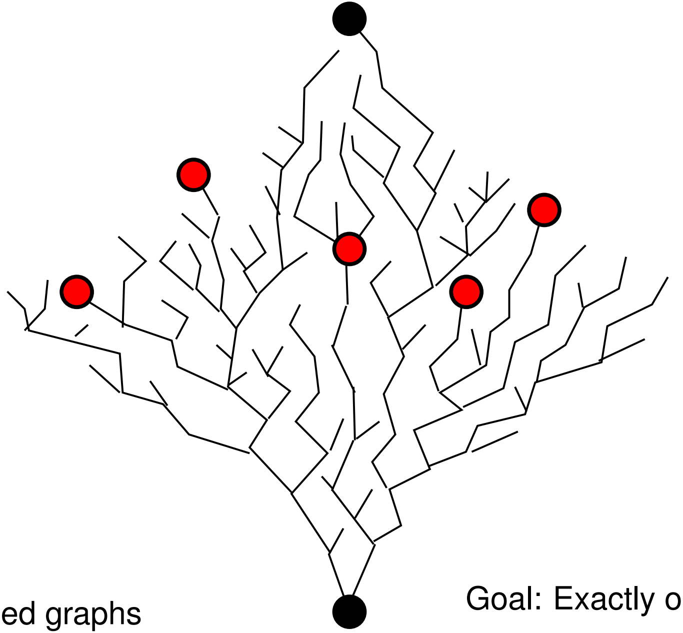


Unlabeled graphs

Goal: Exactly one path.



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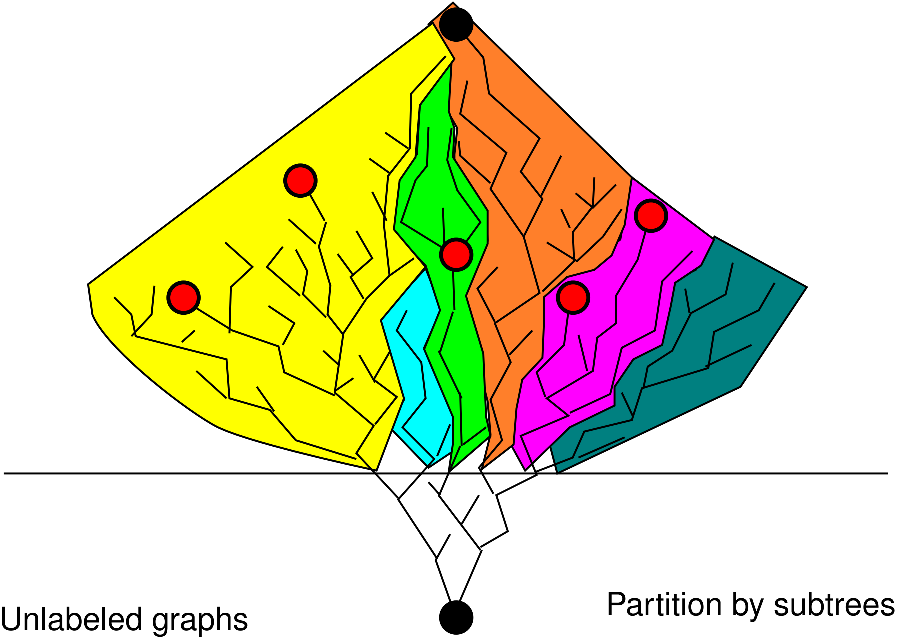
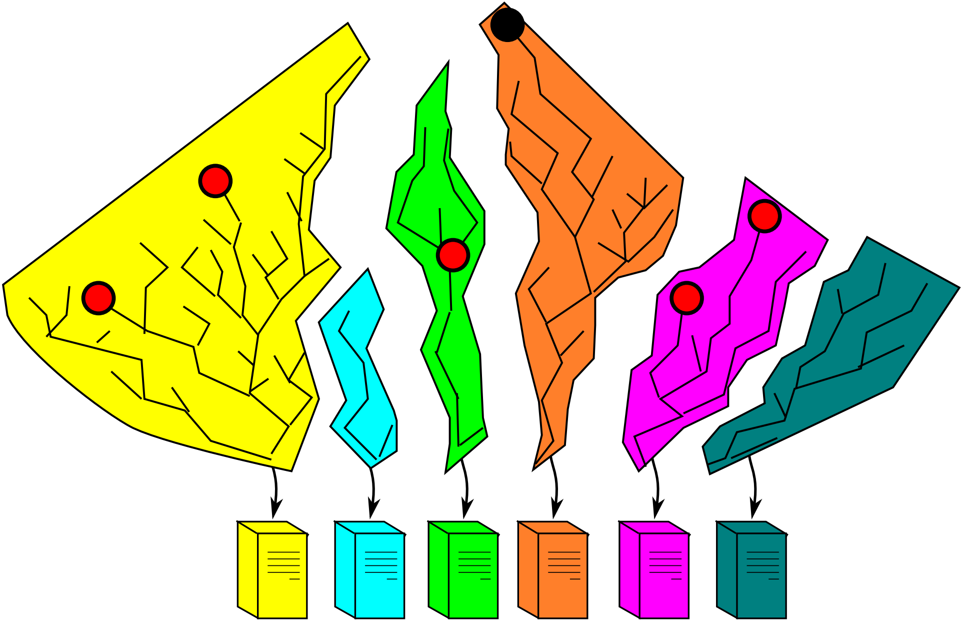


Unlabeled graphs

Partition by subtrees.



Parallelize!

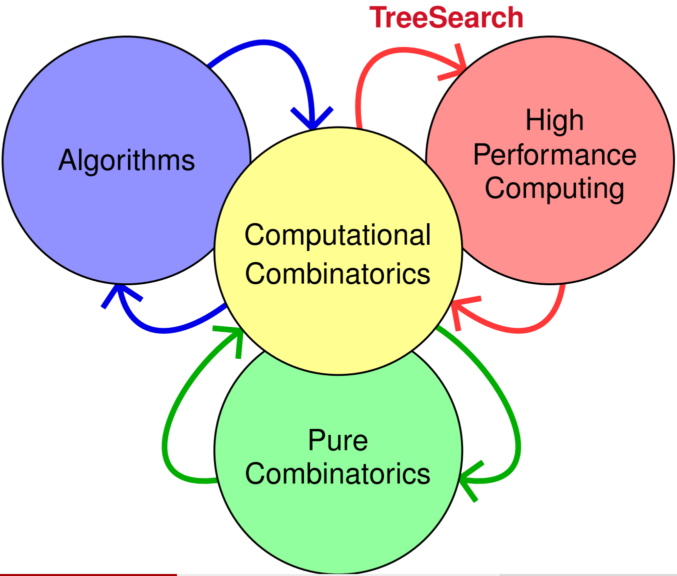
Implementation

The **TreeSearch** library enables parallelization in the Condor scheduler.

Executes on the **Open Science Grid**, a collection of supercomputers around the country.

Open Science Grid

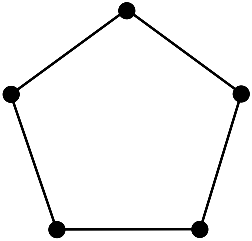
Computational Combinatorics

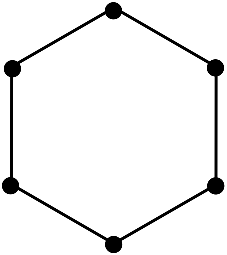


H-Saturated Graphs

Definition A graph G is ***H*-saturated** if

- G does not contain H as a subgraph. (***H*-free**)
- For every $e \in E(\overline{G})$, $G + e$ contains H as a subgraph.


5-cycle


6-cycle

Example: $H = K_3$ where K_r is the **complete graph** on r vertices.

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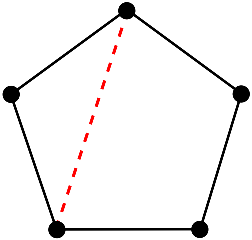
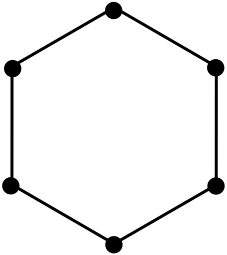


5-cycle
is K_3-saturated



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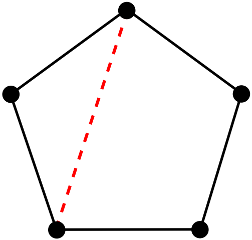

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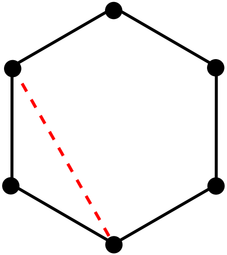

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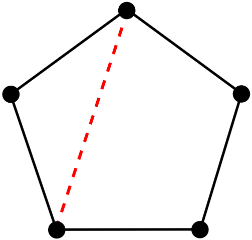

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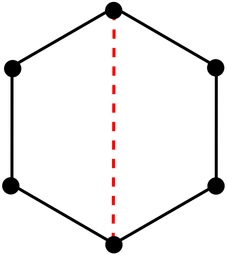

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Turán's Theorem

Theorem (Turán, 1941) Let $r \geq 3$. If G is K_r-saturated on n vertices, then G has **at most** $\left(1 - \frac{1}{r-1}\right) \frac{n^2}{2}$ edges (asymptotically).

