Card Shuffling

Applications

Previous Work

New Results

Mixing Times of Self-Organizing Lists and Biased Permutations

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Joint work with:

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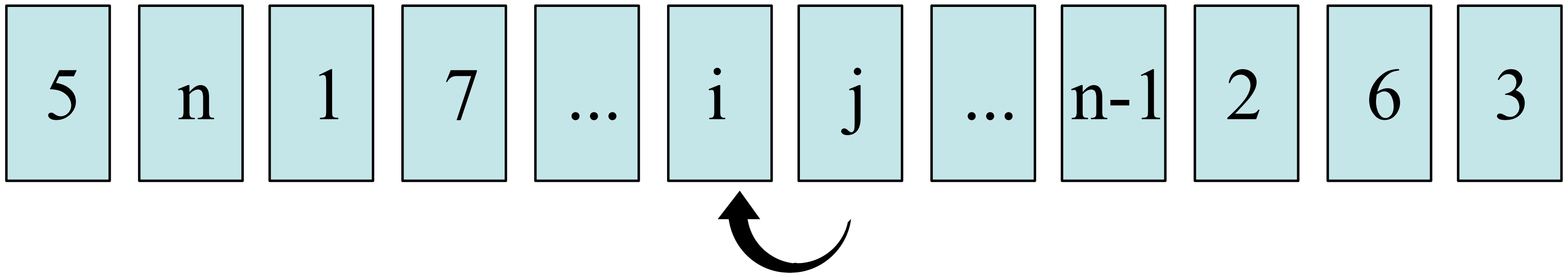
Outline

- Introduction and Background on Markov chains
    - Example: Card Shuffling
- Biased Permutations
    - Application: Self-Organizing Lists
    - Previous work
    - New results

Card Shuffling

- pick a pair of adjacent cards uniformly at random
- put j ahead of i with probability 1/2



Markov Chains

Card Shuffling

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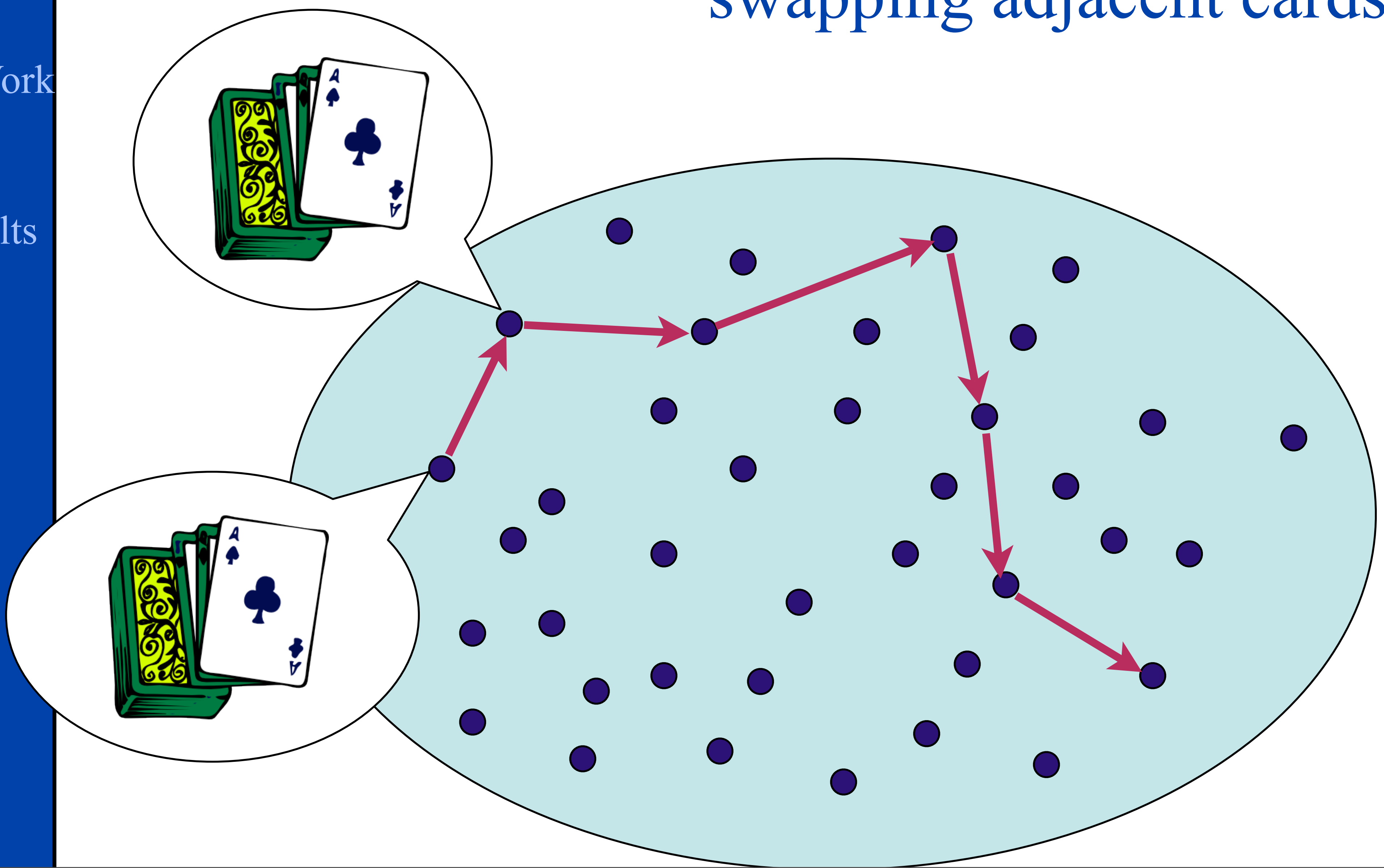
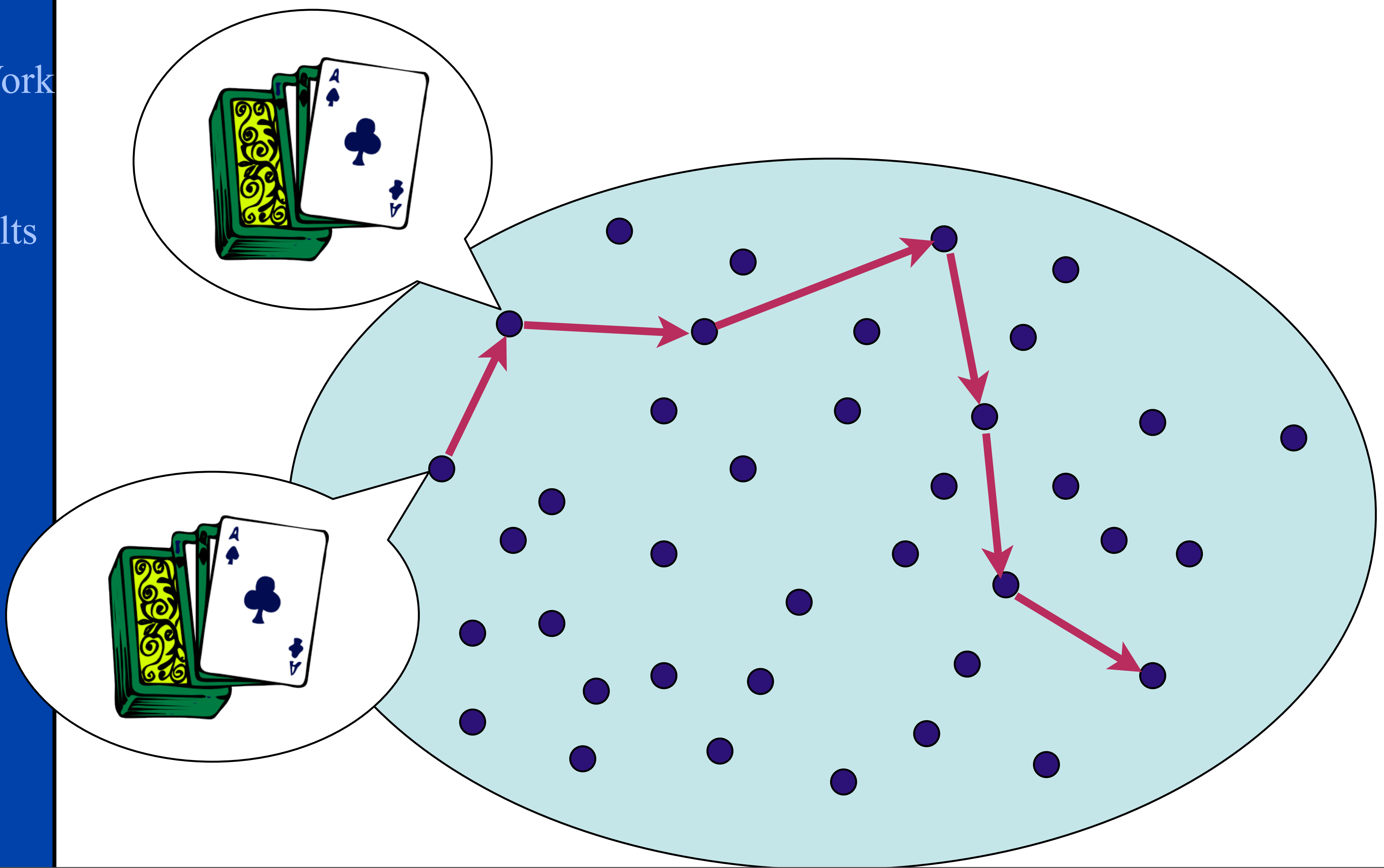
1. It is a Markov chain!

i.e. a random walk on the graph G=(Ω , E):
where (x,y) ∈ E iff can get from x to y by
swapping adjacent cards



Markov Chains

1. It is a Markov chain!
2. It is *ergodic*: i.e. *aperiodic* (not bipartite) and *irreducible* (connected)



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Using Markov Chains

Thm: Any finite, ergodic Markov chain converges to a unique stationary distribution π

(i.e. for all $x,y \in \Omega$, $\lim_t P^t(x,y) = \pi(y)$)

(or equivalently, for all $x \in \Omega$, $\lim_t P^t(x,*) = \pi$)

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If a distribution π satisfies *detailed balance:*

$$\pi(x)\, P(x,y) = \pi(y)\, P(y,x)$$

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$$\frac{\pi(x)}{\pi(y)} = \frac{P(y,x)}{P(x,y)}$$

Card Shuffling

- pick a pair of adjacent cards uniformly at random
- put j ahead of i with probability 1/2

5 n 1 7 ... i j ... n-1 2 6 3

$$\frac{\pi(x)}{\pi(y)} = \frac{P(y,x)}{P(x,y)} = 1$$

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5	n	1	7	...	i	j	...	n-1	2	6	3

$$\frac{\pi(x)}{\pi(y)} = \frac{P(y,x)}{P(x,y)} = 1$$

Converges to the uniform distribution

Card Shuffling

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5	n	1	7	...	i	j	...	n-1	2	6	3

How long until mixed?

Using Markov Chains

How long before

for all $x \in \Omega$, $P^t(x,*)$ and π are *close*?

