Internal and external cyclidic harmonics

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Introduction

A solution $u(x, y, z)$, $(x, y, z) \in D$, of the Laplace equation

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$$

is called harmonic in D. In 1894 Maxime Bôcher showed that the Laplace equation can be solved by the method of separation of variables in 17 coordinate systems. There are 11 quadratic systems and 6 cyclidic systems.

References

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Spherical coordinates

Coordinates: $r > 0,\ 0 < \theta < \pi,\ -\pi < \phi < \pi.$

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$

Coordinate surfaces:

spheres	$x^2+y^2+z^2=r^2,$
circular cones	$(x^2+y^2)\cot^2\theta=z^2,$
planes	$x\sin\phi=y\cos\phi.$

Separation of variables

Laplace equation $\Delta u = 0$ in spherical coordinates:

$$(r^2 u_r)_r + \frac{1}{\sin\theta}(\sin\theta\, u_\theta)_\theta + \frac{1}{\sin^2\theta} u_{\phi\phi} = 0.$$

Separation of variables $u = u_1(r)u_2(\theta)u_3(\phi)$:

$$\begin{aligned}
&r^2 u_1'' + 2ru_1' - n(n+1)u_1 = 0,\\
&u_2'' + \cot\theta\, u_2' + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right] u_2 = 0,\\
&u_3'' + m^2 u_3 = 0,
\end{aligned}$$

where m, n are separation parameters.

Internal harmonics:

$$G_n^m(x,y,z) = r^n P_n^m(\cos\theta)e^{im\phi}, \quad -n \le m \le n,$$

where P_n^m is an associated Legendre function (Ferrer's function). G_n^m is a harmonic functions in $\mathbb{R}^3$. They are polynomials in x, y, z.

External harmonics:

$$H_n^m(x,y,z) = r^{-n-1} P_n^m(\cos\theta)e^{im\phi}, \quad -n \le m \le n.$$

H_n^m is a harmonic functions in $\mathbb{R}^3 \setminus \{0\}$.

Theorem. Let $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ be such that $\|\mathbf{r}\| < \|\mathbf{r}'\|$. Then

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} G_n^m(\mathbf{r}) \overline{H_n^m(\mathbf{r}')}.$$

This formula may be used to find the Green's function for the ball.

Sphero-conal coordinates

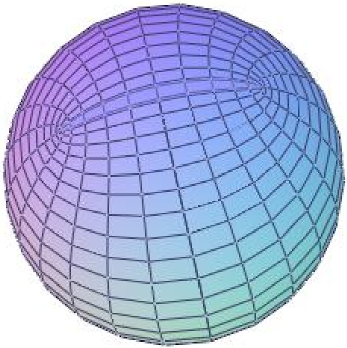
Parameter: $0 < k < 1$, $k' = \sqrt{1-k^2}$.
Coordinates: $r > 0$, $0 < s < 1 < t < k^{-2}$.

$$\begin{aligned} x &= krst, \\ y &= \frac{k}{k'} r\sqrt{1-s}\sqrt{t-1}, \\ z &= \frac{1}{k'} r\sqrt{1-k^2 s}\sqrt{1-k^2 t}. \end{aligned}$$

Coordinate surfaces:

$$\begin{array}{cl} \text{spheres} & x^2+y^2+z^2 = r^2, \\ \text{elliptical cones} & \dfrac{x^2}{s} - \dfrac{y^2}{1-s} - \dfrac{z^2}{k^{-2}-s} = 0, \\ \text{elliptical cones} & \dfrac{x^2}{t} + \dfrac{y^2}{t-1} - \dfrac{z^2}{k^{-2}-t} = 0. \end{array}$$

Sphero-conal coordinates



Separation of variables

Laplace equation in sphero-conal coordinates:

$$\tfrac{1}{4}(t-s)(r^2u_r)_r + \omega(s)(\omega(s)u_s)_s + \omega(t)(\omega(t)u_t)_t = 0,$$

where

$$\omega(s) = |s(1-s)(k^{-2}-s)|^{1/2}.$$

Separation of variables $u = u_1(r)u_2(s)u_3(t)$:

$$r^2u_1'' + 2ru_1' - n(n+1)u_1 = 0,$$

and $v = u_2, u_3$ satisfy Lamé's equation

$$v'' + \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{s-k^{-2}}\right)v' + \frac{k^{-2}h - n(n+1)s}{4s(s-1)(s-k^{-2})}v = 0.$$

where n, h are separation parameters.