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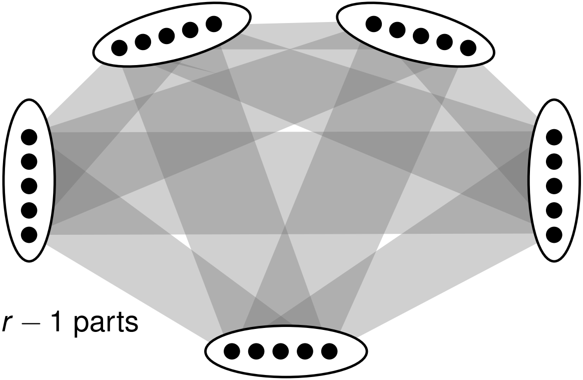

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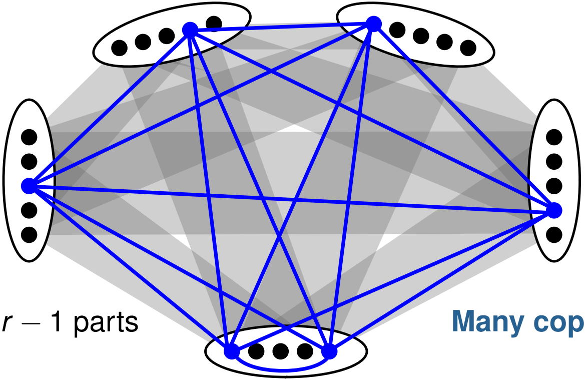

$r - 1$ parts

Many copies of K_r!

Erdős, Hajnal, and Moon

Theorem (Erdős, Hajnal, Moon, 1964) Let $r \geq 3$. If G is K_r-saturated on n vertices, then G has **at least** $\binom{r-2}{2} + (r-2)(n-r+2)$ edges.

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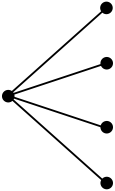
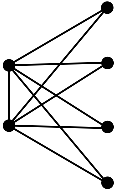
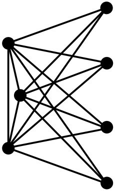


1-book



2-book



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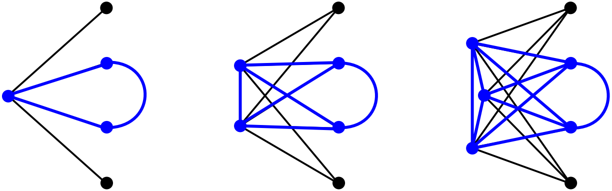

1-book

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Exactly one copy of K_r!

Uniquely H-Saturated Graphs

The Turán graph has **many** copies of K_r when an edge is added.

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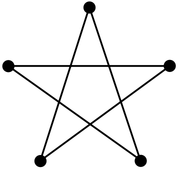
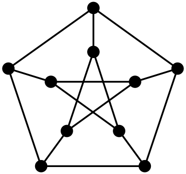


C_5



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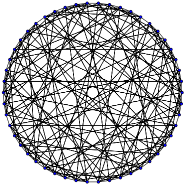

Hoffman–
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?

57-Regular
Order 3250

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Theorem (Wenger, 2010)
For $k \in \{6, 7, 8\}$, no uniquely C_k-saturated graph exists.

Conjecture (Wenger, 2010)
For $k \geq 9$, no uniquely C_k-saturated graph exists.

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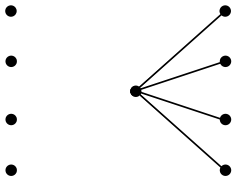

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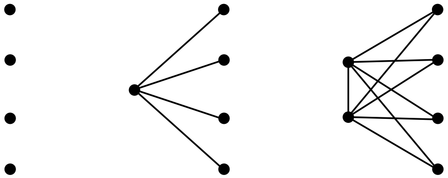

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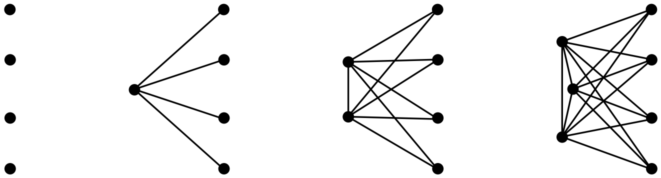

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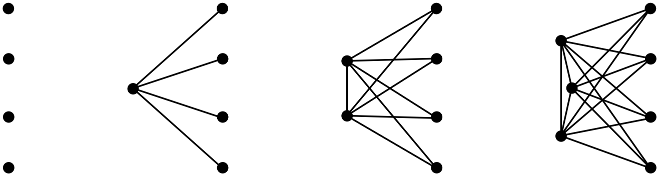

Call uniquely K_r-saturated graphs without a dominating vertex

***r*-primitive**.

r-Primitive Graphs

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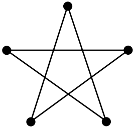

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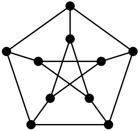

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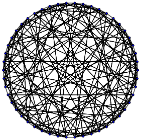

Hoffman–Singleton

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57-Regular
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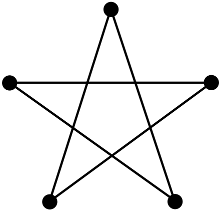
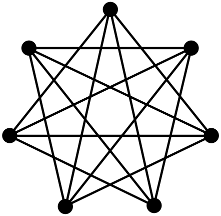


$\overline{C_5}$



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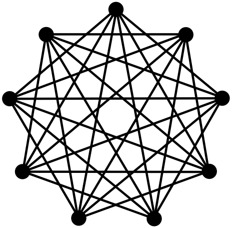

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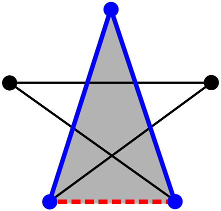

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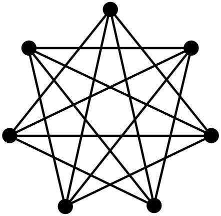

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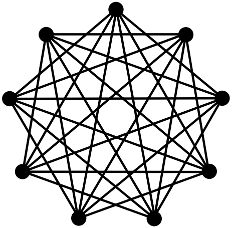

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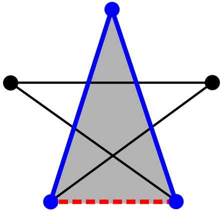

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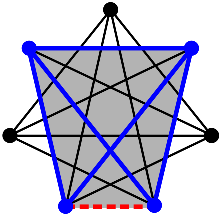

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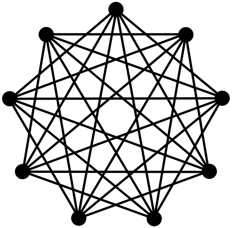

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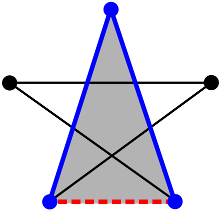

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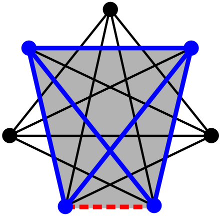

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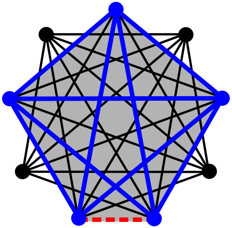

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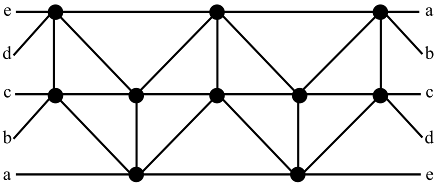
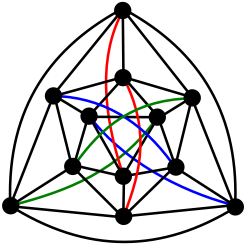


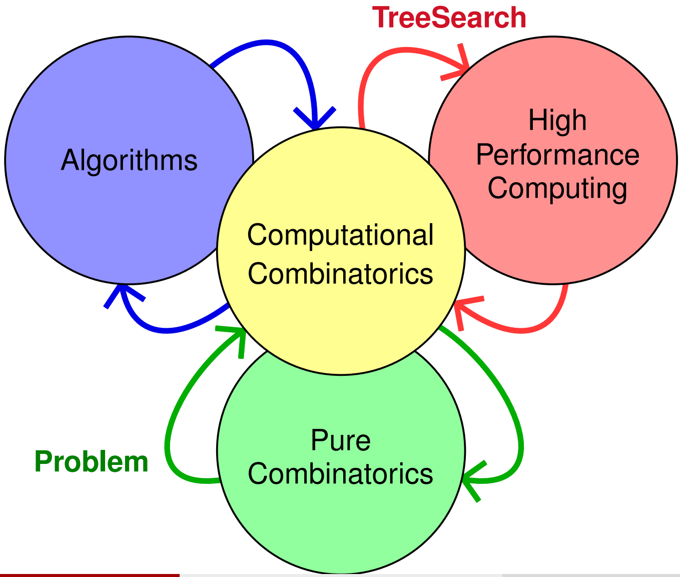
10 vertices



12 vertices

Previously known 4-primitive graphs (Collins, Cooper, Kay, 2010)

Computational Combinatorics



TreeSearch

Algorithms

High Performance Computing

Computational Combinatorics

Pure Combinatorics

Problem

The Problem

Goal: Characterize uniquely K_r-saturated graphs.

First Step: Reduce to r-primitive graphs.