Measure closeness using *Total Variation Distance*:

$$| P^t(x,*), \pi |_{TV} = \tfrac{1}{2} \, || P^t(x,*), \pi ||_1$$
$$= \tfrac{1}{2} \sum_{y \in \Omega} | P^t(x,y) - \pi(y)|$$

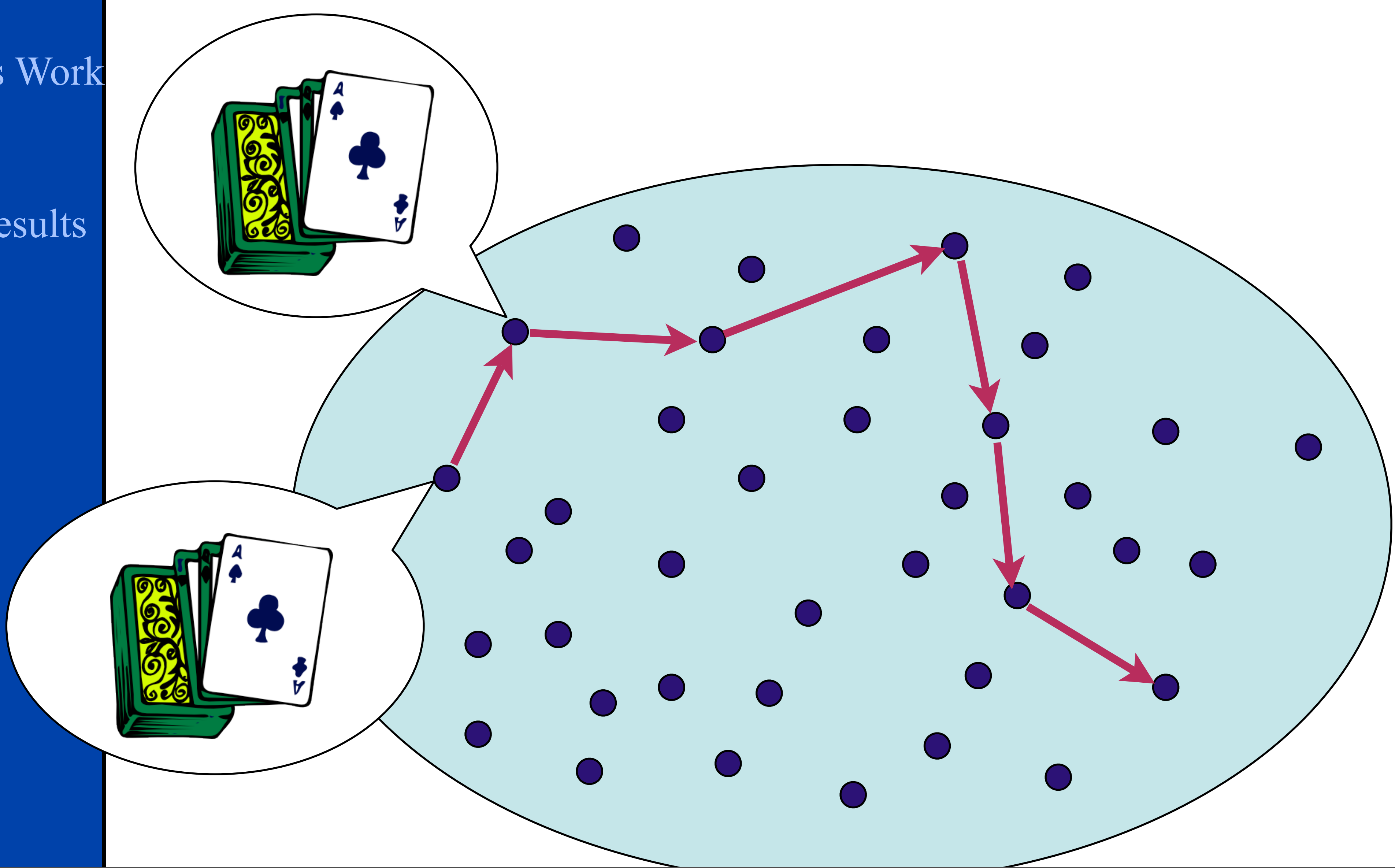
The time it takes for a Markov chain to converge within ε of π is called its *mixing time:*

$$\tau(\varepsilon) = \max_{x \in \Omega} \min \{ t : | P^t(x,*), \pi |_{TV} < \varepsilon \}$$

Using Markov Chains

We generally want the mixing time to be poly(n), where in this case n = # cards

whereas $|\Omega| = n!$

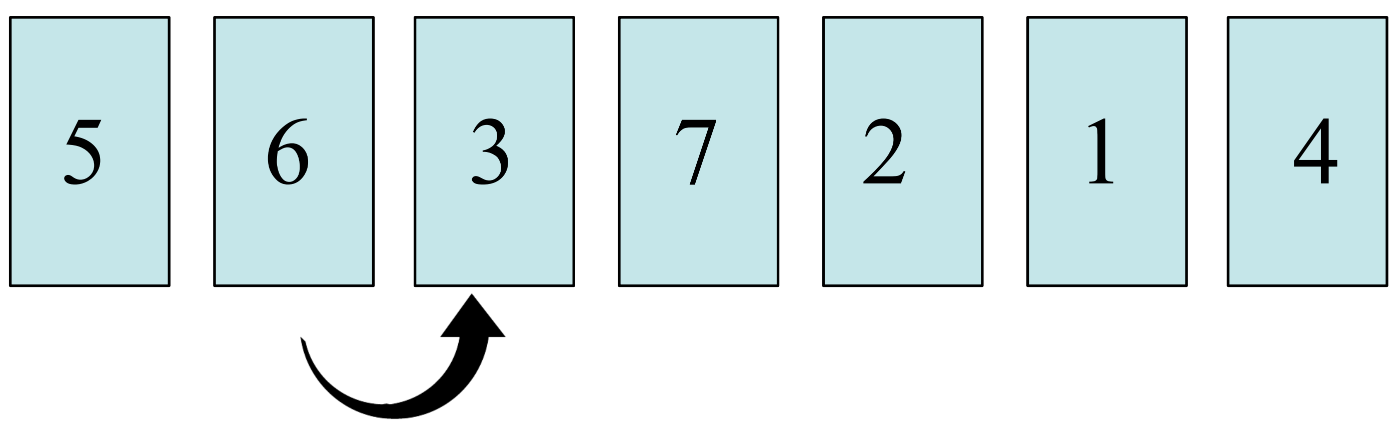


Previous work: Uniform sampling

How long does it take to mix?

$\tau = O(n^3 \log n)$ - Diaconis and Shashahani (1981),
Diaconis and Saloff-Coste (1993)

$\tau = \Omega(n^3 \log n)$ - Wilson (2004)



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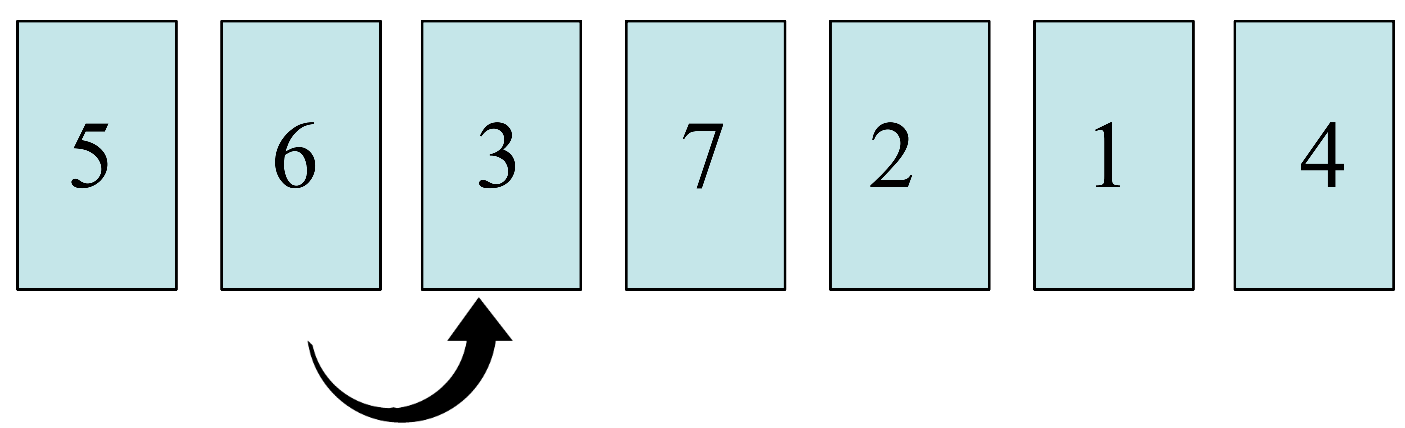

Coupling

Using Markov Chains

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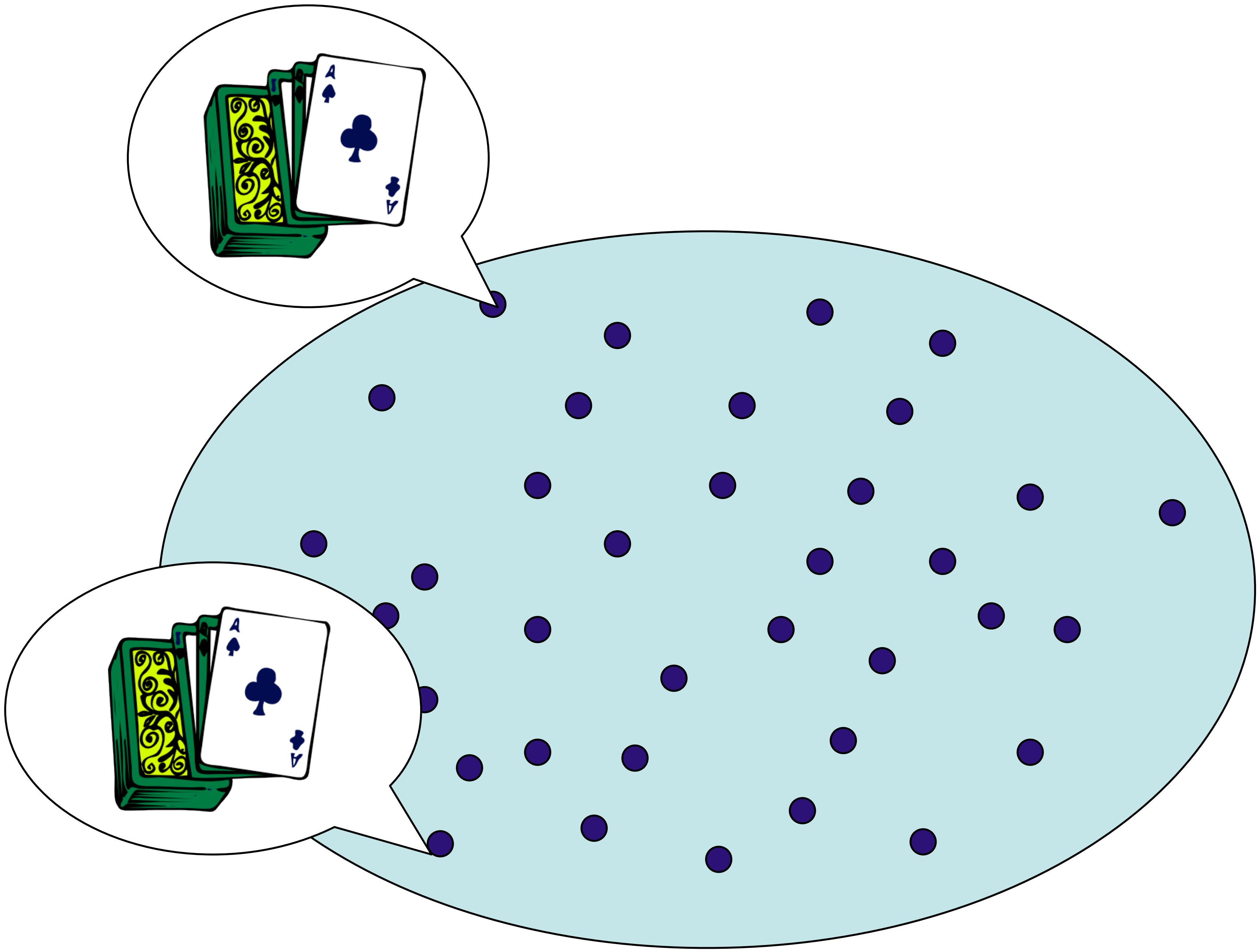