Internal sphero-conal harmonics:

$$G_n^m(x,y,z) = r^n E_n^m(s) E_n^m(t).$$

where E_n^m are Lamé polynomials. G_n^m is a harmonic polynomial.

External sphero-conal harmonics:

$$H_n^m(x,y,z) = r^{-n-1} E_n^m(s) E_n^m(t).$$

H_n^m is harmonic in $\mathbb{R}^3 \setminus \{0\}$.

Expansion of reciprocal distance

Let $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ be such that $\|\mathbf{r}\| < \|\mathbf{r}'\|$. Then

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_n^m G_n^m(\mathbf{r}) H_n^m(\mathbf{r}').$$

Toroidal coordinates

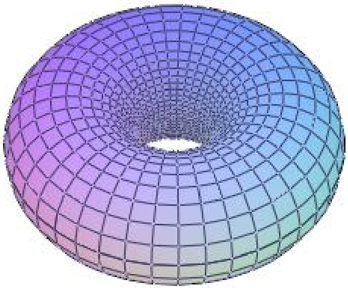
Coordinates: $0 < \sigma < \infty$, $-\pi < \psi, \phi < \pi$.

$$
\begin{aligned}
x &= \frac{\sinh\sigma\cos\phi}{\cosh\sigma - \cos\psi}, \\
y &= \frac{\sinh\sigma\sin\phi}{\cosh\sigma - \cos\psi}, \\
z &= \frac{\sin\psi}{\cosh\sigma - \cos\psi}.
\end{aligned}
$$

Coordinate surfaces:

tori	$(1 + x^2 + y^2 + z^2)^2 = 4(x^2 + y^2)\coth^2\sigma$
spherical bowls	$(z - \cot\psi)^2 + x^2 + y^2 = \dfrac{1}{\sin^2\psi}$,
planes	$x\sin\phi = y\cos\phi$.

Toroidal coordinates



Separation of variables

Laplace equation:

$$\left(\frac{\sinh\sigma\, u_\sigma}{\cosh\sigma-\cos\psi}\right)_\sigma+\left(\frac{\sinh\sigma\, u_\psi}{\cosh\sigma-\cos\psi}\right)_\psi+\frac{u_{\phi\phi}}{(\cosh\sigma-\cos\psi)\sinh\sigma}=0.$$

Separation of variables:

$$u=\sqrt{\cosh\sigma-\cos\psi}\, u_1(\sigma)u_2(\psi)u_3(\phi),$$

$$\begin{aligned}
&\frac{1}{\sinh\sigma}\left(\sinh\sigma\, u_1'\right)'-\left(n^2-\frac{1}{4}+\frac{m^2}{\sinh^2\sigma}\right)u_1=0,\\
&u_2''+n^2u_2=0,\\
&u_3''+m^2u_3=0.
\end{aligned}$$

where m, n are separation parameters.

Internal and external toroidal harmonics

Internal toroidal harmonics: $m, n \in \mathbb{Z}$

$$G_n^m(x,y,z) = \sqrt{\cosh\sigma - \cos\psi}\, Q_{n-\frac{1}{2}}^m(\cosh\sigma) e^{in\psi} e^{im\phi},$$

where Q is the associated Legendre function. These are harmonic functions in $\mathbb{R}^3$ except for the z-axis.

External sphero-conal harmonics:

$$H_n^m(x,y,z) = \sqrt{\cosh\sigma - \cos\psi}\, P_{n-\frac{1}{2}}^m(\cosh\sigma) e^{in\psi} e^{im\phi}.$$

These are harmonic function in $\mathbb{R}^3$ except for the unit circle $z = 0$, $x^2 + y^2 = 1$.

Expansion of reciprocal distance

Let $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ be such that $\sigma' < \sigma$. Then

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\pi} (-1)^m \frac{\Gamma(n - m + \frac{1}{2})}{\Gamma(n + m + \frac{1}{2})} G_n^m(\mathbf{r}) \overline{H_n^m(\mathbf{r}')}.$$

Stereographic projection

Let x_0, x_1, x_2, x_3 be cartesian coordinates in $\mathbb{R}^4$. We consider the stereographic projection $P : \mathbb{S}^3 \setminus \{1, 0, 0, 0)\} \to \mathbb{R}^3$ given by

$$P(x_0, x_1, x_2, x_3) = \frac{1}{1 - x_0}(x_1, x_2, x_3).$$

The inverse map is

$$P^{-1}(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + 1}(x^2 + y^2 + z^2 - 1, 2x, 2y, 2z).$$

Example: The intersection of the quadric hypersurface

$$x_1^2 + x_2^2 = \tanh^2 \sigma$$

with $\mathbb{S}^3$ is mapped to the torus

$$(1 + x^2 + y^2 + z^2)^2 = 4(x^2 + y^2) \coth^2 \sigma.$$

The stereographic projection maps a quadric surface to a cyclidic surface.

