Problems of Enumeration and Realizability on Matroids, Simplicial Complexes, and Graphs

Yvonne Kemper
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In Honor of a Diagram
Wait, what were those things again?

Definition
A graph \( G = (V, E) \) is a set of vertices \( V = \{v_1, \ldots, v_n\} \) and a set of edges \( E = \{v_iv_j : v_i, v_j \in V\} \).

Example
Here is a graph!

\[ G = (V, E) = (\{1, 2, 3, 4\}, \{12, 13, 14, 23, 24\}) \]
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graphs
Wait, what were those things again?

**Definition**

A matroid $M = (E(M), \mathcal{I}(M))$ consists of a ground set $E(M)$ and a family of subsets $\mathcal{I}(M) \subseteq 2^{E(M)}$ called independent sets such that

1. $\emptyset \in \mathcal{I}$;

2. if $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$; and

3. if $I, J \in \mathcal{I}$, and $|J| < |I|$, then there exists some $e \in I \setminus J$ such that $J \cup \{e\} \in \mathcal{I}$.
Example

Here is a graph matroid!

\[ M = (E(M), I(M)) = (\{1, 2, 3, 4, 5\}, \{\emptyset, 1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 124, 125, 134, 135, 145, 234, 235, 245\}) \]
Wait, what were those things again?

Example
Here is a graph matroid!

\[ M = (E(M), \mathcal{I}(M)) \]
\[ = (\{1, 2, 3, 4, 5\}, \{\emptyset, 1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 124, 125, 134, 135, 145, 234, 235, 245\}) \]
In Honor of a Diagram

graphs
matroids
In Honor of a Diagram

graphs
matroids
simplicial complexes
Definition
An *(abstract) simplicial complex* $\Delta$ on a *vertex set* $V$ is a set of subsets of $V$. These subsets are called the *faces* of $\Delta$, and we require that

(1) for all $v \in V$, $\{v\} \in \Delta$, and

(2) for all $F \in \Delta$, if $G \subseteq F$, then $G \in \Delta$. 
Example

Here is a graph matroid simplicial complex!

\[ \Delta = (V, F) = (\{1, 2, 3, 4\}, \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24\}) \]
Example

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graphs

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problems on matroids, simplicial complexes, and graphs

Lookin' good!

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The Problems Three

- \textit{h}-Vectors of Small Matroids
- Flows on Simplicial Complexes
- Polytopal Embeddings of Cayley Graphs
$h$-Vectors of Small Matroids
Definition
The **dimension** of a face $F$ is $|F| - 1$, and the **dimension** of $\Delta$ is $d = \max\{|F| : F \in \Delta\} - 1$.

Definition
A simplicial complex is **pure** if all maximal elements of $\Delta$ have the same cardinality.
In this case, a **facet** is a maximal face, a **ridge** is a face of one dimension lower.
The \textbf{f-vector} of a simplicial complex $\Delta$, $\dim \Delta = d - 1$, is \[
f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta)),\]
where $f_i(\Delta) := |\{F \in \Delta : \dim F = i\}|$. 
The \textit{f-vector} of a simplicial complex $\Delta$, $\dim \Delta = d - 1$, is

$$f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta)),$$

where $f_i(\Delta) := |\{ F \in \Delta : \dim F = i \}|$.

The \textit{h-vector}, $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$, is given by:

$$\sum_{j=0}^{d} h_j(\Delta) \lambda^j = \sum_{i=0}^{d} f_{i-1}(\Delta) \lambda^i (1 - \lambda)^{d-i}.$$
Characterizations of $f$- and $h$-Vectors

Definition
Given two integers $k, i > 0$, write

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Define

$$k^{(i)} = \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \cdots + \binom{n_j}{j+1}.$$

Theorem (Schützenberger, Kruskal, Katona)
A vector $(1, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{Z}^{d+1}$ is the $f$-vector of some $(d-1)$-dimensional simplicial complex $\Delta$ if and only if

$$0 < f_{i+1} \leq f^{(i+1)}_i, \quad 0 \leq i \leq d - 2.$$
Question

Can we characterize subclasses of simplicial complexes?