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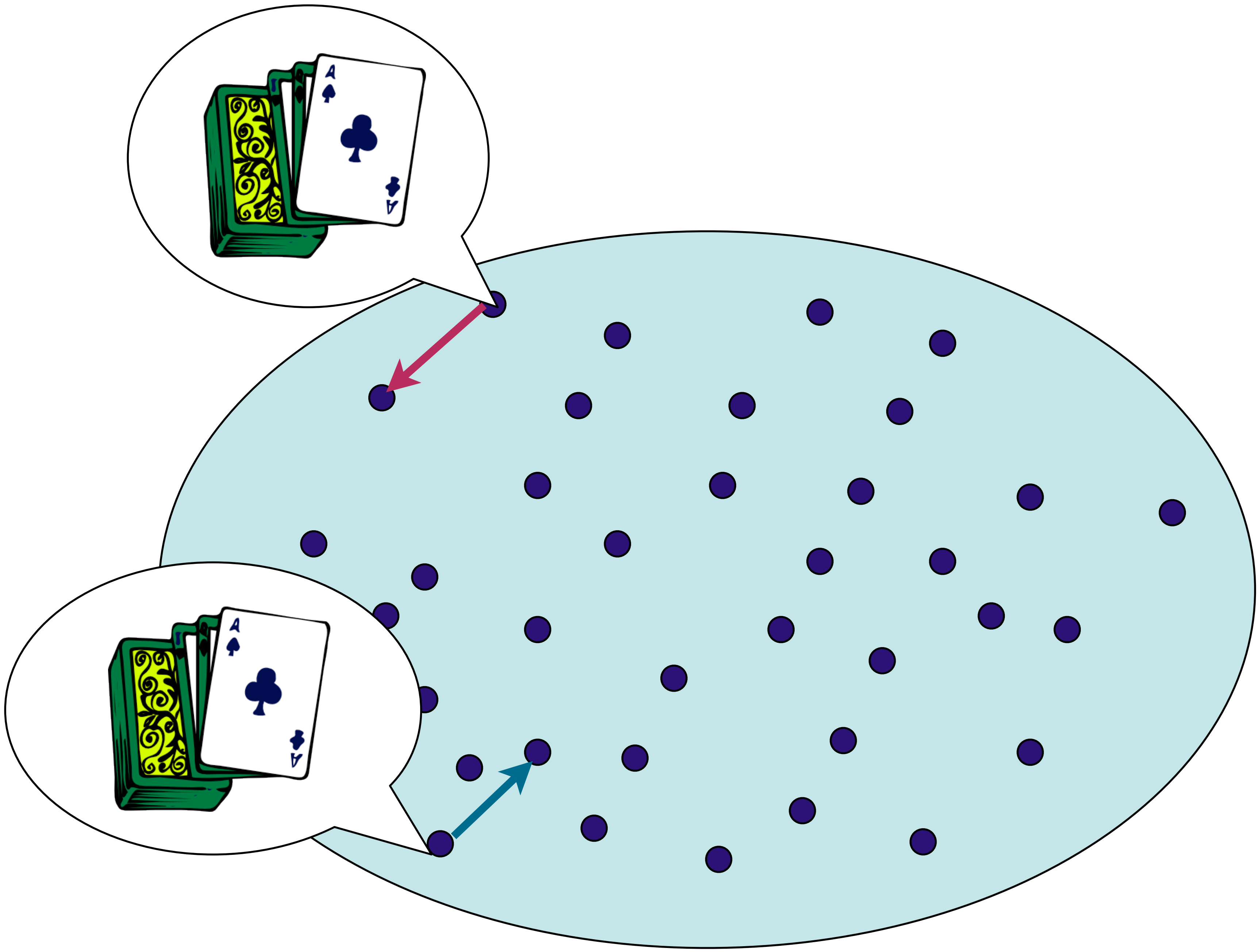

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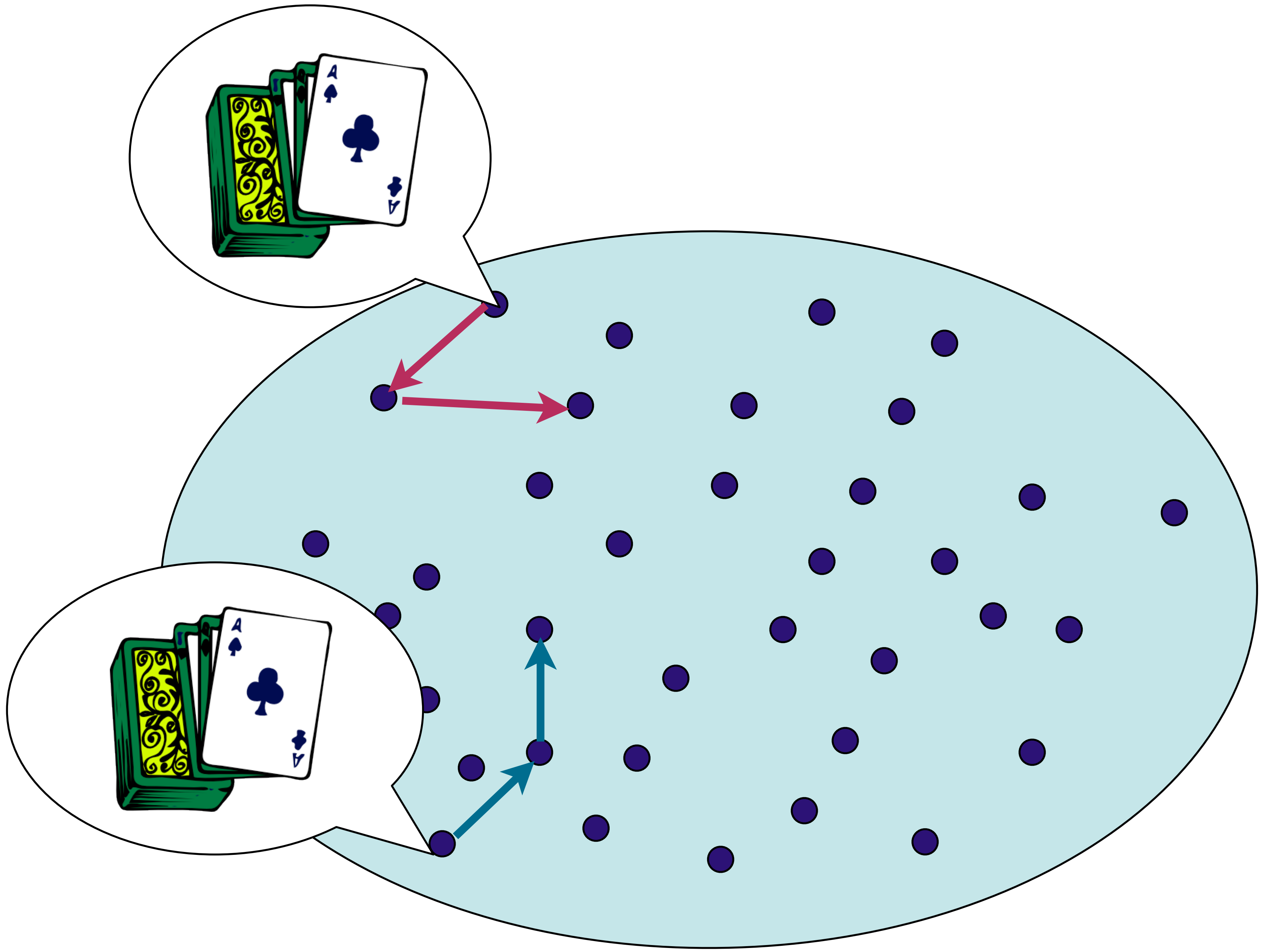

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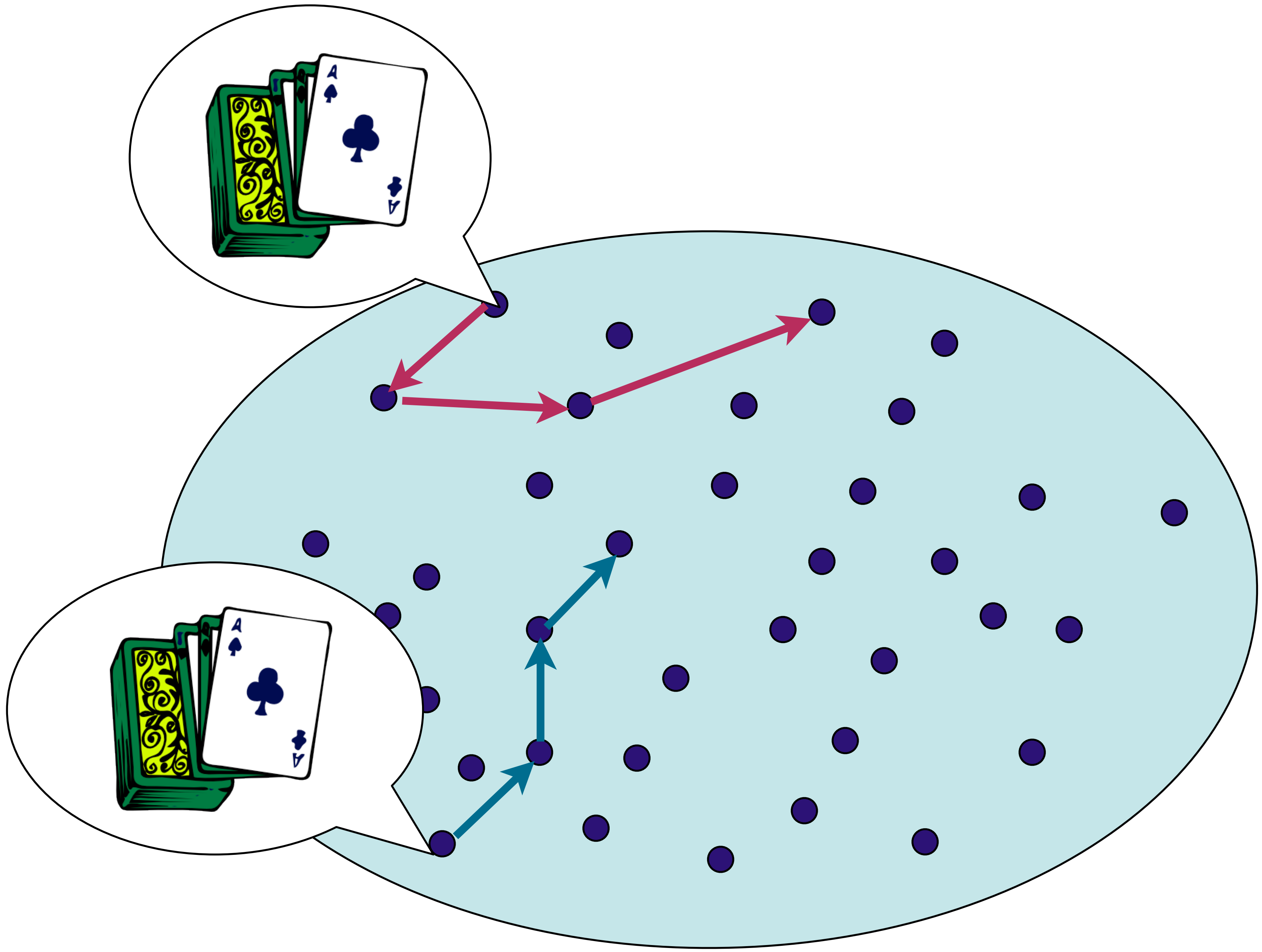