Theorem. Let D be an open subset of $\mathbb{S}^3$ not containing $(1,0,0,0)$, let $E=\{(rx_0,rx_1,rx_2,rx_3):r>0,(x_0,x_1,x_2,x_3)\in D\}$, and let $F=P(D)$ be the stereographic image of D. Let the function $U:E\to\mathbb{R}$ be homogeneous of degree $-\frac{1}{2}$ or $-\frac{3}{2}$, and let $w:F\to\mathbb{R}$ satisfy $U=w\circ P$ on D. Then U is harmonic on E if and only if $w(x,y,z)(x^2+y^2+z^2+1)^{-1/2}$ is harmonic on F.

Sphero-conal coordinates on $\mathbb{R}^4$

Coordinates: $r > 0$, $a_0 < s_1 < a_1 < s_2 < a_2 < s_3 < a_3$,

$$x_j^2 = r^2 \frac{\prod_{i=1}^{4}(s_i - a_j)}{\prod_{j \neq i=0}^{4}(a_i - a_j)}.$$

Coordinate surfaces:

$$r^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

and

$$\sum_{j=0}^{4} \frac{x_j^2}{s_i - a_j} = 0 \quad \text{for } i = 1, 2, 3.$$

Sphero-conal coordinates are orthogonal.

Assume a harmonic function of the form

$$U(x_0, x_1, x_2, x_3) = w_0(r)w_1(s_1)w_2(s_2)w_3(s_3).$$

Then

$$w_0'' + \frac{4}{r}w_0' + \frac{4\lambda_0}{r^2}w_0 = 0$$

and, for $w = w_1, w_2, w_3$,

$$\prod_{j=0}^{3}(s - a_j)\left[w'' + \frac{1}{2}\sum_{j=0}^{3}\frac{1}{s - a_j}w'\right] + \left[\sum_{i=0}^{2}\lambda_i s^{2-i}\right]w = 0.$$

This is a Fuchsian equation with five regular singular points $a_0, a_1, a_2, a_3, \infty$ with exponents 0 and $1/2$ at a_0, a_1, a_2, a_3. The separation parameters are $\lambda_0, \lambda_1, \lambda_2$.

Five-cyclide coordinate system on $\mathbb{R}^3$

Sphero-conal coordinates s_1, s_2, s_3 form a coordinate system for the intersection of the hypersphere $\mathbb{S}^3$ with the positive cone in $\mathbb{R}^4$. Using the stereographic projection P we project these coordinates to $\mathbb{R}^3$. We obtain an orthogonal coordinate system for the set

$$T = \{(x, y, z) : x, y, z > 0,\ x^2 + y^2 + z^2 > 1\}.$$

Explicitly,

$$x = \frac{x_1}{1 - x_0}, \quad y = \frac{x_2}{1 - x_0}, \quad z = \frac{x_3}{1 - x_0},$$

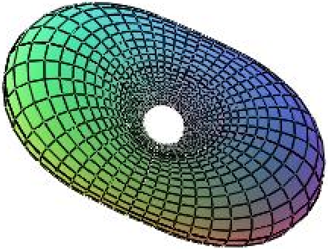
where

$$x_j^2 = \frac{\prod_{i=1}^3 (s_i - a_j)}{\prod_{j \neq i=0}^3 (a_i - a_j)}, \quad j = 0, 1, 2, 3.$$

Coordinate surfaces:

$$\frac{(x^2 + y^2 + z^2 - 1)^2}{s - a_0} + \frac{4x^2}{s - a_1} + \frac{4y^2}{s - a_2} + \frac{4z^2}{s - a_3} = 0.$$

Coordinate surface $s_2 = const$



Theorem. Let $w_1 : (a_0, a_1) \to \mathbb{C}$, $w_2 : (a_1, a_2) \to \mathbb{C}$, $w_3 : (a_2, a_3) \to \mathbb{C}$ be solutions of the Fuchsian equation

$$\prod_{j=0}^{3}(s-a_j)\left[w'' + \frac{1}{2}\sum_{j=0}^{3}\frac{1}{s-a_j}w'\right] + \left(\frac{3}{16}s^2 + \lambda_1 s + \lambda_2\right)w = 0,$$

where λ_1, λ_2 are given (separation) constants. Then the function

$$u(x,y,z) = (x^2+y^2+z^2+1)^{-1/2}w_1(s_1)w_2(s_2)w_3(s_3)$$

is a harmonic function on the set T.

