Phase Transition in Highly Edge-Connected Graphs

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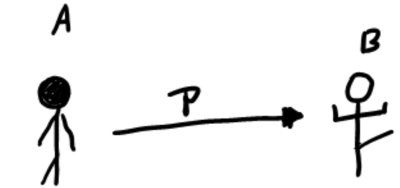
May 20, 2013

infected 1-to-1

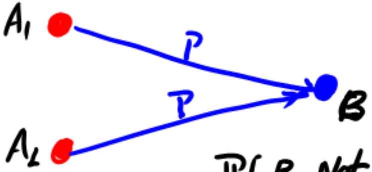


infected — un-infected

A infects B with prob. p

A does not infect B with prob $q = 1 - p$

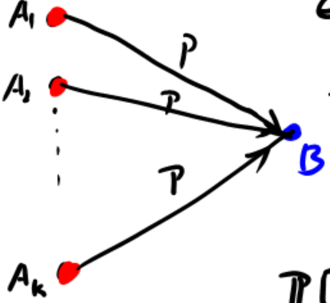
infected 2-to-1



$$\mathbb{P}[B \text{ Not infected}] = q^2 = (1-p)^2$$

$$\mathbb{P}[B \text{ infected}] = 1-(1-p)^2$$

infected k-to-1

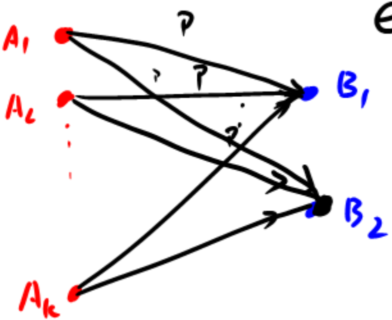


each A_i infects B independently

$\mathbb{P}[B$ Not infected$]$
$= (1-p)^k$

$\mathbb{P}[B$ infected$] = 1-(1-p)^k$

infected k-to-2



each B_i to be infected independently.

$\mathbb{P}[B_i \text{ Not infected}] = (1-p)^k$

Definition (Reed-Frost (RF) Process)

Given a population V, the RF process is a Markov chain with states (I_t, S_t):

- *Start with an initial infected set I_0 and the corresponding susceptible set $S_0 = V - I_0$;*

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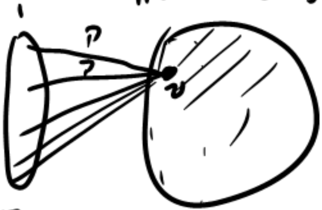
- *the set I_t of infected individuals are assumed to be removed by time step $t+1$;*
- *The process stops at the first time $I_t = \emptyset$.*

RF-Diagram

$$\mathbb{P}[v \in I_{t+1}] = 1-(1-p)^{|I_t|}$$

I_0 I_1

Initial infected population



I_t S_t

$v \in S_t$ to be infected will be only dept on. I_t

a few remarks

The formula $\mathbb{P}[v \in I_{t+1}] = 1 - (1-p)^{|I_t|}$ gives the following properties.

1. Each infected individual *infects* each susceptible individual independently with probability p at each time step.

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The formula $\mathbb{P}[v \in I_{t+1}] = 1 - (1-p)^{|I_t|}$ gives the following properties.

1. Each infected individual *infects* each susceptible individual independently with probability p at each time step.
2. Every individuals knows all other individuals, that is, the ground graph is a complete graph K_n.

Definition (r-bootstrap percolation)

Given a graph $G = (V, E)$, consider a deterministic process (I_t, S_t):

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$$I_{t+1} = \{v \in S_t \ : \ d_{I_t}(v) \geq r\} \text{ and } S_{t+1} = S_t - I_{t+1}.$$

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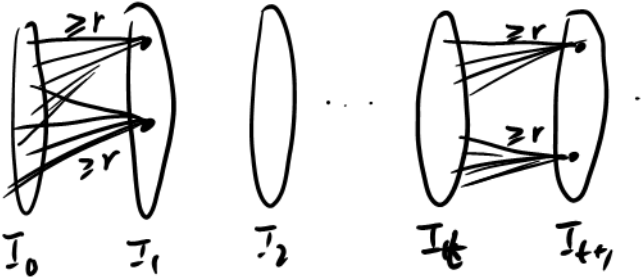
Diagram of Bootstrap Percolation



Theorem (Barbour and Mollison)