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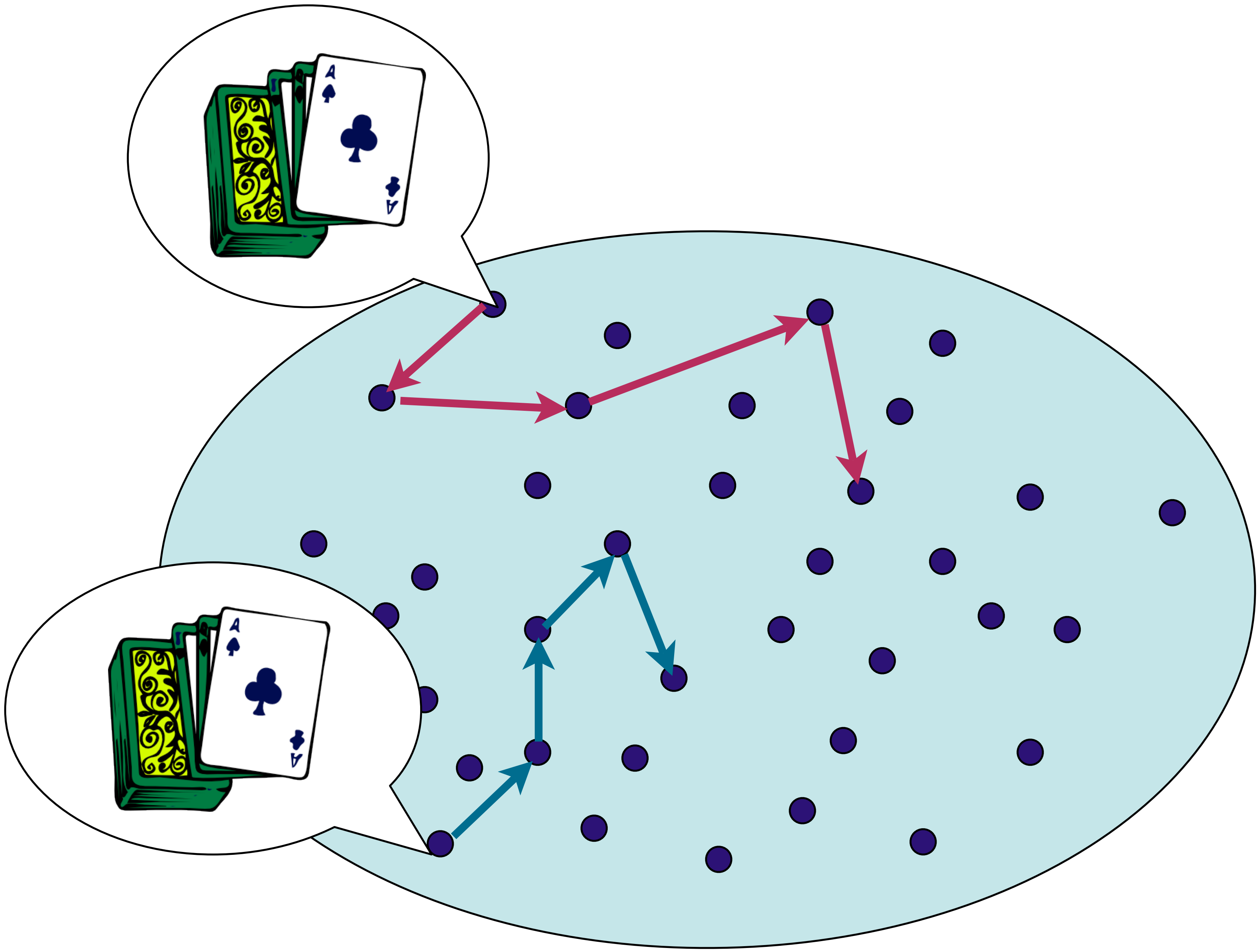

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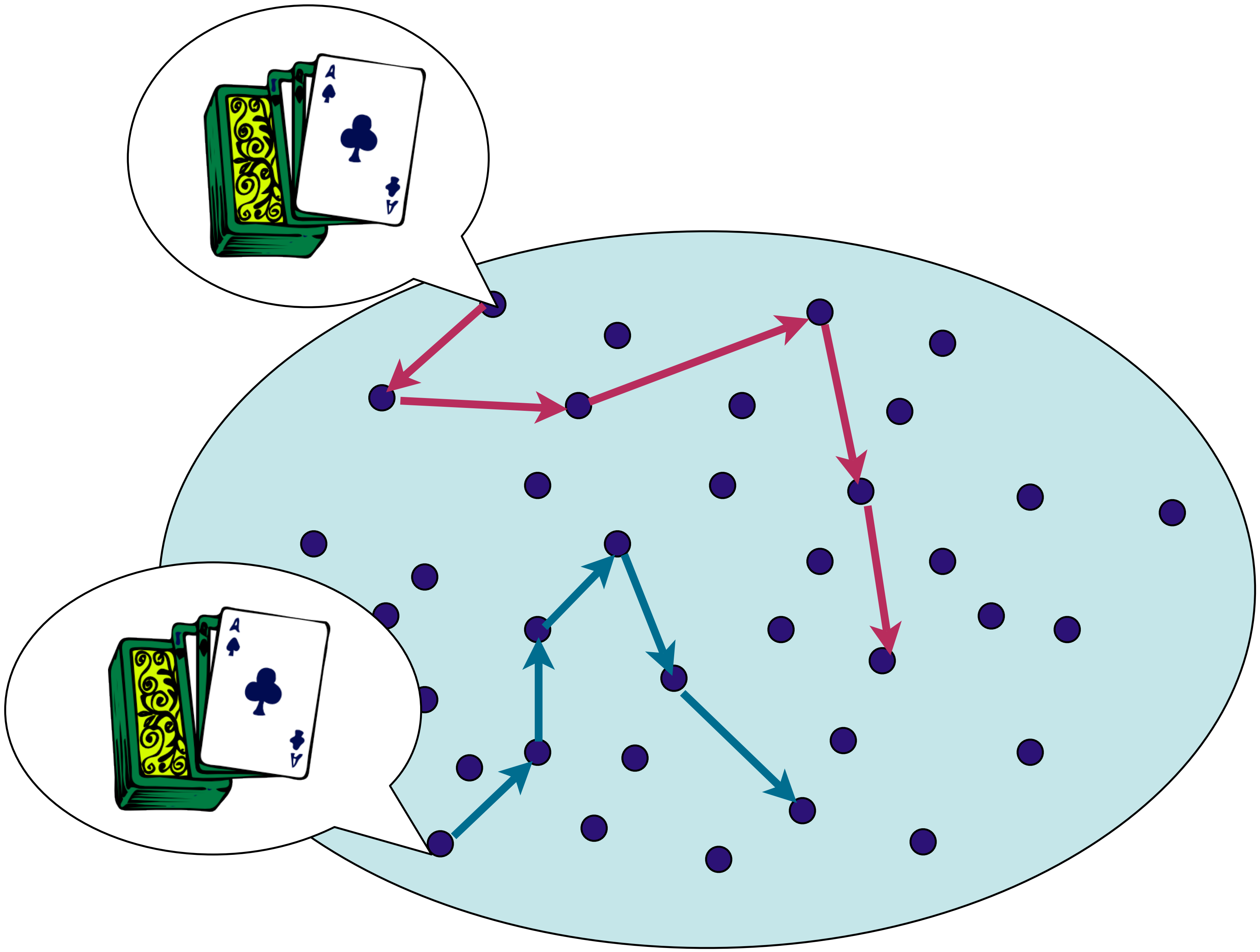

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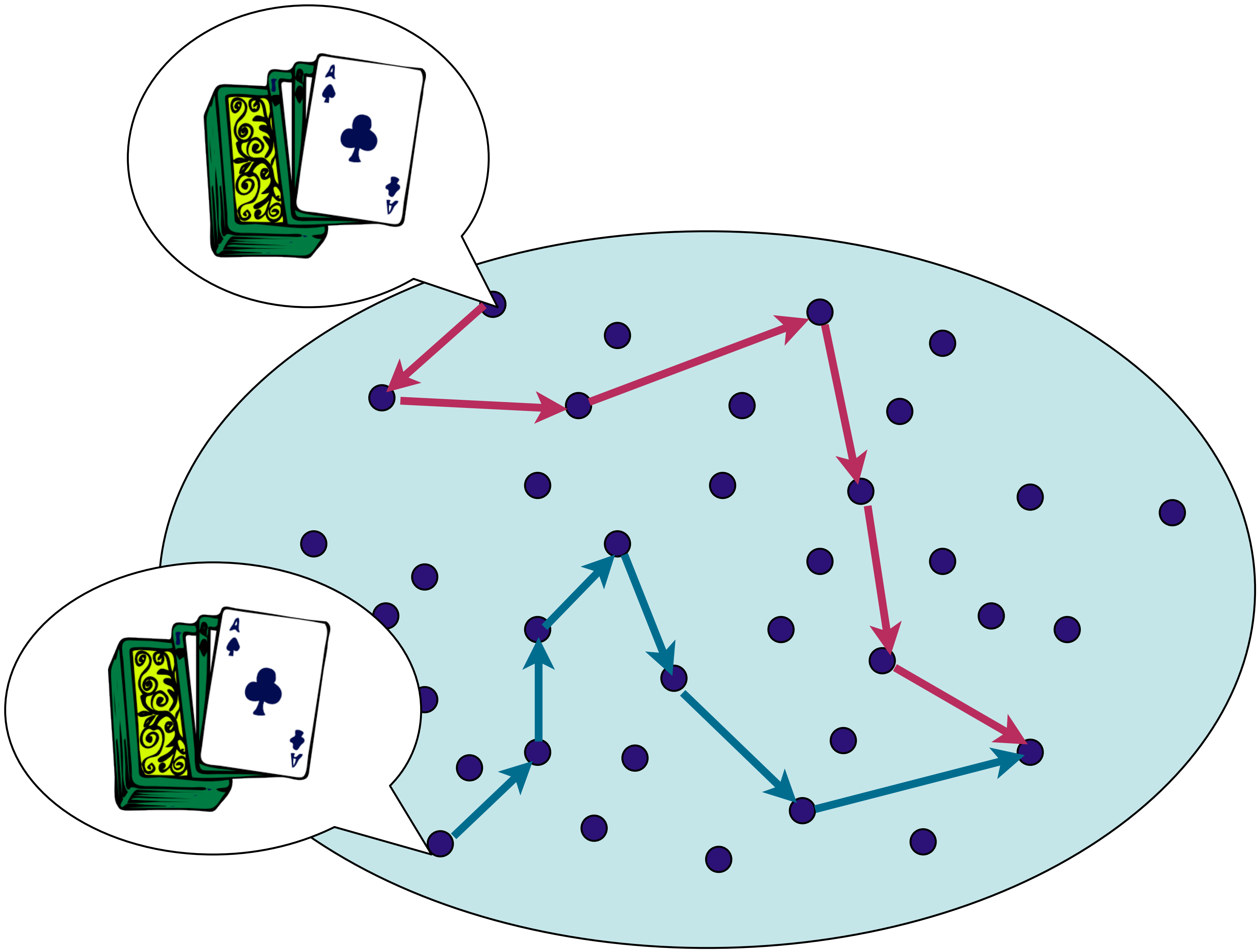