A two-parameter eigenvalue problem

In order to introduce internal cyclidic ring harmonics we have to consider the following two-parameter eigenvalue problem.

$$\prod_{j=0}^{3}(s_1 - a_j)\left[w_1'' + \frac{1}{2}\sum_{j=0}^{3}\frac{1}{s_1 - a_j}w_1'\right] + \left(\frac{3}{16}s_1^2 + \lambda_1 s_1 + \lambda_2\right)w_1 = 0,$$

$$\prod_{j=0}^{3}(s_3 - a_j)\left[w_3'' + \frac{1}{2}\sum_{j=0}^{3}\frac{1}{s_3 - a_j}w_3'\right] + \left(\frac{3}{16}s_3^2 + \lambda_1 s_3 + \lambda_2\right)w_3 = 0.$$

We add boundary conditions: w_1 is analytic at a_0, a_1, and w_3 is analytic at a_2, a_3.

Klein's oscillation theorem and completeness theorem

Theorem. For every $\mathbf{n} = (n_1, n_3) \in \mathbb{N}_0^2$, there exists a uniquely determined eigenvalue $(\lambda_{1,\mathbf{n}}, \lambda_{2,\mathbf{n}}) \in \mathbb{R}^2$ admitting an eigenfunction $w_1 = E_{1,\mathbf{n}}(s_1)$ with exactly n_1 zeros in (a_0, a_1) and an eigenfunction $w_3 = E_{3,\mathbf{n}}(s_3)$ with exactly n_3 zeros in (a_2, a_3).

Theorem. The double sequence of functions

$$E_{1,\mathbf{n}}(s_1)E_{3,\mathbf{n}}(s_3), \quad \mathbf{n} \in \mathbb{N}_0^2,$$

when properly normalized forms an orthonormal basis in an appropriate Hilbert space.

Internal cyclidic ring harmonics

Let $E_{2,\mathbf{n}}(s_2)$, $a_1 < s_2 < a_2$, be a solution of the Fuchsian equation which is analytic at a_1.
For $\mathbf{n} = (n_1, n_3) \in \mathbb{N}_0^2$ we define the **internal 5-cyclidic harmonic**

$$G_{\mathbf{n}}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E_{1,\mathbf{n}}(s_1) E_{2,\mathbf{n}}(s_2) E_{3,\mathbf{n}}(s_3).$$

The function $(x^2 + y^2 + z^2 + 1)^{1/2} G_n(x, y, z)$ is invariant under inversion

$$\sigma_0(x, y, z) = (x^2 + y^2 + z^2)^{-1}(x, y, z)$$

and under reflections at the coordinate planes

$$\sigma_1(x, y, z) = (-x, y, z),\ \sigma_2(x, y, z) = (x, -y, z),\ \sigma_3(x, y, z) = (x, y, -z).$$

The region interior to the cyclidic ring surface $s_2 = d_2 \in (a_1, a_2)$ is

$$D = \{(x, y, z) \in \mathbb{R}^3 : s_2 < d_2\}.$$

The function $G_{\mathbf{n}}$ is harmonic in D.

Theorem. Let e be a function defined on the boundary ∂D. Suppose that the function $(x^2+y^2+z^2+1)^{1/2}e(x,y,z)$ is invariant under σ_i, $i=0,1,2,3$. Then

$$u(x,y,z)=\sum_{\mathbf{n}} c_{\mathbf{n}} G_{\mathbf{n}}(x,y,z),$$

is harmonic in D and assumes the values e on the boundary, where

$$c_{\mathbf{n}}=\frac{1}{4\pi\omega(d_2)\{E_{2,\mathbf{n}}(d_2)\}^2}\int_{\partial D}\frac{e}{h_2}G_{\mathbf{n}}(\mathbf{r})\,dS(\mathbf{r}),$$

$$\omega(s)=|(s-a_0)(s-a_1)(s-a_2)(s-a_3)|^{1/2},$$

$$16\{h_2(\mathbf{r})\}^2=\frac{(\|\mathbf{r}\|^2-1)^2}{(d_2-a_0)^2}+\frac{4x^2}{(d_2-a_1)^2}+\frac{4y^2}{(d_2-a_2)^2}+\frac{4z^2}{(d_2-a_3)^2}.$$