Example

- Cohen-Macaulay complexes
- Flag complexes
- Shifted complexes
- Independence complexes of matroids
Let $M$ be given by:

$$I(M) = \{\emptyset,\ 1, 2, 3, 4, 5,\ 12, 13, 14, 15, 23,\ 24, 25, 34, 35, 45,\ 124, 125, 134, 135,\ 145, 234, 235, 245\}$$

Then: $f(M) = (1, 5, 10, 8)$ and $h(M) = (1, 2, 3, 2)$. 
O-Sequences

- A non-empty set of monomials $\mathcal{M}$ is a **multicomplex** if
  
  \[ m \in \mathcal{M} \text{ and } n|m \Rightarrow n \in \mathcal{M}. \]

- A sequence $h = (h_0, h_1, \ldots, h_d)$ of integers is an **O-sequence** if there exists a multicomplex with precisely $h_i$ monomials of degree $i$.

- An O-sequence is **pure** if all maximal elements have the same degree.

Example: Let $\mathcal{M} = \{1, x_1, x_2, x_1x_2, x_2^2, x_2^1x_2, x_2^2x_1, x_2^1x_2^2, x_2^2x_1x_2 \}$. Then, the corresponding (pure) O-sequence is: $O(\mathcal{M}) = (1, 2, 3, 2)$. 

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A non-empty set of monomials $\mathcal{M}$ is a **multicomplex** if

$$m \in \mathcal{M} \text{ and } n|m \Rightarrow n \in \mathcal{M}.$$ 

A sequence $h = (h_0, h_1, \ldots, h_d)$ of integers is an **$O$-sequence** if there exists a multicomplex with precisely $h_i$ monomials of degree $i$.

An $O$-sequence is **pure** if all maximal elements have the same degree.

**Example**

Let $M = \{1, x_1, x_2, x_1x_2, x_1^2, x_2^2, x_1x_2^2, x_1^2x_2\}$. Then, the corresponding (pure) $O$-sequence is:

$$O(M) = (1, 2, 3, 2).$$
Stanley’s Conjecture

Conjecture (Stanley, 1977)

*The $h$-vector of a matroid complex is a pure $O$-sequence.*

Little progress was made for twenty years, but since 1997, the conjecture has been proved for matroids which are:

- of rank 4 (Klee, Samper),
- of rank less than or equal to 3 (Stokes, Há et al.),
- cographic (Biggs, Merino),
- lattice-path (Schweig),
- cotransversal (Oh),
- paving (Merino, et al.).
Theorem (De Loera, K., Klee)

- Let $M$ be a matroid of rank 2. Then $h(M)$ is a pure $O$-sequence.

- Let $M$ be a matroid of corank 2. Then $h(M)$ is a pure $O$-sequence.

- Let $M$ be a matroid of rank $d \geq 4$. Then, the subsequence $(1, h_1(M), h_2(M), h_3(M))$ of $h(M)$ is a pure $O$-sequence.
  - Let $M$ be a matroid of rank 3. Then $h(M)$ is a pure $O$-sequence.

- Let $M$ be a matroid on at most 9 elements. Then $h(M)$ is a pure $O$-sequence.
An Experimental Result: Matroids on at Most Nine Elements

- Royle and Mayhew generated list of all matroids on at most nine elements - why not check them all?
- Used database to generate all $h$-vectors for these matroids.
- Generated list of all possible $O$-sequences of multicomplexes (up to maximal degree 9 on at most 9 variables), then checked that every $h$-vector appeared on this list.
Example: A Multicomplex in Two Variables

1. Pick a point \((a, b) \in \mathbb{Z}^2\) on the hyperplane \(x + y = r\), where \(r\) is the rank of the matroid.
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Let’s say \(r = 3\).
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\((a, b)\) corresponds to the monomial \(x^a y^b\)
Example: A Multicomplex in Two Variables

1. Pick a point \((a, b) \in \mathbb{Z}^2\) on the hyperplane \(x + y = r\), where \(r\) is the rank of the matroid.

2. Add all points in the shadow of \((a, b)\) to the multicomplex.

\[
M = \{1, x, y, xy, y^2, xy^2\}
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3. Repeat as desired to generate all possible \(O\)-sequences (of rank 3 and corank 2).