The RF-process with probability p on K_n can be divided into two subprocesses:

- **BM-1:** *Associate each vertex x with a random set $L(x)$ such that $\mathbb{P}[\, y \in L(x)\,] = p$ for each $y \in N_G(x)$;*
- **BM-2:** *Starting with the initial infected set I_0, we construct a sequence $I_0, I_1, \ldots$ such that $I_{t+1} = \bigcup_{x \in I_t} L(x) - \bigcup_{i=0}^{t} I_i$, and stop when $I_i = \emptyset$.*

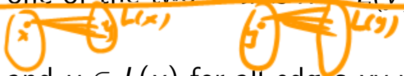
BM-labeling

$\Longrightarrow$ potential problem

$y \in L(x)$ but $x \notin L(y)$

Random Graphs and Bootstrap Percolation

- In BM-1, we may identify two events $x \in L(y)$ and $y \in L(x)$ by setting $\mathbb{P}[x \in L(y) \text{ and } y \in L(x)] = p$ for any edge $xy \in E(G)$, as long as it is independent of all other assignments for edges. The reason is that for any RF process, at most one of the two events $x \in L(y)$ and $y \in L(x)$ holds.



- Identifying $x \in L(y)$ and $y \in L(x)$ for all edges xy with probability p is equivalent to keeping each edge xy in G with probability p, which is exactly the definition of $G(n, p)$.

Theorem (Barbour and Mollison)

The RF-process with probability p on K_n can be divided into two subprocesses:

- **BM-1:** *Generalize a $G(n, p)$.*
- **BM-2:** *Process 1-bootstrap percolation on the generalized (G, n, p) with the initial infected set I_0.*

Definition (Generalized Reed-Frost (GRF) Process)

Given a graph $G = (V, E)$*, consider a Markov Chain process* (I_t, S_t)*:*

- *Starts with the initial set* $I_0 \neq \emptyset$*, and set* $S_0 = V - I_0$*;*

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- *Starts with the initial set $I_0 \neq \emptyset$, and set $S_0 = V - I_0$;*
- *Given (I_t, S_t) is given, for each $v \in S_t$*

$$\mathbb{P}[v \in I_{t+1}] = 1 - (1 - p)^{d_{I_t(v)}}, \text{ and } S_{t+1} = S_t - I_t;$$

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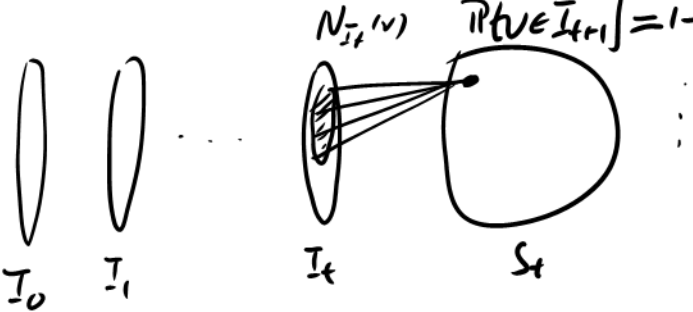
GRF

$\mathbb{P}[v \in I_{t+1}] = 1 - (1-p)^{d_{I_t}(v)}$



percolation on graphs

Definition (Bollobás ...)

Given a graph G and a probability p, a random subgraph H of G is constructed as follows: each edge of G is kept with probability p, or equivalently, each edge of G is deleted with probability $1 - p$, independent from every other edge.

Clearly, a $G(n, p)$ is a $K_n(p)$

Theorem (Barbour and Mollison)

The RF-process with probability p on K_n can be divided into two subprocesses:

- *Generalize a $G(n, p)$.*
- *Process 1-bootstrap percolation on the generalized (G, n, p) with the initial infected set I_0.*

Theorem (Bollobás, Borgs, Chayes, Riordan)

Let (G_n) be a sequence of dense graphs with $|G_n| = n$, let λ_n be the largest eigenvalue of the adjacency matrix of G_n, and let $p_n = \min\{c/\lambda_n, 1\}$.

- *If $c \leq 1$, then all components of $G_n(p_n)$ are of size $o_p(n)$.*
- *If $c > 1$, then the largest component of $G_n(p_n)$ has size $\Theta(n)$ w.h.p..*

$$\underbrace{I_0 \quad I_1 \cdots \quad I_t}_{A_t}$$

Given a GRF process $I_0, I_2, \ldots,$. Let $A_t = \cup_{i=0}^{t} I_t$.