The general case

We call $f(x, y, z)$ of parity $\mathbf{p} = (p_0, p_1, p_2, p_3) \in \{0, 1\}^4$ if

$$f(\sigma_i(x, y, z)) = (-1)^{p_i} f(x, y, z) \quad \text{for } i = 0, 1, 2, 3.$$

If f is any function, we write f as a sum of sixteen functions

$$f = \sum_{\mathbf{p} \in \{0,1\}^4} f_{\mathbf{p}},$$

where $f_{\mathbf{p}}$ is of parity $\mathbf{p}$. Then the solution of the corresponding Dirichlet problem is given by

$$u(x, y, z) = \sum_{\mathbf{n},\mathbf{p}} c_{\mathbf{n},\mathbf{p}} G_{\mathbf{n},\mathbf{p}}(x, y, z).$$

External cyclidic ring harmonics

Let $F_{2,\mathbf{n}}(s_2)$ be the solution to the Fuchsian equation (with $\lambda_j = \lambda_{j,\mathbf{n}}$) on (a_1, a_2) which is analytic at $s_2 = a_2$. We define **external 5-cyclidic harmonics** by

$$H_{\mathbf{n}}(x,y,z) = (x^2+y^2+z^2+1)^{-1/2} E_{1,\mathbf{n}}(s_1) F_{2,\mathbf{n}}(s_2) E_{3,\mathbf{n}}(s_3).$$

Then $H_{\mathbf{n}}$ is harmonic outside each cyclidic ring. The function $(x^2+y^2+z^2+1)^{1/2} H_n(x,y,z)$ is invariant under σ_i, $i = 0,1,2,3$. Moreover,

$$H_{\mathbf{n}}(\mathbf{r}) = O(\|\mathbf{r}\|^{-1}) \quad \text{as } \|\mathbf{r}\| \to \infty.$$

Integral representation

Theorem. Let $d_2 \in (a_1, a_2)$, $\mathbf{n} \in \mathbb{N}_0^2$, $\mathbf{p} \in \{0,1\}^4$. Then

$$H_{\mathbf{n},\mathbf{p}}(\mathbf{r}') = \frac{1}{4\pi\omega(d_2)\{E_{2,\mathbf{n},\mathbf{p}}(d_2)\}^2} \int_{\partial D} \frac{G_{\mathbf{n},\mathbf{p}}(\mathbf{r})}{h_2(\mathbf{r})\|\mathbf{r}-\mathbf{r}'\|}\, dS(\mathbf{r})$$

for all $\mathbf{r}' \in \mathbb{R}^3 \setminus \bar{D}$.

Proof. Let D be an open bounded subset of $\mathbb{R}^3$ with smooth boundary. For $u, v \in C^2(\bar{D})$, Green's formula states that

$$\int_D (u\Delta v - v\Delta u)\, d\mathbf{r} = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS,$$

where $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u on the boundary ∂D of D.

Proof

We apply Green's formula to $u = G = G_{\mathbf{n},\mathbf{p}}$, $v(\mathbf{r}) = \frac{1}{4\pi\|\mathbf{r}-\mathbf{r}'\|}$. Since u, v are harmonic on an open set containing $\bar{D}$ we obtain

$$0 = \int_{\partial D} \left(G \frac{\partial v}{\partial \nu} - v \frac{\partial G}{\partial \nu} \right) dS. \qquad (1)$$

We now use Green's formula a second time. We choose $R > 0$ so large that the ball $B_R(\mathbf{0})$ contains $\mathbf{r}'$ and $\bar{D}$. Then we take $D = B_R(\mathbf{0}) - \bar{D} - B_\epsilon(\mathbf{r}')$ with small radius $\epsilon > 0$. Take $u = H = H_{\mathbf{n},\mathbf{p}}$ and v as before. Note that u, v are harmonic on an open set containing $\bar{D}$. Taking the limit $\epsilon \to 0$, we obtain

$$H(\mathbf{r}') = \int_{\partial B_R(\mathbf{0})} \left(H \frac{\partial v}{\partial \nu} - v \frac{\partial H}{\partial \nu} \right) dS - \int_{\partial D} \left(H \frac{\partial v}{\partial \nu} - v \frac{\partial H}{\partial \nu} \right) dS,$$

where, in the second integral, $\frac{\partial}{\partial \nu}$ denotes the same derivative as before. The first integral tends to 0 as $R \to \infty$. Therefore,

$$H(\mathbf{r}') = -\int_{\partial D} \left(H \frac{\partial v}{\partial \nu} - v \frac{\partial H}{\partial \nu} \right) dS. \qquad (2)$$