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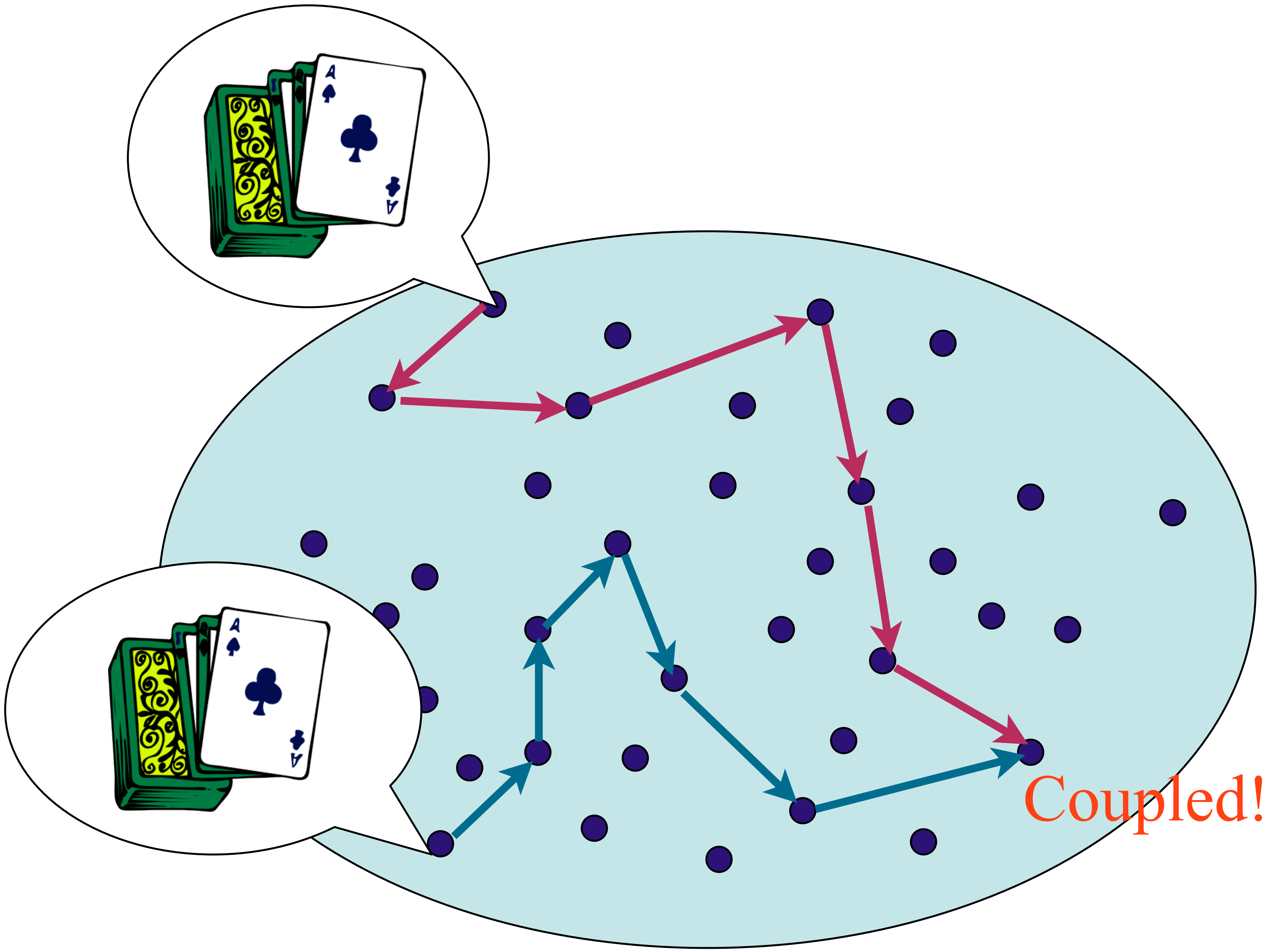

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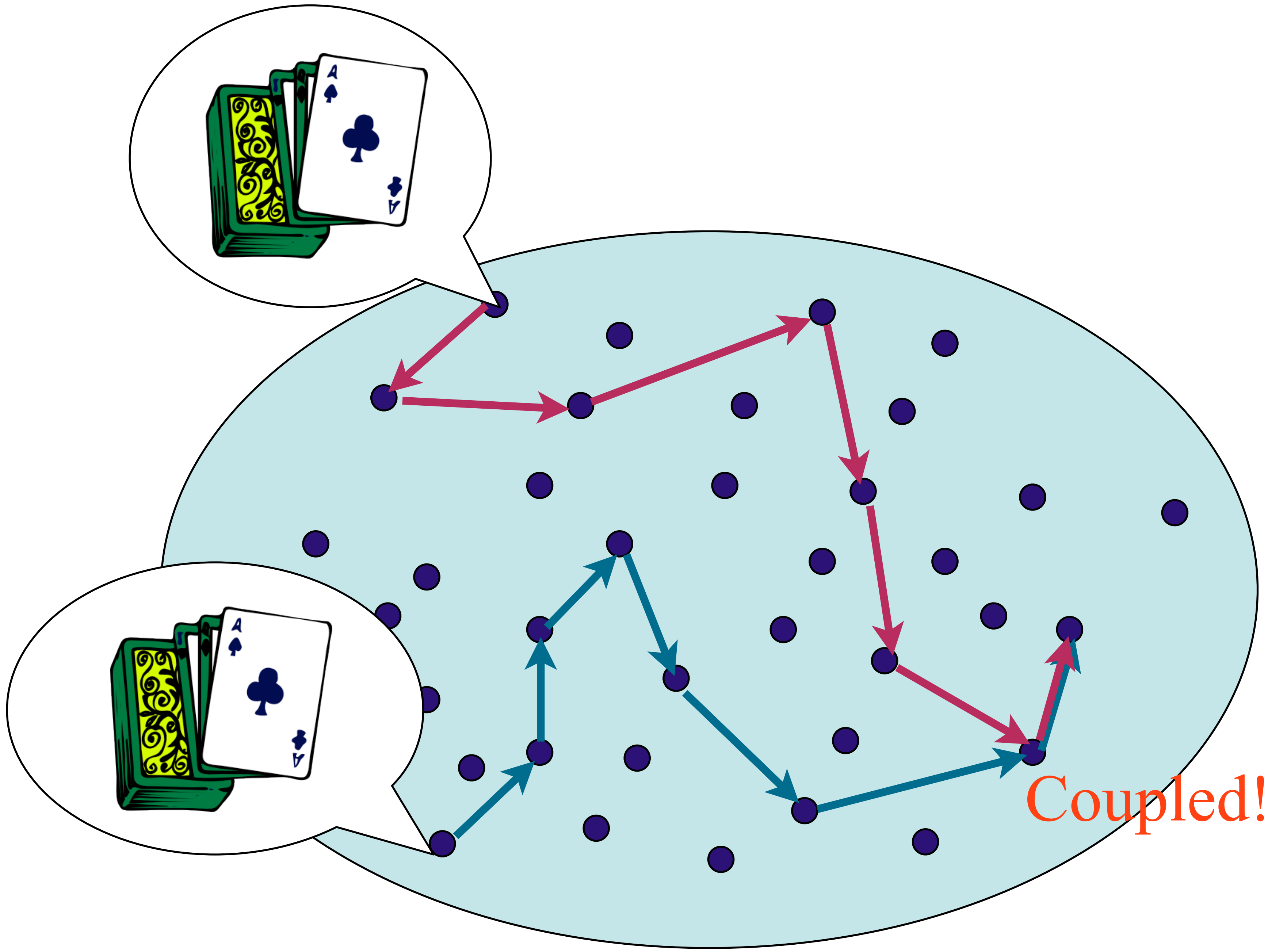

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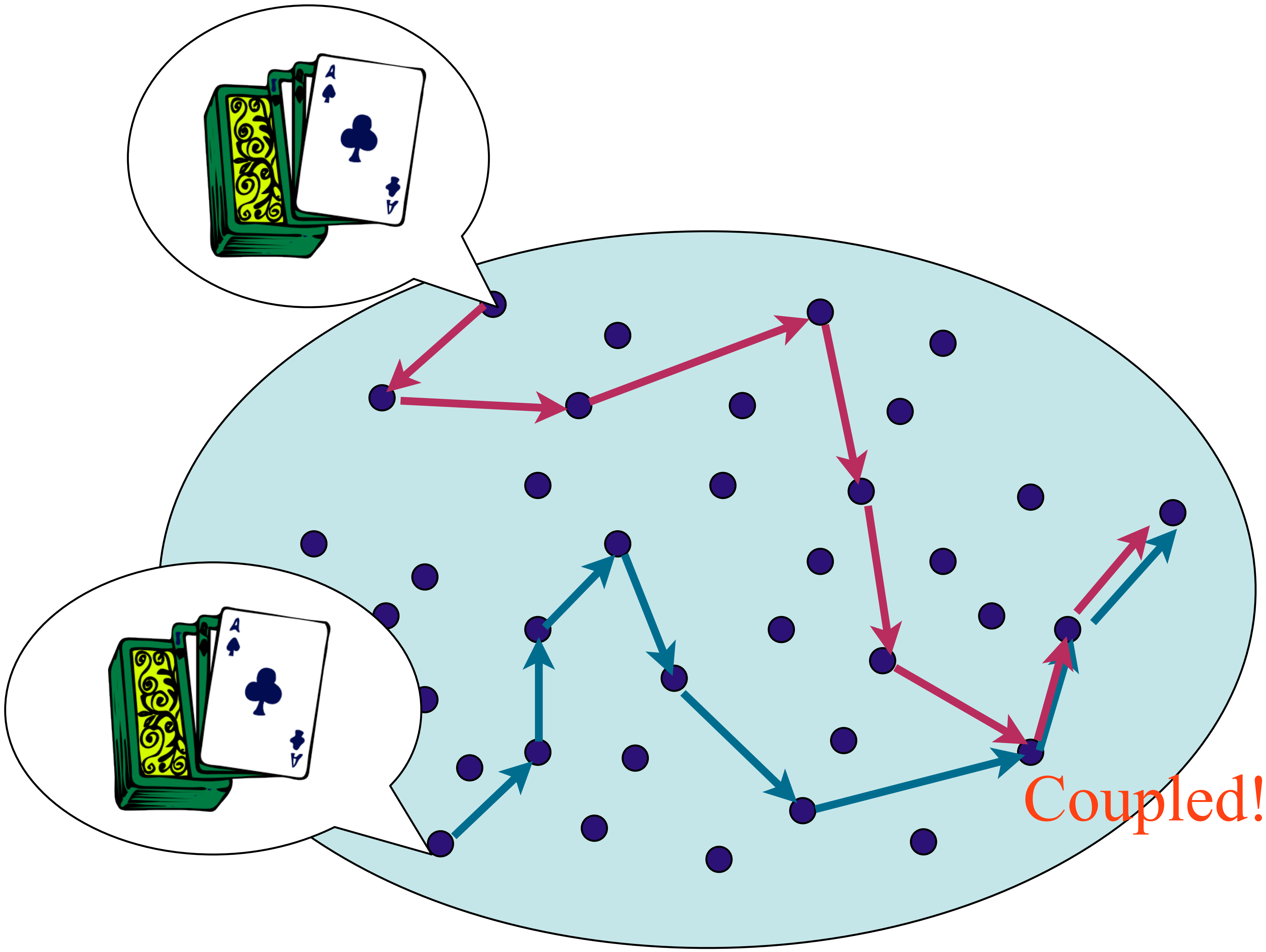