Then, as time t progresses, we obtain an increasing sequence of removed vertex sets $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_t \subseteq \cdots$.

The percolation time $t(G, p, I_0)$ is the least T with $A_T = V(G)$ if such a T exists or ∞ otherwise.

The process time $T(G, p, I_0)$ is the least T with $I_{T+1} = \emptyset$.

Note that *percolation time* $\neq$ *process time*.

percolation time $t(G, p, I_0)$

- structure of G;
- transition probability p;
- initial infected set I_0, in particular, $|I_0| = 1$;
- $t(G, p, I_0)$ is closely related to the diameter of G_p.

We call $(\mathcal{G}, \mathfrak{p}, 1)$ percolation finite if there exists a constant T such that

$$\lim_{n\to\infty} \mathbb{P}[t(G, p_n, I_0) \leq T \mid G \in \mathcal{G}|G| = n \text{ and } |I_0| = 1] \; = \; 1,$$

that is, w.h.p., all graph $G \in \mathcal{G}$ can be percolated within at most T steps.

Theorem (Bollobás, Klee and Lamar)

w.h.p. $diam(G(n,p)) = d$ *if as* $n \to \infty$,

$$2^{d-2}(pn)^{d-1}/n - \log n \to -\infty$$

$$p < \frac{(2^{d-2}\log n)^{\frac{1}{d-1}}}{n^{\frac{d-2}{d-1}}}$$

and

$$2^{d-1}(pn)^{d}/n - \log n \to \infty.$$

$$p \overset{?}{>} \frac{(2^{d-1}\log n)^{\frac{1}{d}}}{n^{\frac{d-1}{d}}}$$

Three classes of dense graphs.

$$\begin{aligned}\mathcal{G}_{\delta,\alpha} &= \{G \mid \delta(G) \ge \alpha|G|\}, \\ \mathcal{G}_{\kappa',\alpha} &= \{G \mid \kappa'(G) \ge \alpha|G|\}, \text{ and} \\ \mathcal{G}_{\kappa,\alpha} &= \{G \mid \kappa(G) \ge \alpha|G|\}.\end{aligned}$$

Clearly, $\mathcal{G}_{\delta,\alpha} \supset \mathcal{G}_{\kappa',\alpha} \supset \mathcal{G}_{\kappa,\alpha}$.

Difference among connectivities

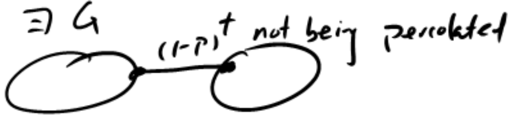
δ

$\kappa'(G) = \kappa(G) \le 1$

$\kappa'(G)$ big

$\kappa(G)$ small

Let G be a graph of order n. If $\delta(G) \geq (n-1)/2$, then $\kappa'(G) = \delta(G)$. So $\mathcal{G}_{\delta,\alpha} = \mathcal{G}_{\kappa',\alpha}$ for $\alpha \geq 1/2$.



Observation: If $\alpha \leq 1/2$ and $\lim_{n\to\infty} \mathfrak{p}_n \neq 1$, then $(\mathcal{G}_{\delta,\alpha}, \mathfrak{p}, 1)$ is not *percolation finite*.

If G has a *bridge uv* and u gets infected at a specific time t_0, then at any time t the probability for v being uninfected is $(1-\mathfrak{p}_n)^{t-t_0}$, which does not converge to 0 unless $\lim_{n\to\infty} \mathfrak{p}_n = 1$. Therefore, if $\lim_{n\to\infty} \mathfrak{p}_n \neq 1$ and $\mathcal{G}$ is a class of graphs such that for any $n \in \mathbb{N}$ there exists a $G \in \mathcal{G}$ of order $|G| \geq n$ with $\kappa'(G) = 1$, then $(\mathcal{G}, \mathfrak{p}, 1)$ is *not percolation finite*.