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Edges and Non-Edges

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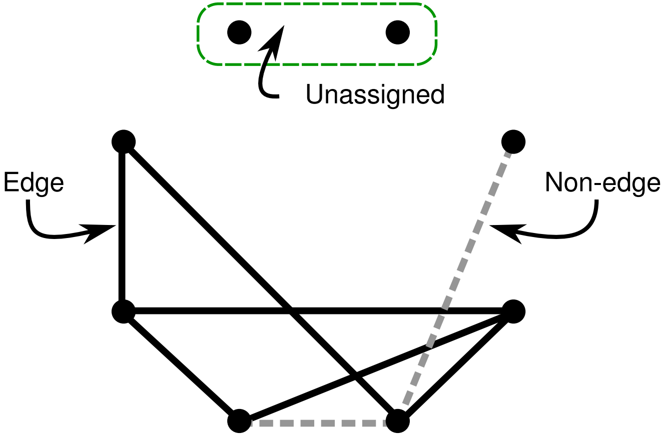

Tricolored graph

Edges, Non-Edges, and Variables

Fix a vertex set $\{v_1, v_2, \ldots, v_n\}$.

For $i, j \in \{1, \ldots, n\}$, let

$$x_{i,j} = \begin{cases} 1 & v_i v_j \in E(G) \\ 0 & v_i v_j \notin E(G) \\ * & v_i v_j \text{ unassigned} \end{cases}.$$

A vector $\mathbf{x} = (x_{i,j} : i, j \in \{1, \ldots, n\})$ is a **variable assignment**.

Symmetries of the System

The constraints

- There is no r-clique in G.
- Every non-edge e of G has exactly one r-clique in $G + e$.

are independent of vertex labels.

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Automorphisms of the tricolored graph define **orbits** on variables $x_{i,j}$.

Orbital Branching

Orbital branching reduces the number of isomorphic duplicates.
(Ostrowski, Linderoth, Rossi, Smriglio, 2007)

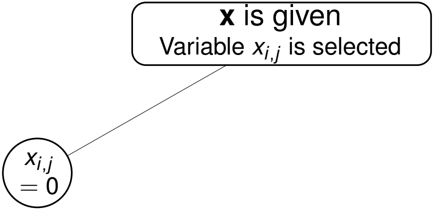
Generalizes **branch-and-bound** strategy from Integer Programming.

Branch-and-Bound

$\mathbf{x}$ is given

Variable $x_{i,j}$ is selected

Branch-and-Bound



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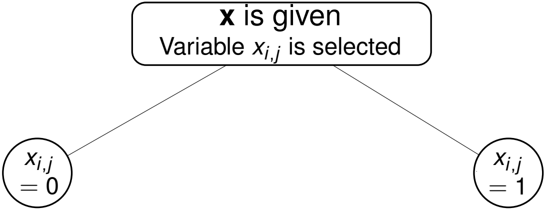

Orbital Branching

x is given
Orbit $\mathcal{O}$ is selected

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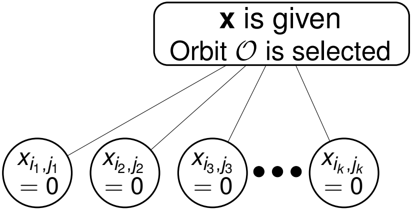

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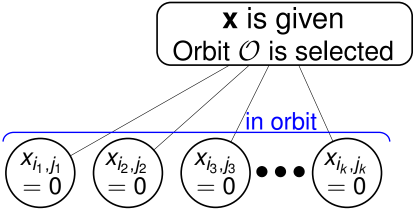

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in orbit

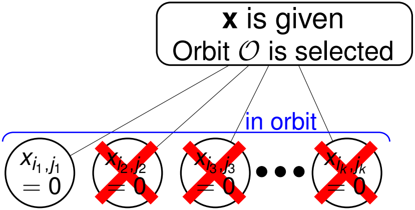
$x_{i_1,j_1} = 0$

$x_{i_2,j_2} = 0$

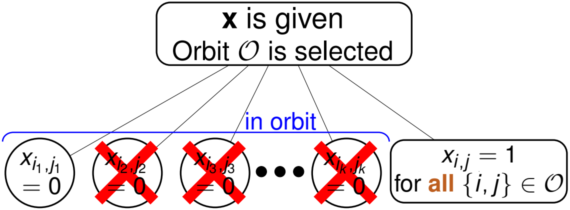
$x_{i_3,j_3} = 0$

$x_{i_k,j_k} = 0$

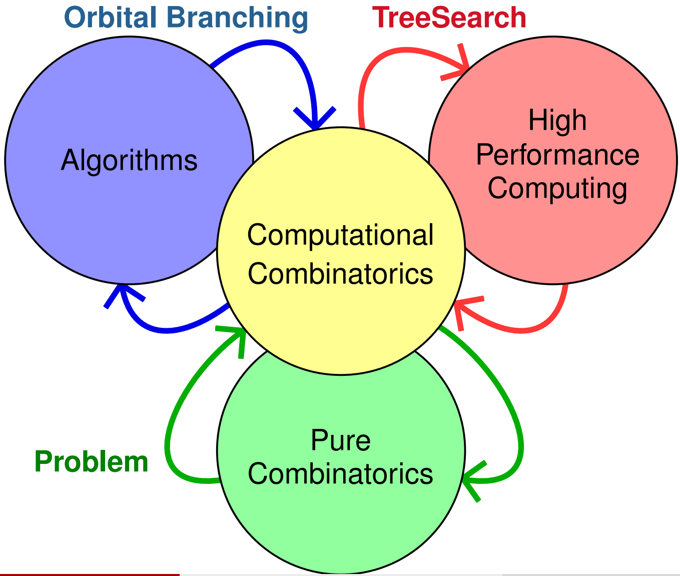
Orbital Branching

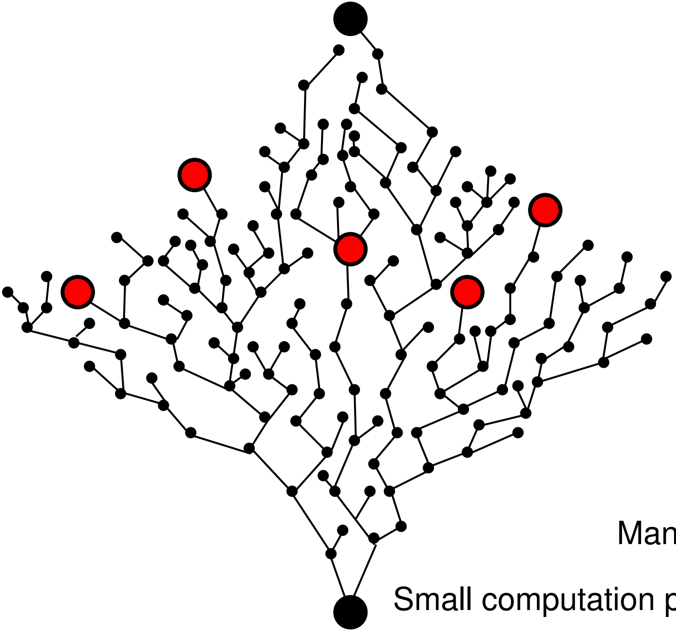


Orbital Branching



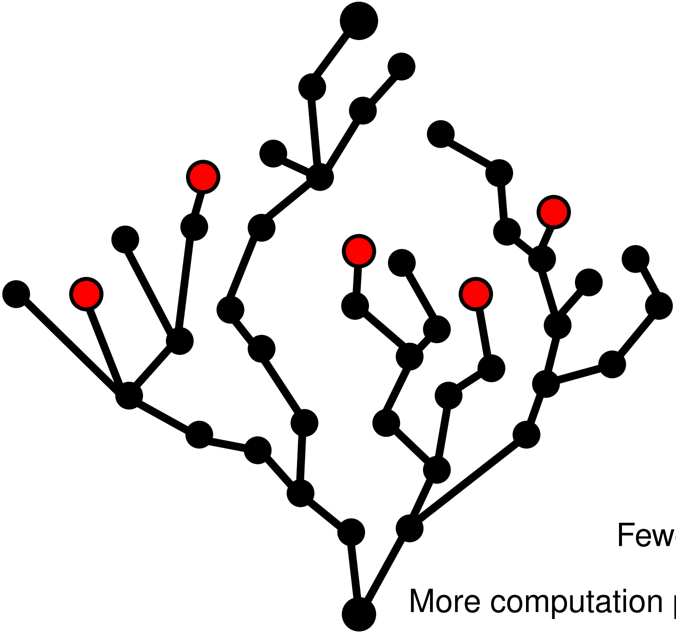
Computational Combinatorics





Many nodes.

Small computation per node.



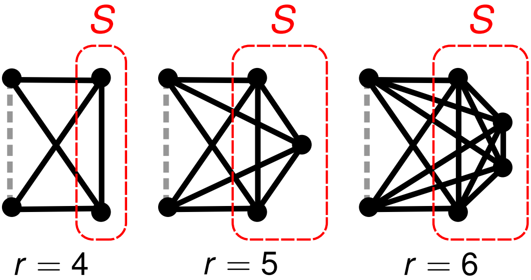
Fewer nodes.

More computation per node.