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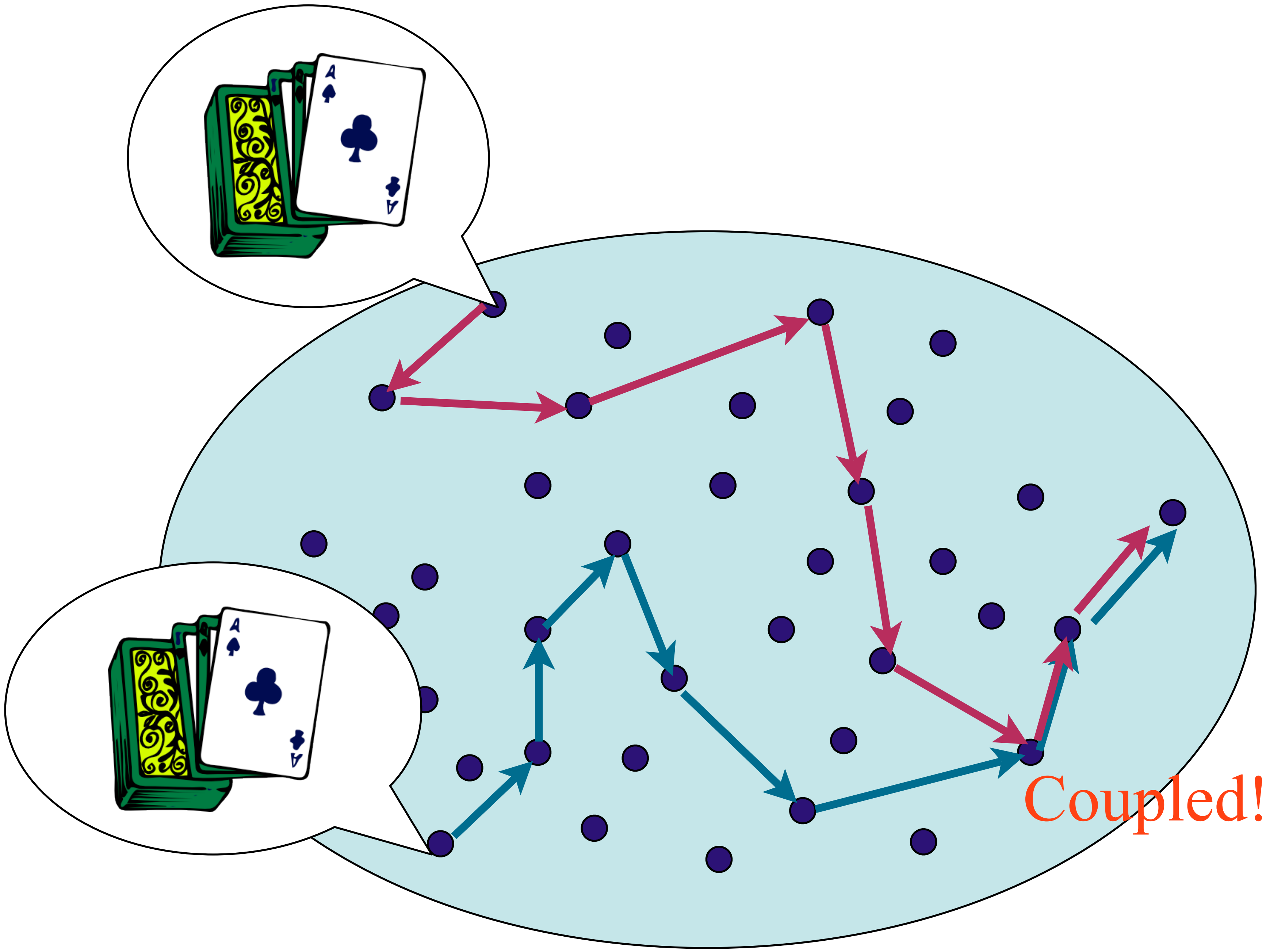

Mixing time $\leq$ *Coupling time*

Previous work: Uniform sampling

How long does it take to mix?

Permutations:

5	6	3	7	2	1	4
0	0	0	0	0	1	0
0	0	0	0	1	1	0
0	0	1	0	1	1	0
0	0	1	0	1	1	1
1	0	1	0	1	1	1
1	1	1	0	1	1	1
1	1	1	1	1	1	1

$$A_{i,j} = 1 \text{ iff } j \leq i$$

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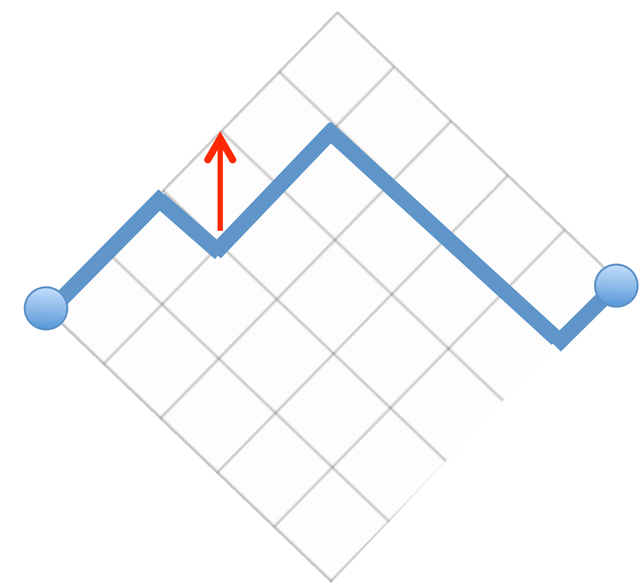
Lattice Paths:



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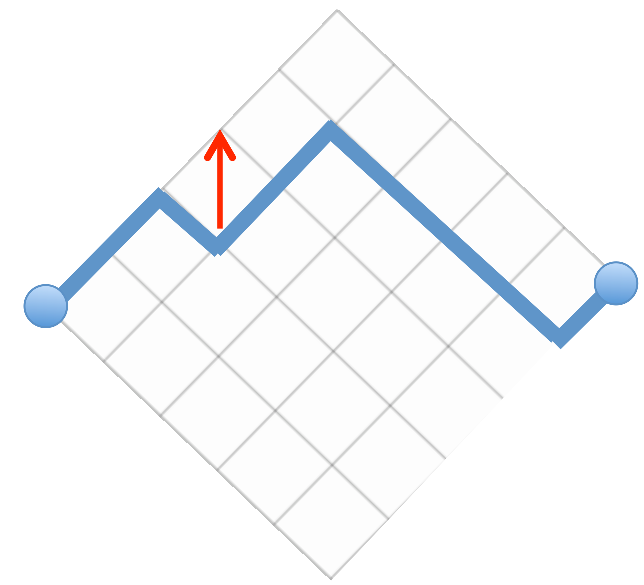

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Coupling time (perm) $\leq$ max *{Coupling time(lattice paths)}*

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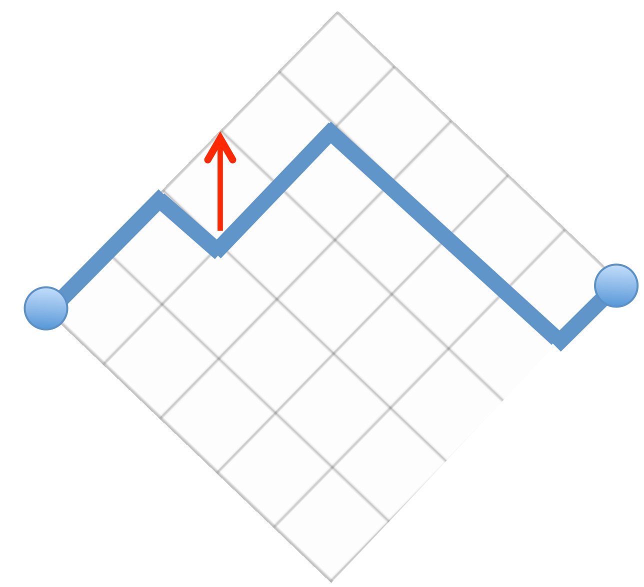

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$\tau = \Theta(n^3 \log n)$ [Wilson]

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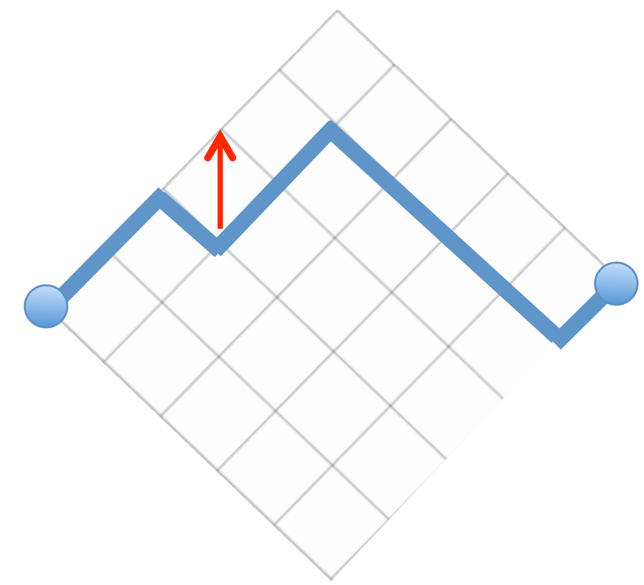

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Biased Permutations

Self-Organizing Lists

Card Shuffling

Applications

Previous Work

New Results

Example: Pizza Delivery

- List of clients and addresses
- O(n) search time

… → Alice → Tony → Sarah → Bob → … → Nancy

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- Each client has a different frequency of ordering (unknown at the beginning)
- Goal: obtain a list with most frequent clients first

Self-Organizing Lists

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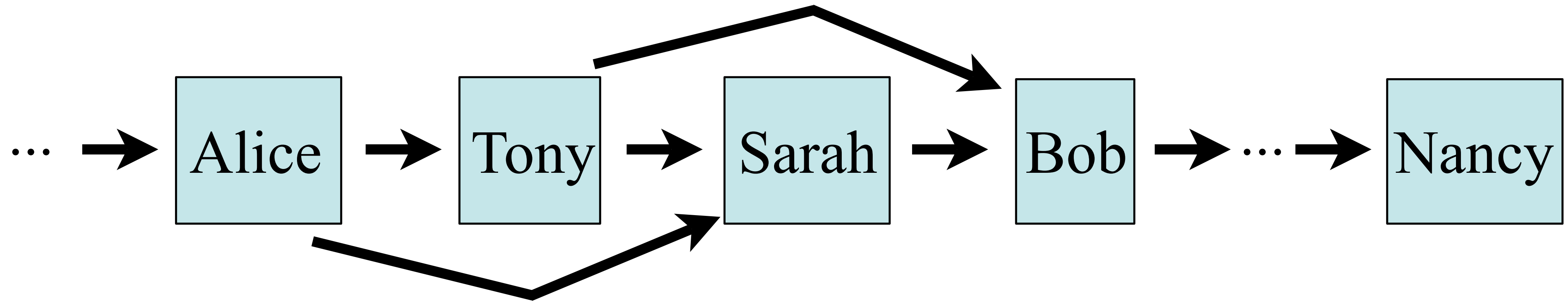

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- Each time a person orders a pizza, move them up in the list.

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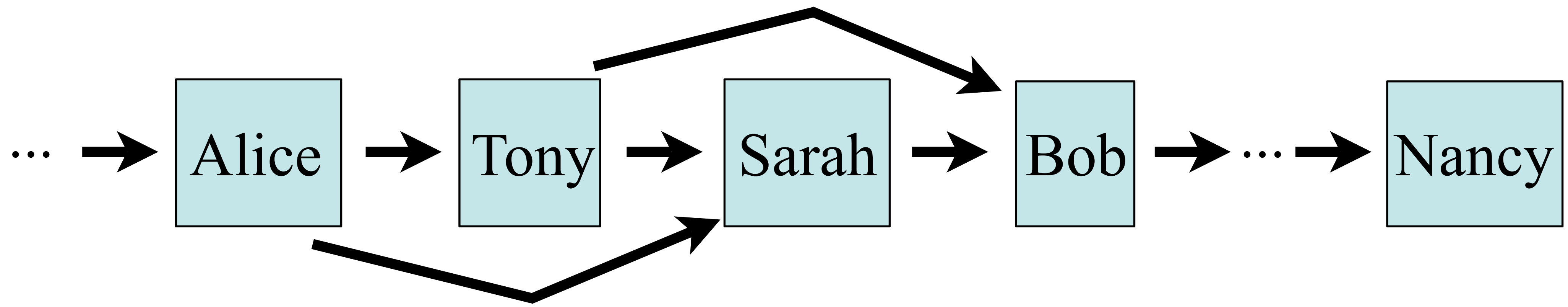

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---- **(Move Ahead One Algorithm)**