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The corresponding \(O\)-sequence is: \((1, 2, 3, 2)\).
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The corresponding \(O\)-sequence is: \((1, 2, 3, 2)\).
Future Questions and Directions

- Cannot extend results directly, but can we use the geometric viewpoint to verify the conjecture for further classes of matroid complexes?

- Use PS-ear decomposability of matroid complexes?

- Characterize $f$- and $h$-vectors for further classes of simplicial complexes, such as matroid polytopes?

- There is a list of all matroids on at most 10 elements – can we find a counterexample here?
Flows on Simplicial Complexes
Definition
A $\mathbb{Z}_q$-flow on an oriented graph $G$ is a vector $x \in \mathbb{Z}_q^E$ such that

$$\sum_{h(e)=v} x_e \equiv \sum_{t(e)=v} x_e \mod q,$$

for all $v \in V$. A $\mathbb{Z}_q$-flow is nowhere-zero if it is fully supported.
Definition

A $\mathbb{Z}_q$-flow on an oriented graph $G$ is a vector $x \in \mathbb{Z}_q^E$ such that

$$\sum_{h(e)=\nu} x_e \equiv \sum_{t(e)=\nu} x_e \mod q,$$

for all $\nu \in V$. A $\mathbb{Z}_q$-flow is nowhere-zero if it is fully supported.

Equivalently, a nowhere-zero $\mathbb{Z}_q$-flow is a fully-supported element of the kernel mod $q$ of the signed incidence matrix of the graph.
(1, 2, 2, 3, 3) is a nowhere-zero \( \mathbb{Z}_5 \)-flow and an element of the kernel (mod 5) of the incidence matrix.
Flows originally defined in the context of electric circuits and networks

Some Previous Work:
- The number of nowhere-zero $\mathbb{Z}_q$-flows on a graph is a polynomial in $q$.
- For a planar graph $G$, $\chi_G(k) = k^{c(G)}(G^*)_*(k)$.
- The Max-Flow/Min-Cut problem of optimization.

Open Questions:
- 5-flow conjecture
- Volumes of flow polytopes
Definition
Let $\partial$ be a boundary map on a $(d - 1)$-dimensional complex $\Delta$ given by:

$$\partial [v_{i_0} \cdots v_{i_r}] = \sum_{j=0}^{r} (-1)^j [v_{i_0} \cdots \hat{v}_{i_j} \cdots v_{i_r}],$$

where $0 \leq r \leq d$. The boundary matrix of $\Delta$ is given by the signs of the ridges in the boundary maps of the facets. We denote this matrix $\partial \Delta$. 
Definition

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The boundary matrix of an oriented graph is equal to its signed incidence matrix.
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The boundary matrix of an oriented graph is equal to its signed incidence matrix.

Definition
A $\mathbb{Z}_q$-flow on a pure simplicial complex $\Delta$ is an element of the kernel mod $q$ of the boundary matrix of $\Delta$. 
Flows on Simplicial Complexes

Example
The surface of a tetrahedron.

An example of a $\mathbb{Z}_4$-flow is: $(1, 3, 1, 3)$. 
Proposition (Beck, K.)

Let $\Delta$ be a triangulation of a manifold, and let $\phi_\Delta(q)$ be the number of nowhere-zero $\mathbb{Z}_q$-flows on $\Delta$. Then,

$$\phi_\Delta(q) = \begin{cases} 
0 & \text{if } \Delta \text{ has boundary;} \\
q - 1 & \text{if } \Delta \text{ is without boundary, } \mathbb{Z}\text{-orientable;} \\
0 & \text{if } \Delta \text{ is without boundary, non-}\mathbb{Z}\text{-orientable, } q \text{ odd;} \\
1 & \text{if } \Delta \text{ is without boundary, non-}\mathbb{Z}\text{-orientable, } q \text{ even.}
\end{cases}$$
Definition
A function $\phi$ in an integer variable $t$ is a **quasipolynomial** if there exists an integer $k > 0$ and polynomials $p_0(t), \ldots, p_{k-1}(t)$ such that

$$\phi(t) = p_j(t) \quad \text{if} \quad t \equiv j \mod k.$$ 

The minimal such $k$ is the **period** of $\phi$. 