K_r-Completions

For every non-edge we add, we add a K_r-**completion**:

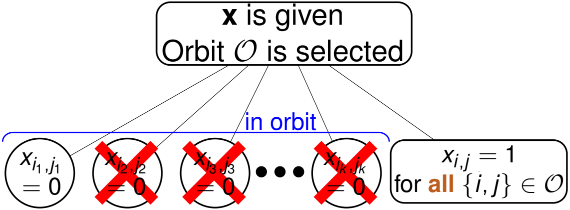
$x_{i,j} = 0$ **if and only if** there exists a set $S \subset [n]$, $|S| = r - 2$, so that $x_{i,a} = x_{j,a} = x_{a,b} = 1$ for all $a, b \in S$.



Orbital Branching with K_r-Completions

$\mathbf{x}$ is given
Orbit $\mathcal{O}$ is selected

Orbital Branching with K_r-Completions



x is given

Orbit $\mathcal{O}$ is selected

in orbit

$x_{i_1,j_1} = 0$

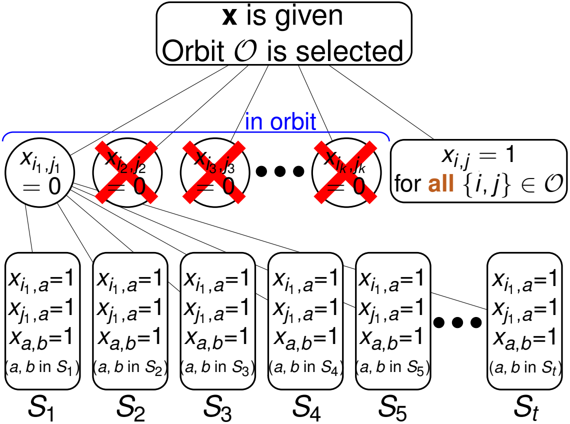
$x_{i_2,j_2} = 0$

$x_{i_3,j_3} = 0$

$x_{i_k,j_k} = 0$

$x_{i,j} = 1$ for **all** $\{i,j\} \in \mathcal{O}$

Orbital Branching with K_r-Completions



x is given

Orbit $\mathcal{O}$ is selected

in orbit

$x_{i_1,j_1} = 0$

$x_{i_2,j_2} = 0$

$x_{i_3,j_3} = 0$

$x_{i_k,j_k} = 0$

$x_{i,j} = 1$ for **all** $\{i,j\} \in \mathcal{O}$

$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_1) S_1

$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_2) S_2

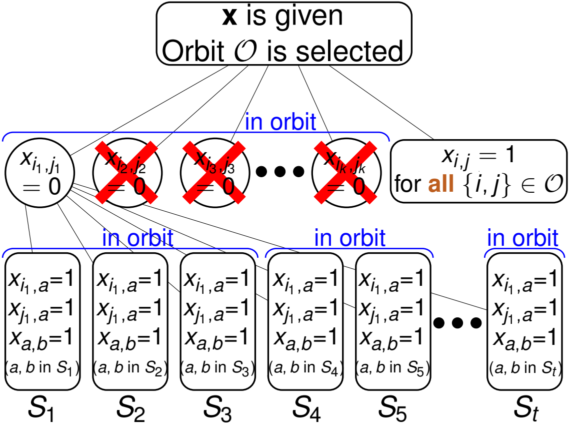
$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_3) S_3

$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_4) S_4

$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_5) S_5

$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_t) S_t

Orbital Branching with K_r-Completions



$\mathbf{x}$ is given

Orbit $\mathcal{O}$ is selected

in orbit

$x_{i_1,j_1} = 0$

$\cancel{x_{i_2,j_2} = 0}$

$\cancel{x_{i_3,j_3} = 0}$

$\cancel{x_{i_k,j_k} = 0}$

$x_{i,j} = 1$ for **all** $\{i,j\} \in \mathcal{O}$

in orbit

in orbit

in orbit

$x_{i_1,a}=1$ $x_{j_1,a}=1$ $x_{a,b}=1$ (a, b in S_1) — S_1

$x_{i_1,a}=1$ $x_{j_1,a}=1$ $x_{a,b}=1$ (a, b in S_2) — S_2

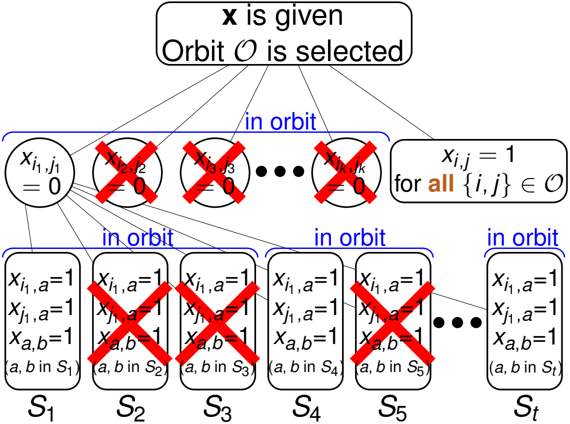
$x_{i_1,a}=1$ $x_{j_1,a}=1$ $x_{a,b}=1$ (a, b in S_3) — S_3

$x_{i_1,a}=1$ $x_{j_1,a}=1$ $x_{a,b}=1$ (a, b in S_4) — S_4

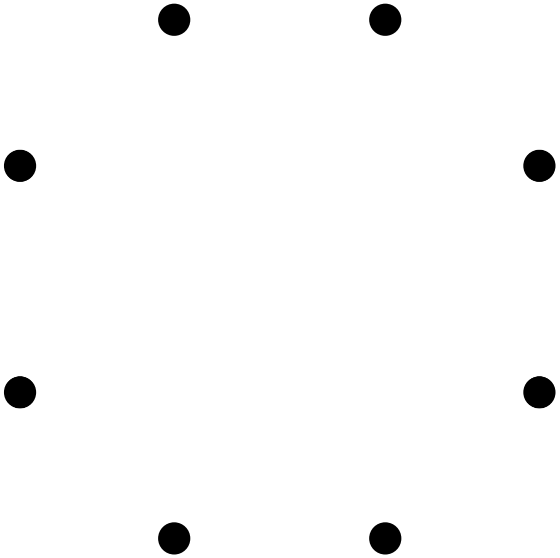
$x_{i_1,a}=1$ $x_{j_1,a}=1$ $x_{a,b}=1$ (a, b in S_5) — S_5

$x_{i_1,a}=1$ $x_{j_1,a}=1$ $x_{a,b}=1$ (a, b in S_t) — S_t

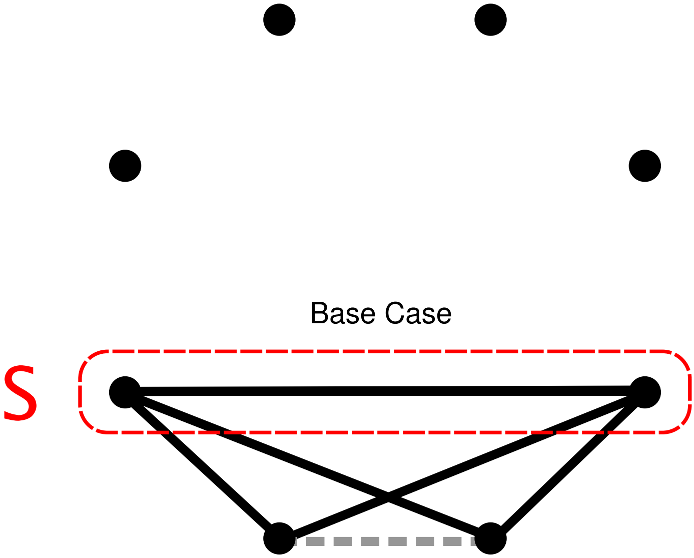
Orbital Branching with K_r-Completions


x is given
Orbit $\mathcal{O}$ is selected
in orbit
$x_{i_1,j_1} = 0$
$x_{i_2,j_2} = 0$
$x_{i_3,j_3} = 0$
$x_{i_k,j_k} = 0$
$x_{i,j} = 1$
for **all** $\{i,j\} \in \mathcal{O}$
in orbit
in orbit
in orbit
$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_1)
$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_2)
$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_3)
$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_4)
$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_5)
$x_{i_1,a}$=1 $x_{j_1,a}$=1 $x_{a,b}$=1 (a, b in S_t)
S_1
S_2
S_3
S_4
S_5
S_t