Theorem

Let $\alpha \in (0,1)$ *and* $\gamma \in (0,1)$ *be two real numbers. Then,* $(\mathcal{G}_{\kappa',\alpha}, \frac{1}{n^\gamma}, 1)$ *is percolation finite.*

$\kappa'(G) \geq \alpha n.$

$p \geq \frac{1}{n^\gamma}$

Parameters

$$f(x) = \frac{1}{2}\left(x - \frac{1}{\lfloor 1/x \rfloor + 1}\right) \quad \text{and} \quad g(x) = \frac{x + \frac{1}{\lfloor 1/x \rfloor + 1}}{2 - \left(x + \frac{1}{\lfloor 1/x \rfloor + 1}\right)}.$$

It is not difficult to see that, for any $x \in (0, 1/2]$, we have $f(x), g(x) \in (0, 1)$, and the following inequality.

$$\lfloor 1/g(x) \rfloor < \lfloor 1/x \rfloor \qquad \text{for all } x \in (0, \tfrac{1}{2}]. \tag{1}$$

for each $\alpha \in (0, \frac{1}{2}]$ we define an increasing sequence $\Theta(\alpha)$: $\alpha_0 := \alpha$, $\alpha_1 = g(\alpha_0) \ldots \alpha_{i+1} = g(\alpha_i) \ldots$. Let $\ell(\alpha)$ be the least index such that $\alpha_{\ell(\alpha)} > 1/2$. By (1), we have $\ell(x) \leq \lfloor 1/\alpha \rfloor - 1$. Let $\beta_i = f(\alpha_i)$ for each $i = 0, 1, \ldots, \ell(\alpha)$ and $\beta = \min\{\beta_0, \beta_1, \ldots, \beta_{\ell(\alpha)}\}$. For any $\alpha \in (0, 1/2]$, $\varepsilon \in (0, 1]$, a positive number c and a positive integer n, let

$$\begin{aligned}
N(\alpha, \varepsilon) &= \left(\frac{1}{\beta}\right)^{\frac{6}{\varepsilon}} ((2k)!)^{\frac{6}{\varepsilon}} \left(\frac{25}{\alpha^6}\left(\frac{2}{\varepsilon} + 3\right)\right)^{\frac{10}{\varepsilon^2}}, \\
\tau(\alpha, \varepsilon) &= 5(2k)!(2/\varepsilon + 3), \\
\eta(\alpha, \varepsilon, c, n) &= 6n^2 (2/\alpha)^{k-1} e^{-\frac{cn^{\varepsilon/2}}{k!}}, \quad \text{and} \\
\xi(\alpha, \varepsilon, c, n) &= e^{-\frac{cn^{\varepsilon/2}}{k!}},
\end{aligned}$$

where $k = \lfloor \frac{1}{\alpha} \rfloor + 1$, i.e. $\alpha \in (\frac{1}{k}, \frac{1}{k-1}]$.

Theorem

Let $\alpha, \epsilon \in (0,1)$ and $n \geq N(\alpha, \varepsilon)$ be an integer, and let $c > 0$ be a constant satisfying $p = c/n^{1-\varepsilon} < 1$. Then for any graph G of order n with $\delta(G) \geq \alpha n$ and $\kappa'(G) \geq n^{1-\varepsilon/2}$,

$$\mathbb{P}[\, t(G, p, 1) > \tau(\alpha, \epsilon) \,] < \eta(\alpha, \varepsilon, c, n).$$

induction

Our proof involves induction on $\lfloor 1/\alpha \rfloor$:

- We first show that it is true for $\lfloor 1\alpha \rfloor = 2$.
- Assume that the result is true for $\lfloor \alpha \rfloor = k$. We show that it is true for $\lfloor 1/\alpha \rfloor = k + 1$.

i.e.. we divide $(0, 1] = (1/2, 1] \cup (1/3, 1/2] \cup (1/4, 1/3] \cdots$.

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- The result holds for graphs G with $\delta(G) \geq n/2$, i.e. $\alpha \geq 1/2$.
- w.h.p. $I_{t+1} \neq$ if $S_t \neq \emptyset$.
- There is a constant $\tau := \tau(\alpha, \epsilon)$ and $\xi := \xi(\alpha, \epsilon)$ such that w.h.p. either $S_{t+\tau} = \emptyset$ or $|A_{t+\tau} - A_t| \geq \xi n$.

Suppose $\delta(G) \geq \alpha n$ and $\alpha < 1/2$.

If the statement 3 in the previous slide is not true, than G can decomposed into vertex disjoint subgraphs G_1, G_2, $\ldots$, G_m, and H such that

- $\delta(G_i) \geq \beta|G_i|$ with $\lfloor 1/\beta \rfloor < \lfloor 1/\alpha \rfloor$, so we can use induction on $\lfloor 1/\alpha \rfloor$.
- $|H|$ is small.

Thank you