Example
Let $\phi(t)$ be defined for $t \in \mathbb{Z}$ as follows:

$$\begin{align*}
\phi(t) &= \begin{cases} 
t^2 + 1 & \text{if } t \equiv 0 \mod 5 \\
t - 4 & \text{if } t \equiv 1, 3 \mod 5 \\
3t^3 + 1 & \text{if } t \equiv 2 \mod 5 \\
0 & \text{if } t \equiv 4 \mod 5
\end{cases}.
\end{align*}$$

Then $\phi(t)$ is a quasipolynomial with period 5.
**Definition**

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**Example**

Let $\phi(t)$ be defined for $t \in \mathbb{Z}$ as follows:

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\phi(t) = \begin{cases} 
    t^2 + 1 & \text{if } t \equiv 0 \mod 5 \\
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    3t^3 + \frac{1}{2}t & \text{if } t \equiv 2 \mod 5 \\
    0 & \text{if } t \equiv 4 \mod 5.
\end{cases}
$$

Then $\phi(t)$ is a quasipolynomial with period 5.
Results

Theorem (Beck, K.)

The number $\phi_\Delta(q)$ of nowhere-zero $\mathbb{Z}_q$-flows on $\Delta$ is a quasipolynomial in $q$. Furthermore, there exists a polynomial $p(x)$ such that $\phi_\Delta(k) = p(k)$ for all integers $k$ that are relatively prime to the period of $\phi_\Delta(q)$.

Theorem (Beck, K.)

- Let $q$ be a sufficiently large prime number, and let $\Delta$ be a simplicial complex of dimension $d$. Then the number $\phi_\Delta(q)$ of nowhere-zero $\mathbb{Z}_q$-flows on $\Delta$ is a polynomial in $q$ of degree $\dim_{\mathbb{Q}}(\tilde{H}_d(\Delta; \mathbb{Q}))$.
- In particular, $\phi_\Delta(q) = (-1)^{|E(M)|-rk(M)} \text{Tr}_M(0, 1 - q)$, where $M$ is the matroid given by the columns of $\partial \Delta$. 
Example

Let $K$ be the Klein bottle. Then:

$$H_2(K; \mathbb{Z}_q) = \begin{cases} \mathbb{Z}_2 & \text{q even} \\ 0 & \text{q odd.} \end{cases}$$

Therefore:

$$\phi_K(q) = \begin{cases} 1 & \text{q even} \\ 0 & \text{q odd;} \end{cases}$$

$\phi_K(q)$ is a quasipolynomial with period 2.
Definition
A matrix is *totally unimodular* (TU) iff every subdeterminant is 0, 1, or −1.

Fact
*If the boundary matrix of a simplicial complex $\Delta$ is TU, then $\phi_\Delta(q)$ has period 1.*
The Period of the Flow Quasipolynomial

**Definition**
A matrix is **totally unimodular** (TU) iff every subdeterminant is 0, 1, or −1.

**Fact**
*If the boundary matrix of a simplicial complex $\Delta$ is TU, then $\phi_\Delta(q)$ has period 1.*

**Theorem (Dey, Hirani, Krishnamoorthy)**
*For a finite simplicial complex $\Delta$ of dimension greater than $d - 1$, the boundary matrix $[\partial_d]$ is totally unimodular if and only if $H_{d-1}(L, L_0)$ is torsion-free for all pure subcomplexes $L_0, L$ in $\Delta$ of dimensions $d - 1$ and $d$ respectively, where $L_0 \subset L$.***
Definition

A **convex ear decomposition** of a pure rank-\(d\) simplicial complex \(\Delta\) is an ordered sequence \(\Sigma_1, \Sigma_2, \ldots, \Sigma_n\) (the **ears**) of pure rank-\(d\) subcomplexes of \(\Delta\) such that