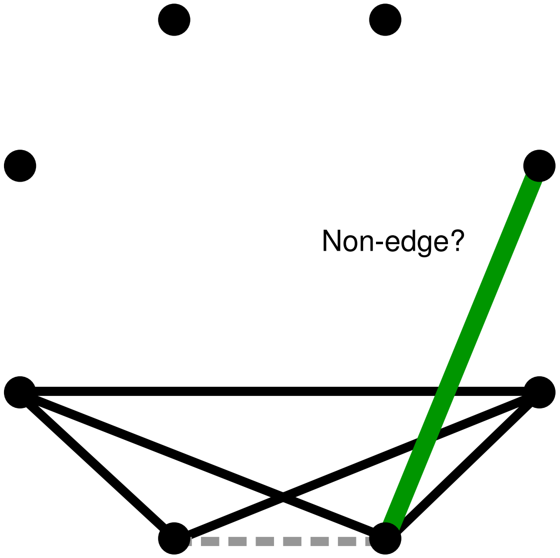
Orbital Branching with K_r-Completions



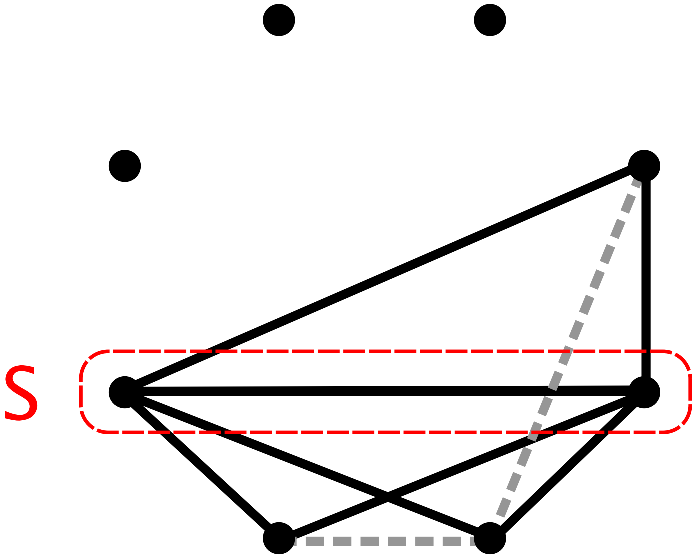
Orbital Branching with K_r-Completions



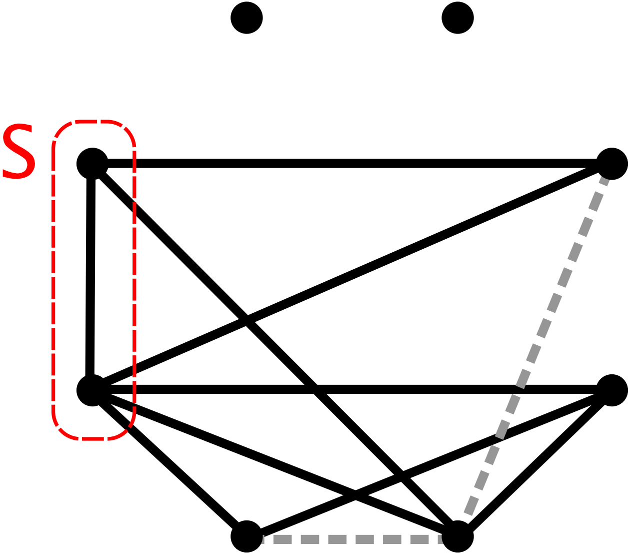
Orbital Branching with K_r-Completions



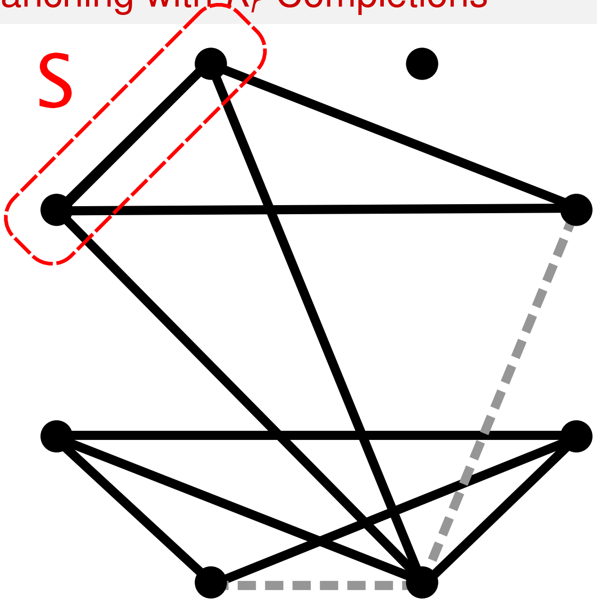
Orbital Branching with K_r-Completions



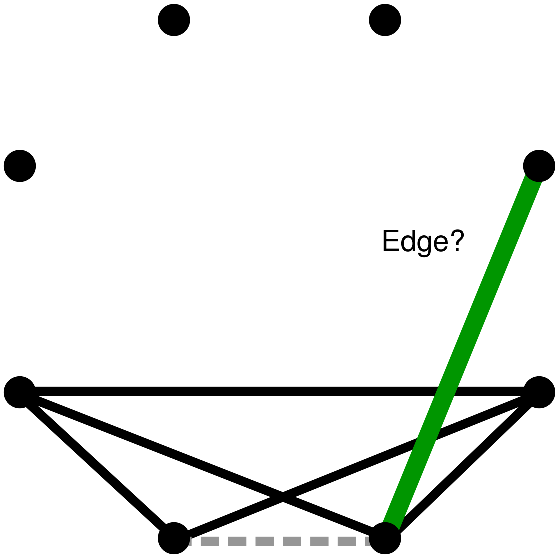
Orbital Branching with K_r-Completions



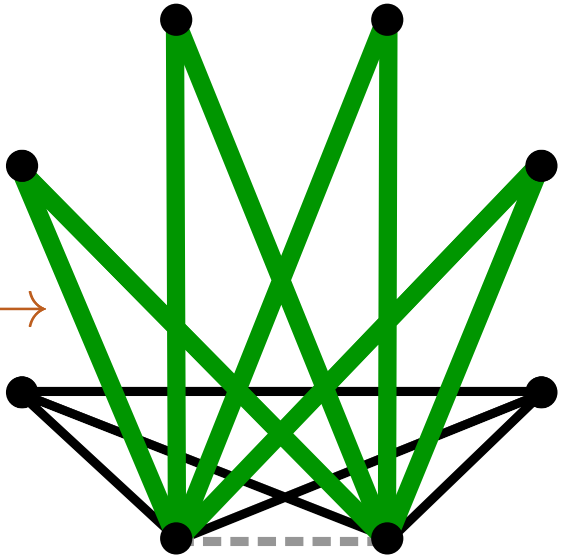
Orbital Branching with K_r-Completions



Orbital Branching with K_r-Completions

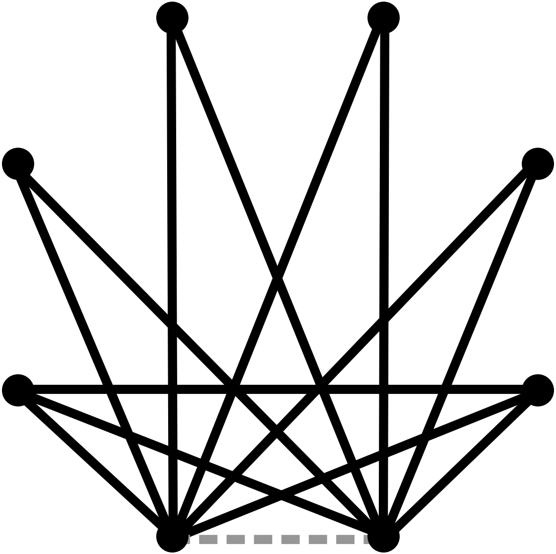


Orbital Branching with K_r-Completions

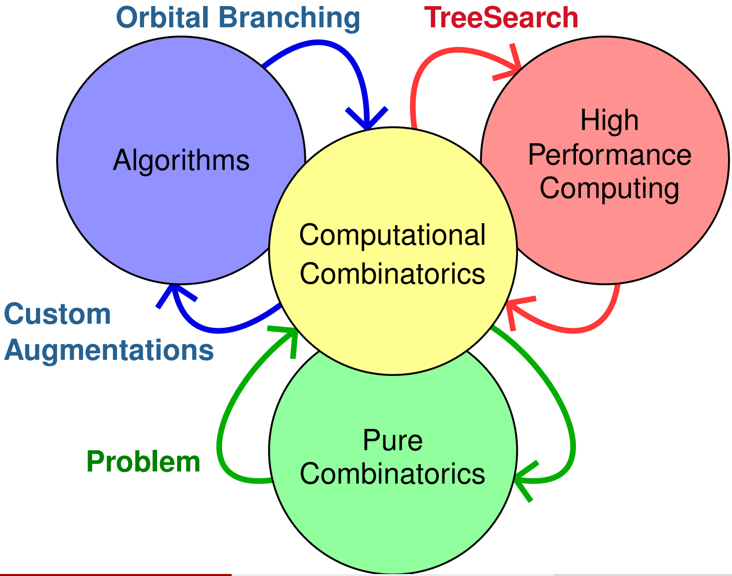


in orbit →

Orbital Branching with K_r-Completions



Computational Combinatorics



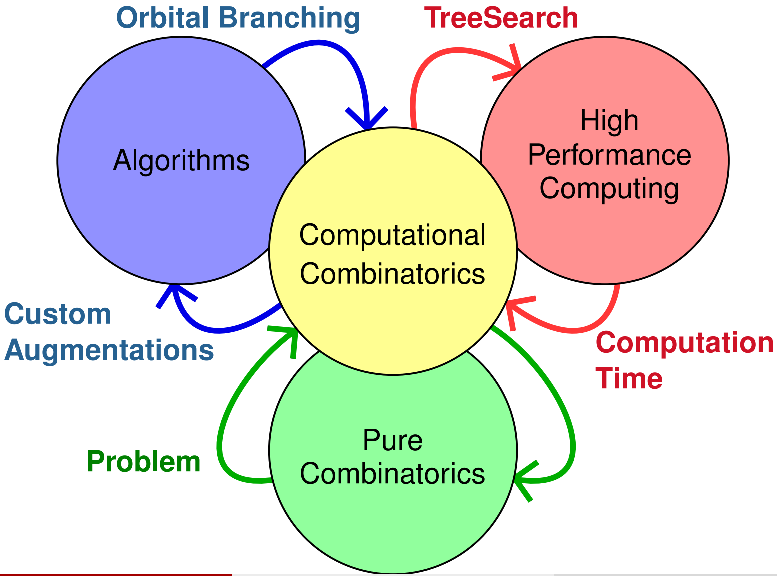
Exhaustive Search Times

n	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$
10	0.10 s	0.37 s	0.13 s	0.01 s	0.01 s
11	0.68 s	5.25 s	1.91 s	0.28 s	0.09 s
12	4.58 s	1.60 m	25.39 s	1.97 s	1.12 s
13	34.66 s	34.54 m	6.53 m	59.94 s	20.03 s
14	4.93 m	10.39 h	5.13 h	20.66 m	2.71 m
15	40.59 m	23.49 d	10.08 d	12.28 h	1.22 h
16	6.34 h	1.58 y	1.74 y	34.53 d	1.88 d
17	3.44 d			8.76 y	115.69 d
18	53.01 d				
19	2.01 y				
20	45.11 y				

Total CPU times using Open Science Grid.