Self-Organizing Lists

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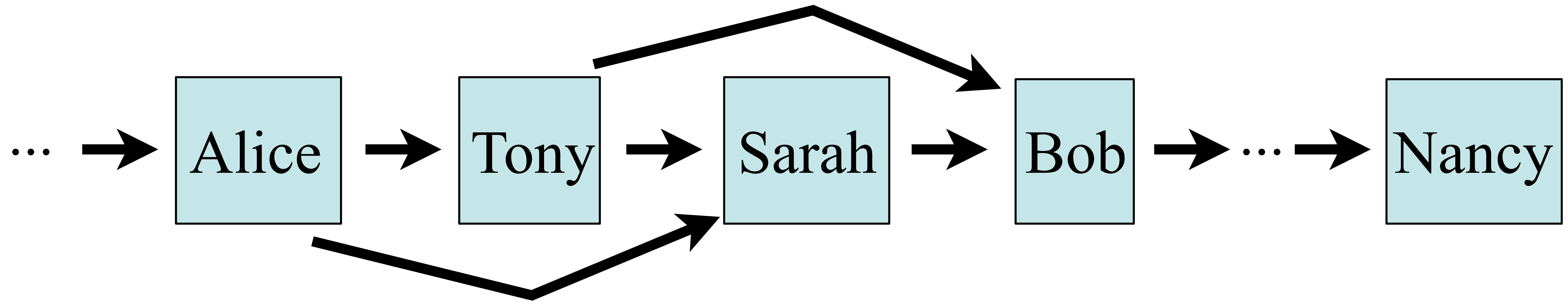

Move Ahead One Algorithm <=> Nearest Neighbor transpositions

Self-Organizing Lists

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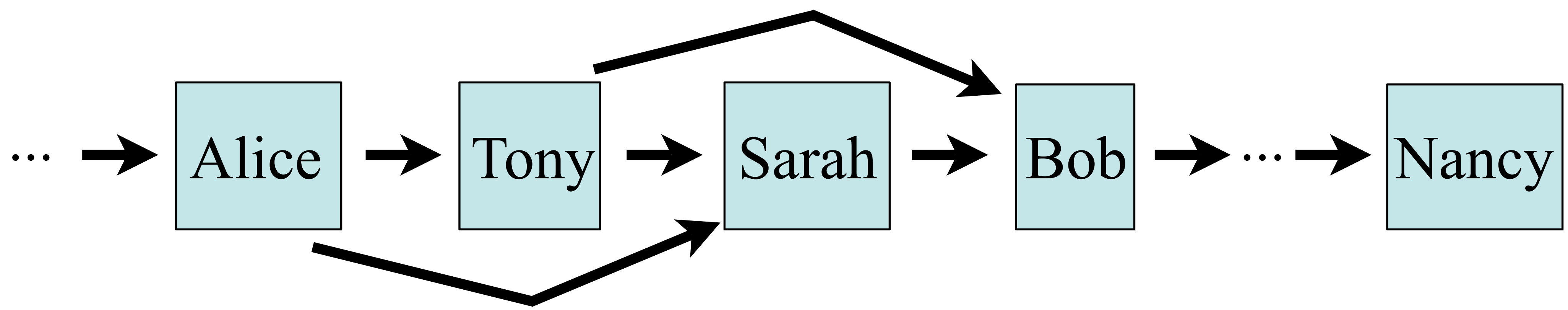

Move Ahead One Algorithm <=> Nearest Neighbor transpositions

How long does it take the list to get organized?

= Mixing Time!

Biased Card Shuffling

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- pick a pair of adjacent cards uniformly at random
- put j ahead of i with probability $p_{j,i} = 1 - p_{i,j}$

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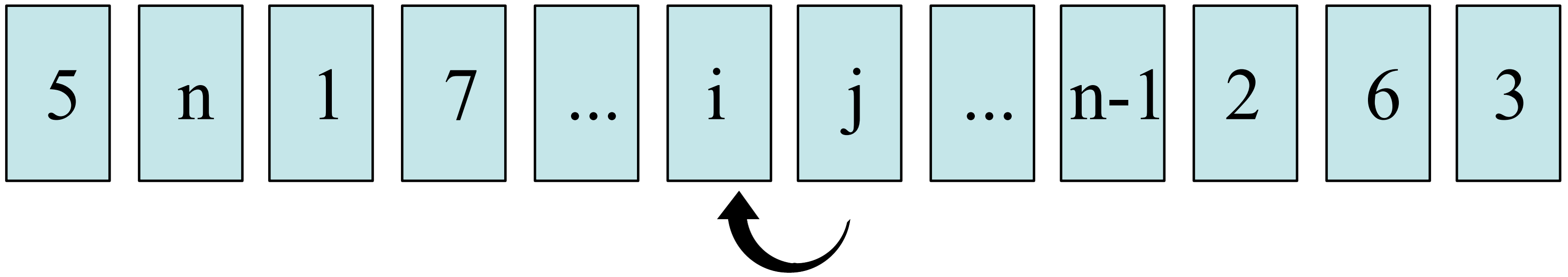

Positively biased:

We make the assumption that

$$p_{i,j} \geq 1/2 \quad \forall\, i < j$$

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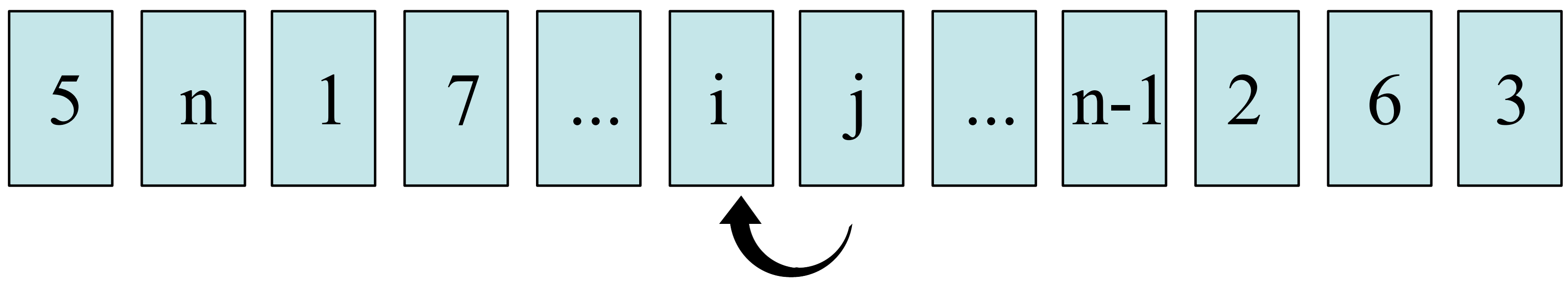

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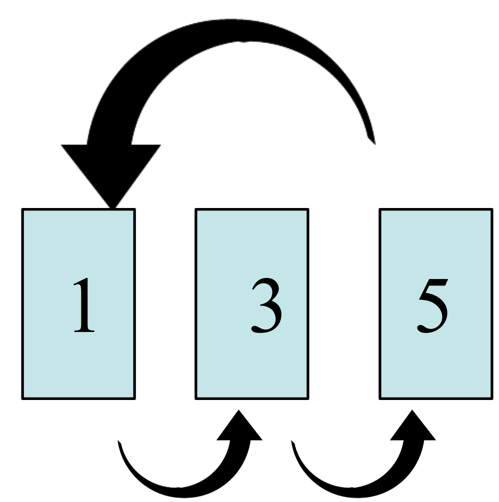

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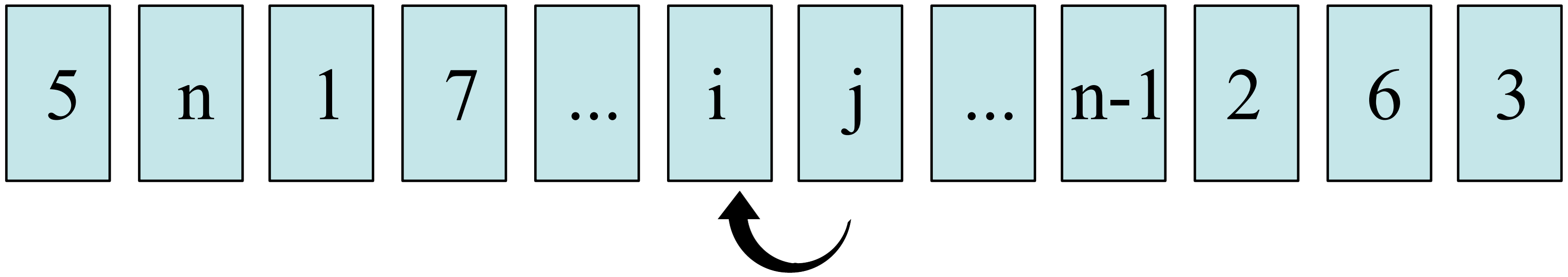

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π favors increasing permutations

Biased Card Shuffling

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Converges to: $$\pi(\sigma) = \prod_{\substack{i<j:\\ \sigma(i)>\sigma(j)}} \frac{p_{ij}}{p_{ji}} \,/\, Z$$

If $p_{i,j} \geq 1/2 \quad \forall\, i < j$, then

π favors increasing permutations

Previous Work

Conjecture:

If the $\{p_{ij}\}$ are positively biased, then M is rapidly mixing.

Fill (2003): Gap problem

- Conjecture:

  If $\{p_{ij}\}$ satisfies a "monotonicity" condition, then the spectral gap is max. when $p_{ij}=1/2$ $\forall$ i,j

- proved this conjecture for $n \leq 4$

Benjamini, Berger, Hoffman, Mossel (2005):

If $p_{i,j} = p > 1/2 \quad \forall\ i < j$, then $\theta(n^2)$ steps

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If $p_{i,j} = 1/2$ or $1 \quad \forall \ i<j$, then $O(n^3 \log n)$ steps

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Simple Markov chain M_{NN}:

- pick a pair of adjacent elements (i,j) u.a.r.
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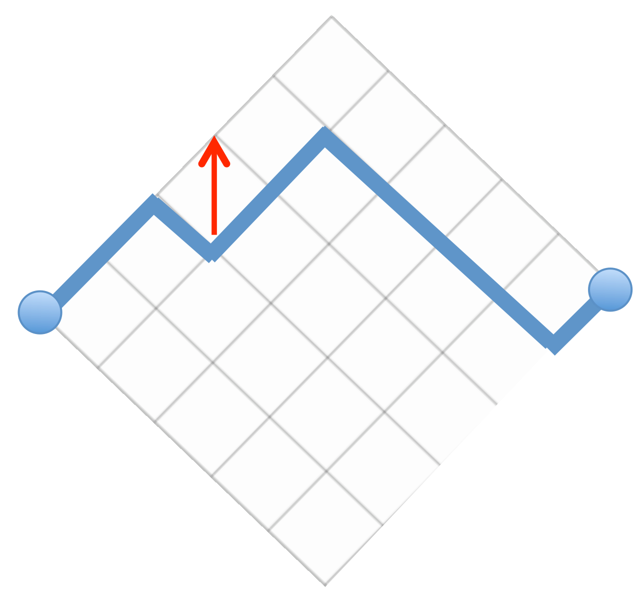

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$\tau = \Theta(n^2)$ Benjamini et al.

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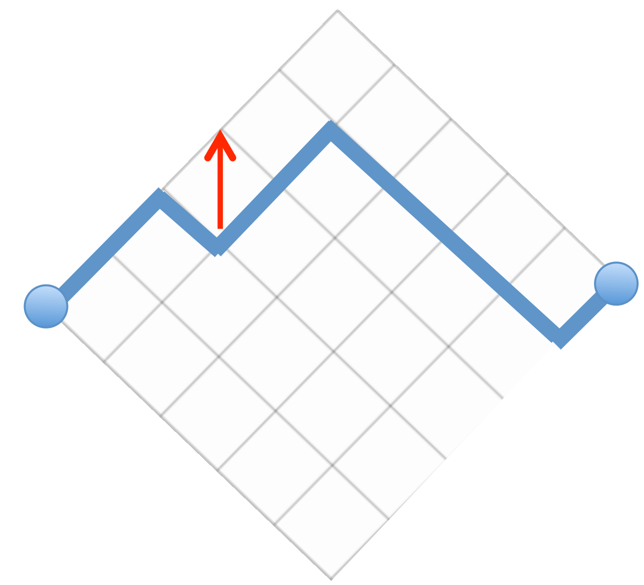

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Our Results

- Two classes - M rapidly mixing

  1. $p_{i,j} = r_i \geq 1/2 \quad \forall\, i < j$
  2. $\{p_{i,j}\}$ have *tree structure*

- Thm: M is not always rapidly mixing.