1. \(\Sigma_1\) is the boundary complex of a simplicial \(d\)-polytope, while for each \(i = 2, \ldots, n\), \(\Sigma_i\) is a \((d-1)\)-ball which is a (proper) sub-complex of the boundary complex of a simplicial \(d\)-polytope, and

2. For \(i \geq 2\), \(\Sigma_i \cap \left( \bigcup_{j=1}^{i-1} \Sigma_j \right) = \partial \Sigma_i\).
CED But Not TU

$\phi(q) = (q - 1)(q - 2)$

BUT the period is still equal to 1 – perhaps there is hope...?
CED But Not TU

$\phi_\Delta(q) = (q - 1)(q - 2)$

BUT the period is still equal to 1 – perhaps there is hope...?
CED, Not TU, and $p > 1$

$$\phi_{\Delta}(q) = q^3 - 7q^2 + 15q - 8 - \gcd(2, q)$$
Open Questions and Future Directions

- **Topological Conditions**
  - Necessary and/or sufficient topological conditions for period equal to 1?
  - Necessary and/or sufficient topological conditions for period greater than 1?

- **The Period of the Flow Quasipolynomial**
  - Is there a bound for quasipolynomials from modular flows?
  - Can we find a subcomplex that guarantees a period greater than 1 – or is there always the possibility of period collapse?

- **Constructions preserving/leading to polynomiality**
  - Families of simplicial complexes with $p = 1$?
  - Relationship between $\phi_G(q)$ of a graph $G$ and $\phi_{\Delta(G)}(q)$?
Polytopal Embeddings of Cayley Graphs
Cayley Graphs

Definition
Let $\Gamma$ be a group, and $\Delta$ a set of generators of $\Gamma$. The **Cayley color graph**, $C(\Gamma, \Delta)$, of $(\Gamma, \Delta)$ is a directed, edge-colored graph such that:

- its vertices are the elements of $\Gamma$, and
- there is directed edge colored $h$ from $g_1$ to $g_2$ if there exists a generator $h \in \Delta$ such that $g_1 h = g_2$.

If we forget the colors and directions of the edges of $C(\Gamma, \Delta)$, we have the **Cayley graph**, $G(\Gamma, \Delta)$.

Remark
A group will have many Cayley graphs, which depend on the representation that is used.
An Example

Say we have a representation of a group
\[ \Gamma = \langle x, y \mid xy = yx, \ x^3 = y^2 = 1 \rangle. \] The Cayley color graph is:

![Cayley color graph](image-url)
An Example

Say we have a representation of a graph $\Gamma = \langle x, y \mid xy = yx, x^3 = y^2 = 1 \rangle$. The Cayley color graph is:

![Cayley color graph]

where $x$ and $y$ are the generators of the group, and $x^3 = y^2 = 1$ are the relations.
An Example

Say we have a representation of a graph
\[ \Gamma = \langle x, y \mid xy = yx, x^3 = y^2 = 1 \rangle. \] The Cayley graph is:

![Cayley graph diagram]

\[ y \quad \quad x \quad \quad yx \quad \quad yx^2 \]

\[ 1 \quad \quad x \quad \quad x^2 \]
Definition
The **genus** of a graph is the minimal genus of all orientable surfaces in which $G$ can be embedded.

Definition
The **genus** of a group $\Gamma$, $\gamma(\Gamma)$, is the minimal genus among the genera of all possible Cayley graphs of $\Gamma$. 
Definition
The *genus* of a graph is the minimal genus of all orientable surfaces in which $G$ can be embedded.

Definition
The *genus* of a group $\Gamma$, $\gamma(\Gamma)$, is the minimal genus among the genera of all possible Cayley graphs of $\Gamma$.