($\approx 8.83 \times 10^{18}$ connected graphs of order 20)

Computational Combinatorics



⟵ clique size ⟶

⟵ vertices ⟶

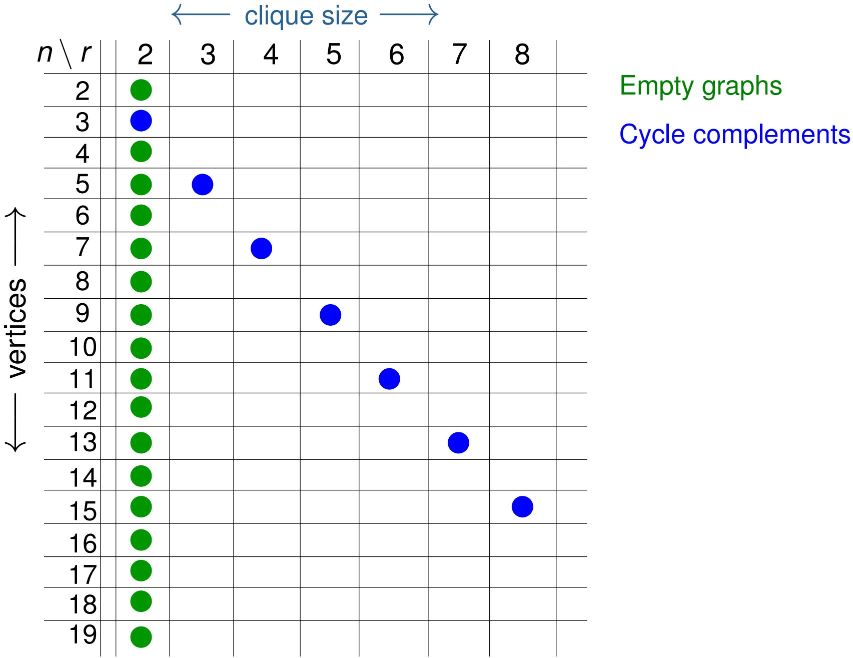
$n \setminus r$		2	3	4	5	6	7	8	
2									
3									
4									
5									
6									
7									
8									
9									
10									
11									
12									
13									
14									
15									
16									
17									
18									
19									

⟵ clique size ⟶

⟵ vertices ⟶

$n \setminus r$	2	3	4	5	6	7	8
2	●						
3	●						
4	●						
5	●						
6	●						
7	●						
8	●						
9	●						
10	●						
11	●						
12	●						
13	●						
14	●						
15	●						
16	●						
17	●						
18	●						
19	●						

Empty graphs



$n \setminus r$	2	3	4	5	6	7	8
2	🟢						
3	🔵						
4	🟢						
5	🟢	🔵					
6	🟢						
7	🟢		🔵				
8	🟢						
9	🟢			🔵			
10	🟢						
11	🟢				🔵		
12	🟢						
13	🟢					🔵	
14	🟢						
15	🟢						🔵
16	🟢						
17	🟢						
18	🟢						
19	🟢						

🟢 Empty graphs

🔵 Cycle complements

⟵ clique size ⟶

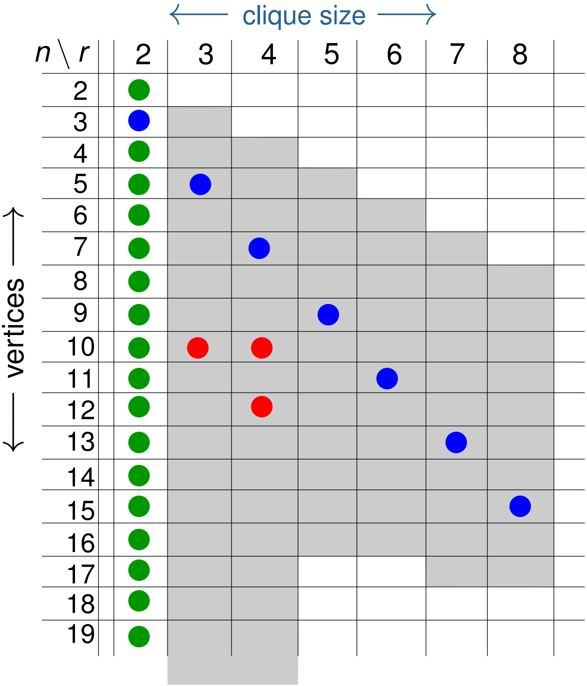
⟵ vertices ⟶

$n \setminus r$	2	3	4	5	6	7	8
2	🟢						
3	🔵						
4	🟢						
5	🟢	🔵					
6	🟢						
7	🟢		🔵				
8	🟢						
9	🟢			🔵			
10	🟢	🔴	🔴				
11	🟢				🔵		
12	🟢		🔴				
13	🟢					🔵	
14	🟢						
15	🟢						🔵
16	🟢						
17	🟢						
18	🟢						
19	🟢						

Empty graphs

Cycle complements

Old examples

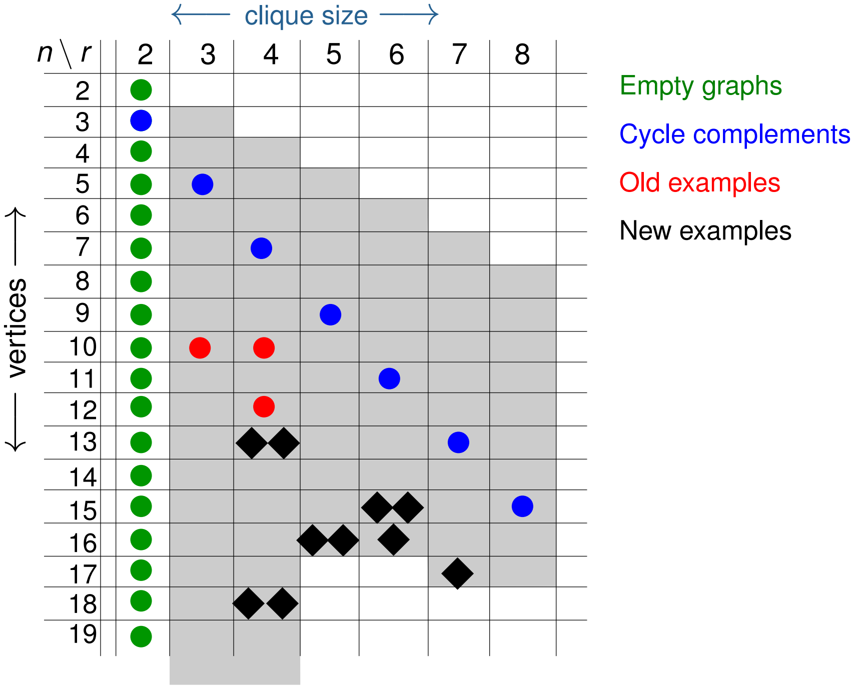


n \ r	2	3	4	5	6	7	8
2	🟢						
3	🔵	▨					
4	🟢	▨	▨				
5	🟢	▨ 🔵	▨	▨			
6	🟢	▨	▨	▨	▨		
7	🟢	▨	▨ 🔵	▨	▨	▨	
8	🟢	▨	▨	▨	▨	▨	▨
9	🟢	▨	▨	▨ 🔵	▨	▨	▨
10	🟢	▨ 🔴	▨ 🔴	▨	▨	▨	▨
11	🟢	▨	▨	▨	▨ 🔵	▨	▨
12	🟢	▨	▨ 🔴	▨	▨	▨	▨
13	🟢	▨	▨	▨	▨	▨ 🔵	▨
14	🟢	▨	▨	▨	▨	▨	▨
15	🟢	▨	▨	▨	▨	▨	▨ 🔵
16	🟢	▨	▨	▨	▨	▨	▨
17	🟢	▨	▨			▨	▨
18	🟢	▨	▨				
19	🟢	▨	▨				