Proof

We now multiply (1) by $F_2(d_2)$, $F_2 = F_{2,\mathbf{n},\mathbf{p}}$, then multiply (2) by $E_2(d_2)$, $E_i = E_{i,\mathbf{n},\mathbf{p}}$, and add these equations. Note that

$$F_2(d_2)G(\mathbf{r}) = E_2(d_2)H(\mathbf{r}), \quad \mathbf{r} \in \partial D.$$

Therefore, we find

$$E_2(d_2)H(\mathbf{r}') = \int_{\partial D} v \left(E_2(d_2)\frac{\partial H}{\partial \nu} - F_2(d_2)\frac{\partial G}{\partial \nu} \right) dS. \qquad (3)$$

The normal derivative and the derivative with respect to s_2 are related by

$$\frac{\partial}{\partial \nu} = \frac{1}{h_2}\frac{\partial}{\partial s_2},$$

where h_2 is the scale factor of the 5-cyclidic coordinate s_2.

Proof

Let $\mathbf{r} \in \partial D \cap R$ with 5-cyclidic coordinates $s_1, s_2 = d_2, s_3$. Then

$$\begin{aligned}
&\left(E_2(d_2)\frac{\partial H}{\partial \nu} - F_2(d_2)\frac{\partial G}{\partial \nu} \right)(\mathbf{r}) \\
&\qquad = E_2(d_2)\frac{\partial(\|\mathbf{r}\|^2+1)^{-1/2}}{\partial \nu} E_1(s_1)F_2(d_2)E_3(s_3) \\
&\qquad\qquad + E_2(d_2)(\|\mathbf{r}\|^2+1)^{-1/2} h_2^{-1} E_1(s_1)F_2'(d_2)E_3(s_3) \\
&\qquad\qquad - F_2(d_2)\frac{\partial(\|\mathbf{r}\|^2+1)^{-1/2}}{\partial \nu} E_1(s_1)E_2(d_2)E_3(s_3) \\
&\qquad\qquad - F_2(d_2)(\|\mathbf{r}\|^2+1)^{-1/2} h_2^{-1} E_1(s_1)E_2'(d_2)E_3(s_3) \\
&\qquad = h_2^{-1}(\|\mathbf{r}\|^2+1)^{-1/2} E_1(s_1)\left\{ E_2(d_2)F_2'(d_2) - E_2'(d_2)F_2(d_2) \right\} E
\end{aligned}$$

When we properly normalize our functions we obtain

$$\left(E_2(d_2)\frac{\partial H}{\partial \nu} - F_2(d_2)\frac{\partial G}{\partial \nu} \right)(\mathbf{r}) = \frac{G(\mathbf{r})}{h_2(\mathbf{r})\omega(d_2)E_2(d_2)}, \qquad (4)$$

which holds for all $\mathbf{r} \in \partial D$. When we substitute (4) in (3) we arrive at the desired integral representation.

Expansion of reciprocal distance

Theorem. Let $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ with 5-cyclidic coordinates s_2, s_2', respectively. If $s_2 < s_2'$ then

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = \pi \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^4} G_{\mathbf{n},\mathbf{p}}(\mathbf{r}) H_{\mathbf{n},\mathbf{p}}(\mathbf{r}').$$

Proof. We pick d_2 such that $s_2 < d_2 < s_2'$, and consider the corresponding cyclidic ring domain D. The function $f(\mathbf{q}) = \|\mathbf{q}-\mathbf{r}'\|^{-1}$ is harmonic on an open set containing $\bar{D}$. Therefore, by the solution formula for the Dirichlet problem we obtained earlier, we have

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = \sum_{\mathbf{n}\in\mathbb{N}_0^2} \sum_{\mathbf{p}\in\{0,1\}^4} c_{\mathbf{n},\mathbf{p}} G_{\mathbf{n},\mathbf{p}}(\mathbf{r}),$$

$$c_{\mathbf{n},\mathbf{p}} = \frac{1}{4\omega(d_2)\{E_{2,\mathbf{n},\mathbf{p}}(d_2)\}^2} \int_{\partial D} \frac{G_{\mathbf{n},\mathbf{p}}(\mathbf{q})}{h_2(\mathbf{q})\|\mathbf{q}-\mathbf{r}'\|}\, dS(\mathbf{q}).$$

Using the expansion theorem, we obtain the desired formula.