Open Problem!
*Classify all finite groups of a particular genus $\gamma$, for all $\gamma > 2$.*/
Finite Groups of Genus 0, 1, and 2

- **Genus 0**: Classified by Maschke (1896)

- **Genus 1**: Classified by Proulx in her thesis (1978)

- **Genus 2**: Just one of them, found by Tucker (1984)

\[ \langle x, y, z \mid x^2 = y^2 = z^2 = 1, \]
\[ (xy)^2 = (yz)^3 = (xz)^8 = 1, \]
\[ y(xz)^4 y(xz)^4 = 1 \rangle \]
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\[ y(xz)^4 y(xz)^4 = 1 \rangle \]
Question

When is there a polyhedral embedding of a planar group?
Definition

- A separator $S$ of a graph $G$ is a subset of the vertices $V$ such that $V \setminus S$ has at least two components.

- A $k$-separator is a separator of cardinality $k$.

- A graph is $k$-connected if there exist no separators of cardinality $\leq k - 1$. 
Question

*When is there a polyhedral embedding of a planar group?*
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When is there a polyhedral embedding of a planar group?

Fact 1: (Steinitz, 1922) A graph is the 1-skeleton of a polyhedron iff it is 3-connected and planar.
Question

When is there a **polyhedral** embedding of a planar group?

- **Fact 1**: (Steinitz, 1922) A graph is the 1-skeleton of a polyhedron iff it is 3-connected and planar.
- **Fact 1+**: (Mani, 1971) Every 3-connected, planar graph $G$ is the 1-skeleton of a polyhedron $P$ such that every automorphism of $G$ is induced by a symmetry of $P$. 
Question

When is there a polyhedral embedding of a planar group?

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Proposition (De Loera)

Let $G(\Gamma, \Delta)$ be a planar Cayley graph for the group $\Gamma$. Then $G(\Gamma, \Delta)$ can be embedded as the 1-skeleton of a polytope.
An Example

Say we have a representation of a graph
\[ \Gamma = \langle x, y \mid xy = yx, \ x^3 = y^2 = 1 \rangle. \] The polytonal embedding of the Cayley color graph is...?

It is 3-connected, and it is clearly planar...
An Example

Say we have a representation of a graph
\[ \Gamma = \langle x, y \mid xy = yx, \ x^3 = y^2 = 1 \rangle. \] The polytonal embedding of the Cayley color graph is:

![Graph Diagram]

1

\[ x \]

\[ y \]

\[ yx \]

\[ yx^2 \]

\[ x^2 \]
A Natural Question

Question

For any group $\Gamma$, and any representation $\Delta$, can we always find a convex polytope such that $G(\Gamma, \Delta)$ is its 1-skeleton?
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For any group $\Gamma$, and any representation $\Delta$, can we always find a convex polytope such that $G(\Gamma, \Delta)$ is its 1-skeleton?

Let’s find out...
One presentation of $Q_8$ is:

$$\Delta = \langle i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} \rangle.$$
The Quaternions: $Q_8$

One presentation of $Q_8$ is:

$$\Delta = \langle i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} \rangle.$$ 

This has the corresponding Cayley color graph:
$Q_8$ has Genus 1

We can embed this Cayley color graph on a Torus:
Question

Can $Q_8$ be embedded as the 1-skeleton of some convex polytope?
A Natural Question: Let’s be specific, here.

**Question**

*Can $Q_8$ be embedded as the 1-skeleton of some convex polytope?*

**Theorem**

*Nope!*
A Natural Question: Let’s be specific, here.

**Question**

*Can $Q_8$ be embedded as the 1-skeleton of some convex polytope?*

**Theorem**

*Nope!*

**A More Honest Theorem**

*There exists no convex polytope $P$ with $G(P)$ equal to the Cayley graph of a minimal presentation of the quaternion group.*
What’s next?

- Are there (infinite) families of groups the minimal presentations of which cannot be embedded as the graphs of convex \( d \)-polytopes?
- Can we use group theory to characterize the embeddability of Cayley graphs?
  - Characterize subgroups that “block” the embedding of the Cayley graphs
  - Show that there exist no such subgroups
- Are there forbidden minor characterizations for the embeddability of Cayley graphs?
- Can we develop constructions that give \( d \)-polytopes with graphs equal to Cayley graphs?
Thank you!