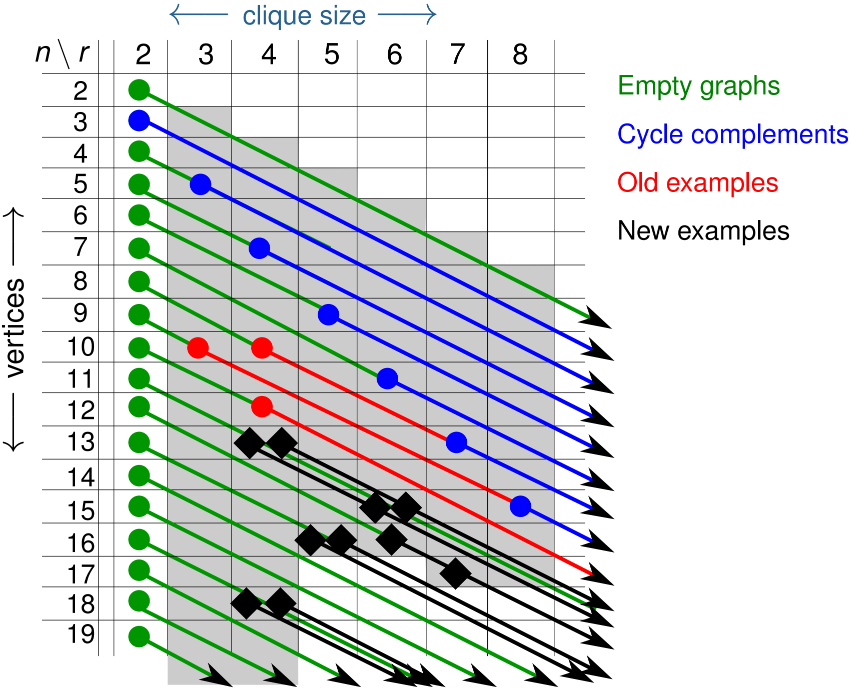
Empty graphs

Cycle complements

Old examples

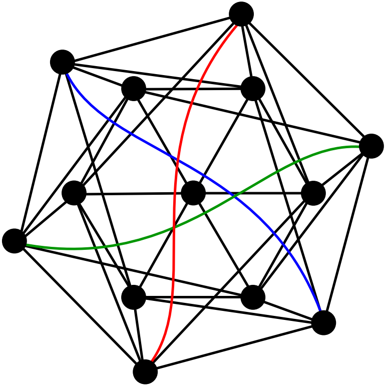


$n \setminus r$	2	3	4	5	6	7	8
2	🟢						
3	🔵						
4	🟢						
5	🟢	🔵					
6	🟢						
7	🟢		🔵				
8	🟢						
9	🟢			🔵			
10	🟢	🔴	🔴				
11	🟢				🔵		
12	🟢		🔴				
13	🟢		◆◆			🔵	
14	🟢						
15	🟢				◆◆		🔵
16	🟢			◆◆	◆		
17	🟢					◆	
18	🟢		◆◆				
19	🟢						

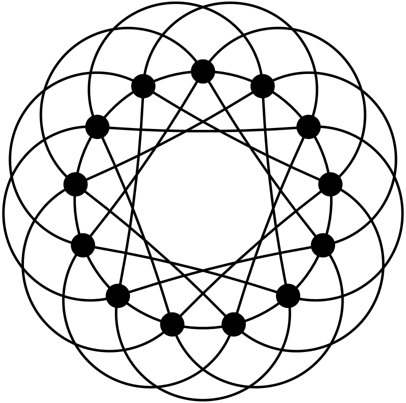

clique size
n \ r
2
3
4
5
6
7
8
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
vertices
Empty graphs
Cycle complements
Old examples
New examples

4-Primitive Graphs

$n = 13$



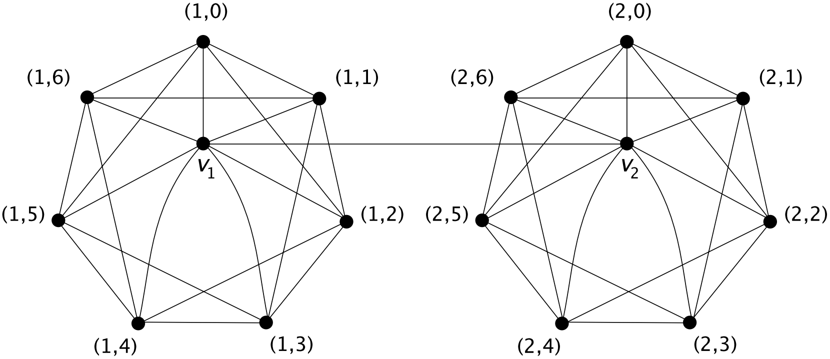
$G_{13}^{(A)}$



Paley(13)

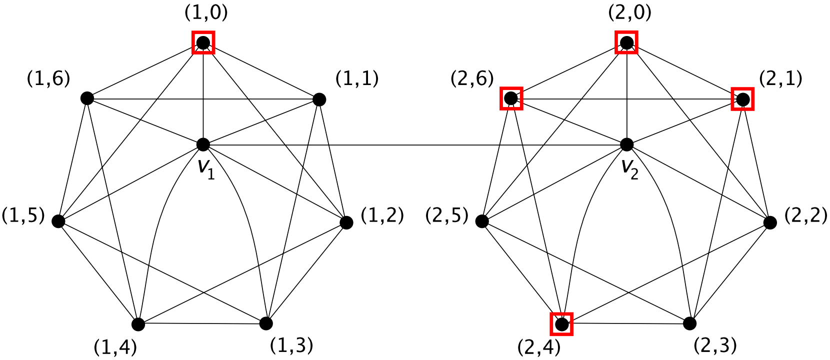
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



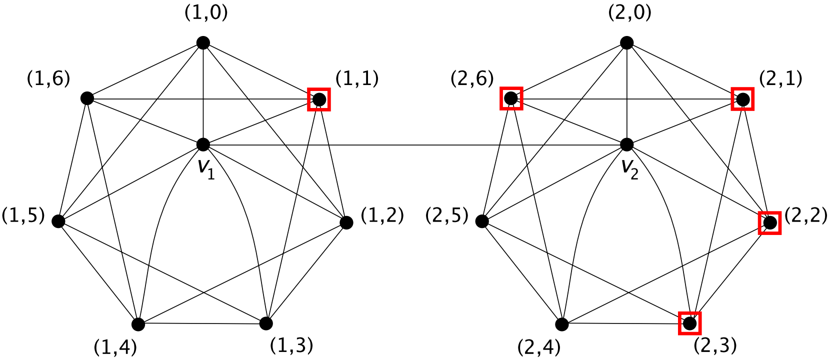
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



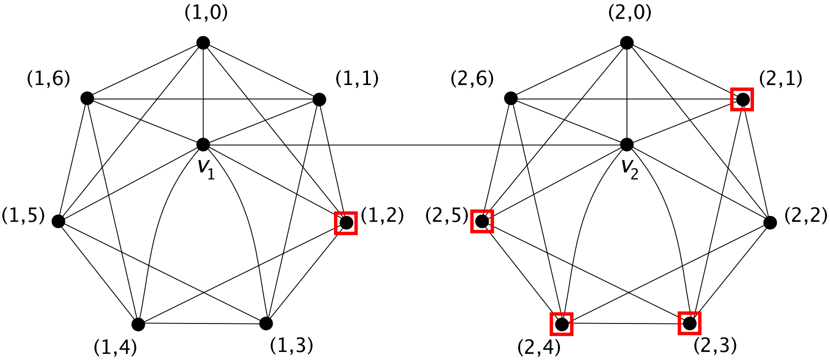
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



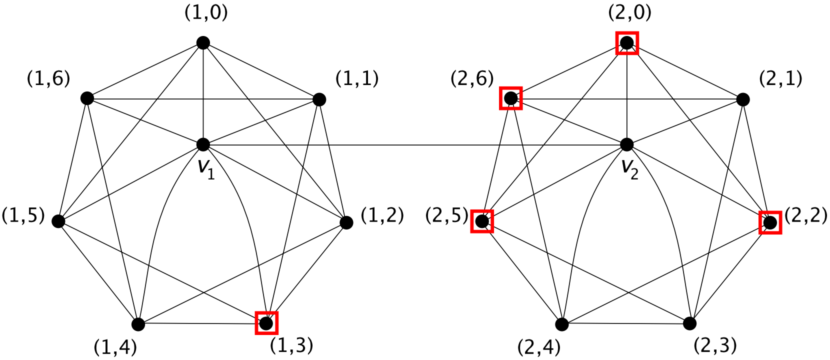
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



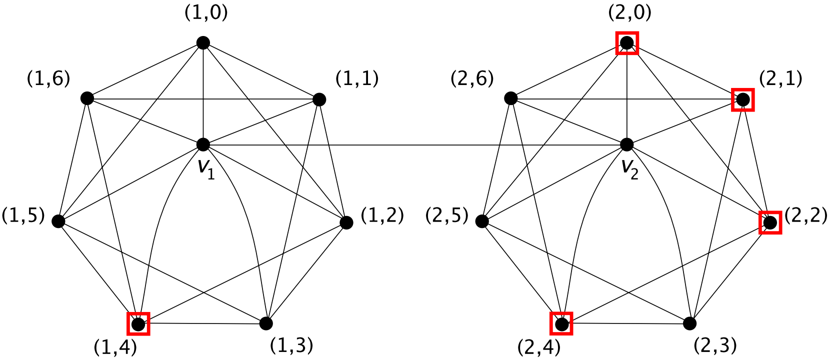
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



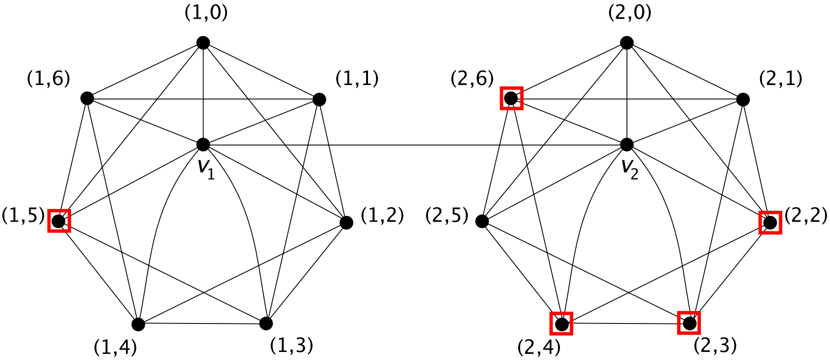
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



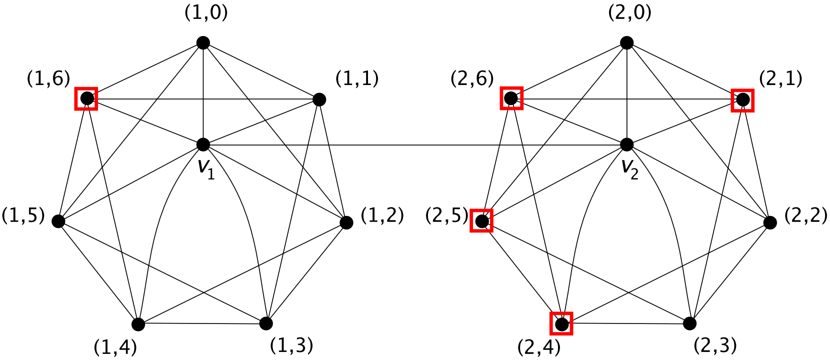
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



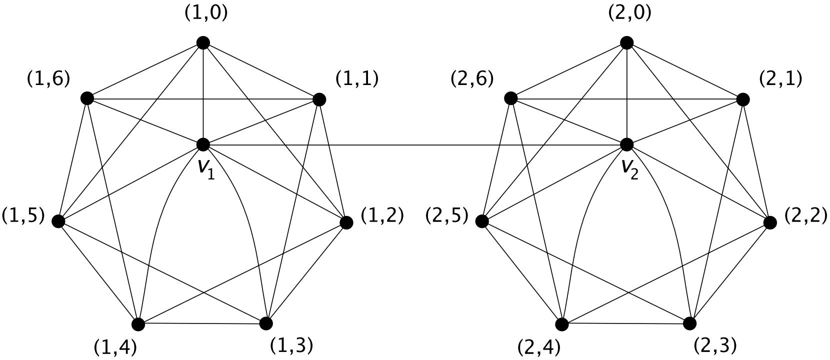
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



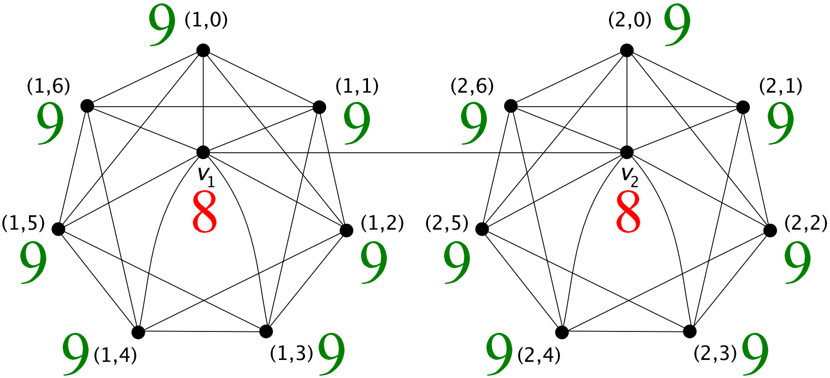
5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



5-Primitive Graph

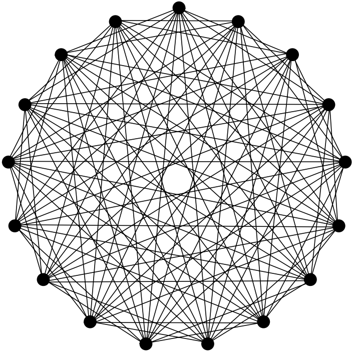
$n = 16 : G_{16}^{(A)}$



Not all *r*-primitive graphs are regular!

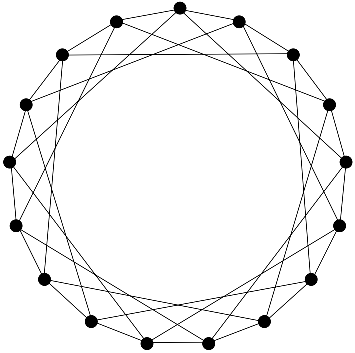
7-Primitive Graph

$n = 17 : G_{17}^{(A)}$



7-Primitive Graph

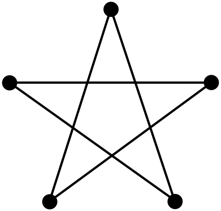
$n = 17 : G_{17}^{(A)}$



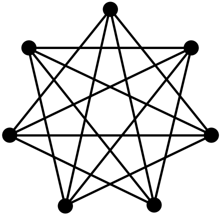
The **Cayley complement** $\overline{C}(\mathbb{Z}_n, S)$ has vertex set $\{0, 1, \ldots, n-1\}$ and an edge *ij* if and only if $|i-j| \pmod{n} \notin S$.

The **Cayley complement** $\overline{C}(\mathbb{Z}_n, S)$ has vertex set $\{0, 1, \ldots, n-1\}$ and an edge *ij* if and only if $|i-j| \pmod{n} \notin S$.

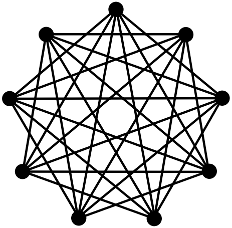
For $r \geq 1$, $\overline{C}(\mathbb{Z}_{2r-1}, \{1\}) \cong \overline{C_{2r-1}}$ is *r*-primitive.



$\overline{C_5}$



$\overline{C_7}$



$\overline{C_9}$

Searching for r-Primitive Cayley Complements

To search for Cayley complements $\overline{C}(\mathbb{Z}_n, S)$ with $|S| = g$:

1. Select a generator set $S = \{a_1 = 1 < a_2 < a_3 < \cdots < a_g\} \subseteq \mathbb{Z}$.

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Used Niskanen and Östergård's *cliquer* software to compute $\omega(G)$.

Two or Three Generators

S	r	n
$\{1,4\}$	7	17
$\{1,6\}$	16	37
$\{1,8\}$	29	65
$\{1,10\}$	46	101
$\{1,12\}$	67	145

$g = 2$

S	r	n
$\{1,5,6\}$	9	31
$\{1,8,9\}$	22	73
$\{1,11,12\}$	41	133
$\{1,14,15\}$	66	211
$\{1,17,18\}$	97	307

$g = 3$

Infinite Families

Conjecture (Hartke, Stolee, 2012) Let $t \geq 1$,

$$n = 4t^2 + 1, \qquad \text{and} \qquad r = 2t^2 - t + 1.$$

The Cayley complement $\overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.

Conjecture (Hartke, Stolee, 2012) Let $t \geq 1$,

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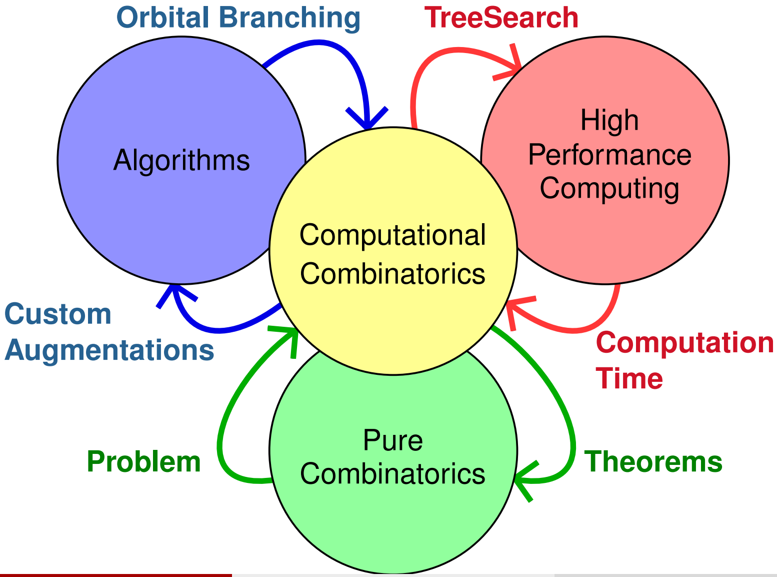
Proof uses **discharging** method.

Computational Combinatorics



What Next?

Technique-specific

1. **Orbital Branching:** Formalize custom augmentations for arbitrary constraint systems. Apply to problems like strongly regular graphs.

2. **Discharging:** Automate process so computer can discover and write proofs.

3. **More Techniques:** Find, Adapt, or Develop.

Computational Combinatorics and the search for Uniquely K_r-Saturated Graphs

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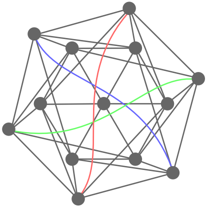

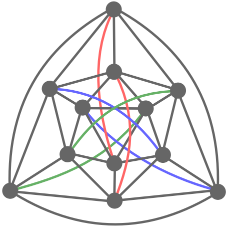

Joint work with Stephen G. Hartke, appeared in *Electronic Journal of Combinatorics*, 2012.