  We identify $\{p_{i,j}\}$ that are positively biased but where M requires exponential time to mix!

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  1. $\boldsymbol{p_{i,j} = r_i \geq 1/2 \quad \forall\, i < j}$

  2\. $\{p_{i,j}\}$ have *tree structure*

- <u>Thm</u>: M is not always rapidly mixing.

We identify $\{p_{i,j}\}$ that are positively biased but where M requires exponential time to mix!

Proof of Thm 1

Thm 1: If $p_{i,j} = r_i \geq 1/2 \quad \forall\, i < j$, then M is rapidly mixing.

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A. Define auxiliary Markov chain M'

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Proof outline:

A. Define auxiliary Markov chain M'
B. Show M' is rapidly mixing
C. Compare the mixing times of M and M'

M' can swap pairs that are not nearest neighbors

- maintain same stationary distribution
- Define the probability of swapping i and j that are not nearest neighbors...

Proof of Thm 1

Thm 1: If $p_{i,j} = r_i \geq 1/2 \quad \forall\, i < j$, then M is rapidly mixing.

M' can swap pairs that are not nearest neighbors

same stationary distribution: $\pi(\sigma) = \prod_{\substack{i<j:\\ \sigma(i)>\sigma(j)}} \frac{p_{ij}}{p_{ji}} \Big/ Z$

Permutation σ:

5	6	3	1	2	7	4

Permutation τ:

5	6	2	1	3	7	4

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$$\frac{P'(\sigma,\tau)}{P'(\tau,\sigma)} = \frac{\pi(\tau)}{\pi(\sigma)} = \frac{p_{2,1}}{p_{1,2}} \frac{p_{2,3}}{p_{3,2}} \frac{p_{1,3}}{p_{3,1}}$$

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Proof of Thm 1

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$$\boxed{P'(\sigma,\tau) = p_{2,3}}$$

Proof of Thm 1

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$P'(\sigma,\tau) = p_{2,3}$

- Can swap i and j across multiple *smaller* elements with probability $p_{i,j}$

Idea: Location of element i is independent of the location of all larger elements!

Inversion Tables

Permutation σ:

5	6	3	1	2	7	4

Inversion Tables

Permutation σ:

5	6	3	1	2	7	4

Inversion Table I_σ:

3 3 2 3 0 0 0

Inversion Tables

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Permutation σ:

5	6	3	1	2	7	4

Inversion Table I_σ:

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$I_\sigma(i)$ = # elements $j > i$ appearing before i in σ

Inversion Tables

Permutation σ:

5	6	3	1	2	7	4

Inversion Table I_σ:

3 3 2 3 0 0 0

$I(2)$

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Inversion Tables

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$I_\sigma(i)$ = # elements $j > i$ appearing before i in σ

- $0 \leq I_\sigma(i) \leq n - i$

Inversion Tables

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$I_\sigma(i)$ = # elements $j > i$ appearing before i in σ

- $0 \leq I_\sigma(i) \leq n - i$
- I is a bijection from S_n to $T = \{(x_1, x_2, \ldots, x_n) : 0 \leq x_i \leq n-i\}$

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We will define M' as a MC on the Inversion Tables.

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What happens when you add 1 to x_i?

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What happens when you add 1 to x_i?

- swap element i with the first $j>i$ to the right

Inversion Tables

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- swap element i with the first $j>i$ to the right
- happens w.p. $1-r_i$

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What happens when you subtract 1 from x_i?

- swap element i with the first $j>i$ to the left

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What happens when you subtract 1 from x_i?

- swap element i with the first $j>i$ to the left
- happens w.p. r_i

Proof of Thm 1

Permutation σ:

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Inversion Table I_σ:

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$I(2)$

Proof outline:

- **A. Define auxiliary Markov chain *M'***
- B. Show *M'* is rapidly mixing
- C. Compare the mixing times of *M* and *M'*

M' samples from $\{(x_1,x_2,\ldots,x_n): 0 \leq x_i \leq n-i\}$:

- choose a column i uniformly
- w.p. r_i: subtract 1 from x_i (if possible)
- w.p. $1-r_i$: add 1 to x_i (if possible)

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- A. Define auxiliary Markov chain *M’*
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M’ is just a cross-product of *n* independent, 1-dimensional random walks

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Idea: Location of element *i* is independent of the location of all larger elements!

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M' is just a cross-product of *n* independent, 1-dimensional random walks → *M'* is rapidly mixing!

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B. Show *M’* is rapidly mixing

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M’ rapidly mixing $\Rightarrow$ *M* is rapidly mixing

Our Results

•Two classes - M rapidly mixing

1. $p_{i,j} = r_i \geq 1/2 \quad \forall\, i < j$
2. $\{p_{i,j}\}$ have *tree structure*

•**Thm 2: *M* is not always rapidly mixing.**

We identify $\{p_{i,j}\}$ where $\{p_{i,j}\}$ are positively biased but M requires exponential time to mix!

Slow Mixing Results

- **<u>Thm 2</u>: *M* is not always rapidly mixing.**

Slow Mixing Results

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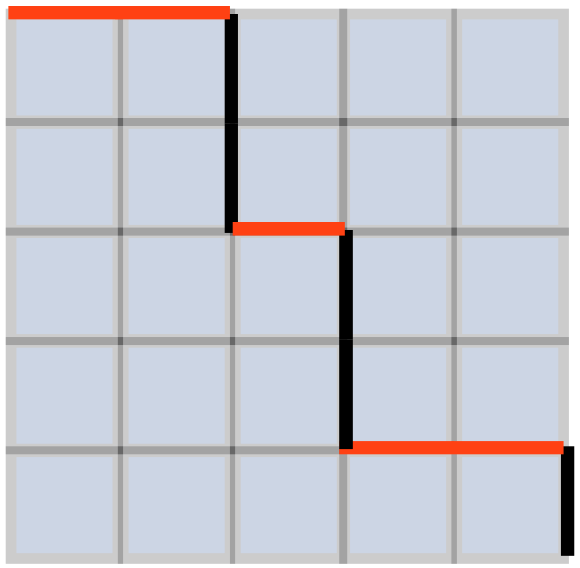
Special Case: Bijection with staircase walks



Slow Mixing Results

•**Thm 2: *M* is not always rapidly mixing.**

Special Case: Bijection with staircase walks

always in order

$$\underbrace{1\ 2\ 3\ \ldots\ \tfrac{n}{2}}\quad \underbrace{\tfrac{n}{2}+1\ \ldots\ n}$$

$$p_{ij} = \begin{cases} 1 \quad \text{if} \quad i < j \le \frac{n}{2} \quad \text{OR} \quad \frac{n}{2} < i < j \\ \end{cases}$$

Permutation σ:

1 2 6 7 3 8 9 4 5 10

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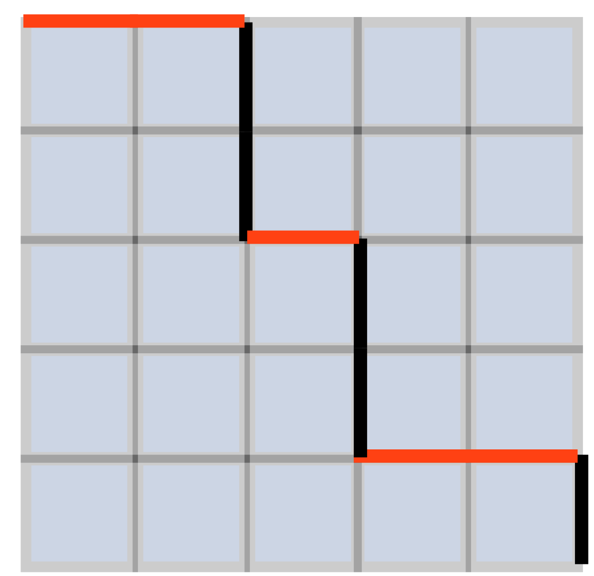
Permutation σ:

1 2 6 7 3 8 9 4 5 10

1 1 0 0 1 0 0 1 1 0

$\longrightarrow$



$\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's

Slow Mixing Results

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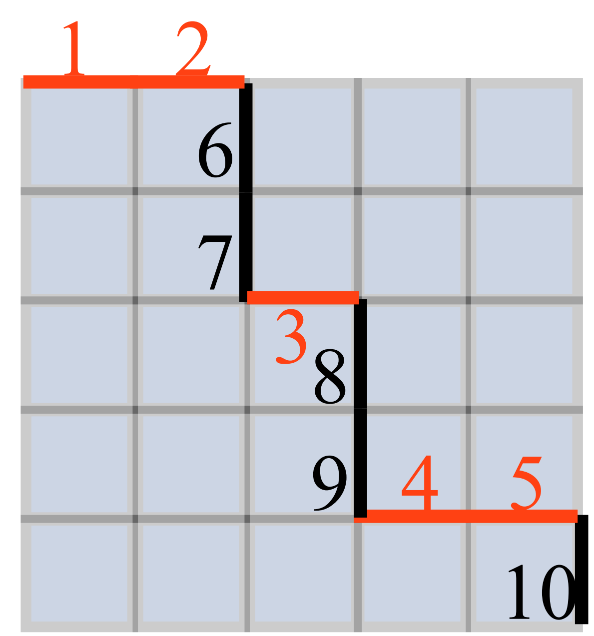
$$p_{ij} = \begin{cases} 1 & \text{if } i < j \le \frac{n}{2} \text{ OR } \frac{n}{2} < i < j \\ ?? & \text{else} \end{cases}$$

<u>Permutation σ:</u>

1 2 6 7 3 8 9 4 5 10

1 1 0 0 1 0 0 1 1 0 $\longrightarrow$



$\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's

Slow Mixing Results

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Special Case: Bijection with staircase walks

always in order

$1\ 2\ 3\ \ldots\ \frac{n}{2}$ | $\frac{n}{2}+1\ \ldots\ n$

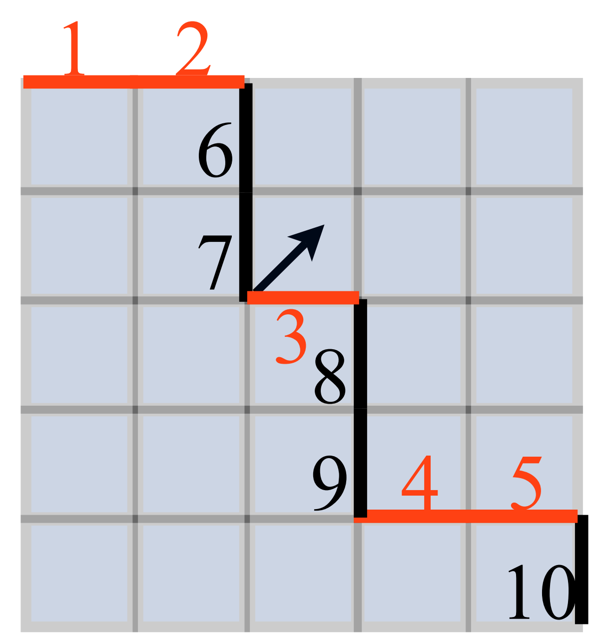
$$p_{ij} = \begin{cases} 1 & \text{if } i < j \le \frac{n}{2} \text{ OR } \frac{n}{2} < i < j \\ ?? & \text{else} \end{cases}$$

Permutation σ:

1 2 6 7 3 8 9 4 5 10

1 1 0 0 1 0 0 1 1 0

p_{37}



$\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's

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•**Thm 2: *M* is not always rapidly mixing.**

Special Case: Bijection with staircase walks

always in order

$\underbrace{1\ 2\ 3\ \ldots\ \tfrac{n}{2}}\quad\underbrace{\tfrac{n}{2}+1\ \ldots\ n}$

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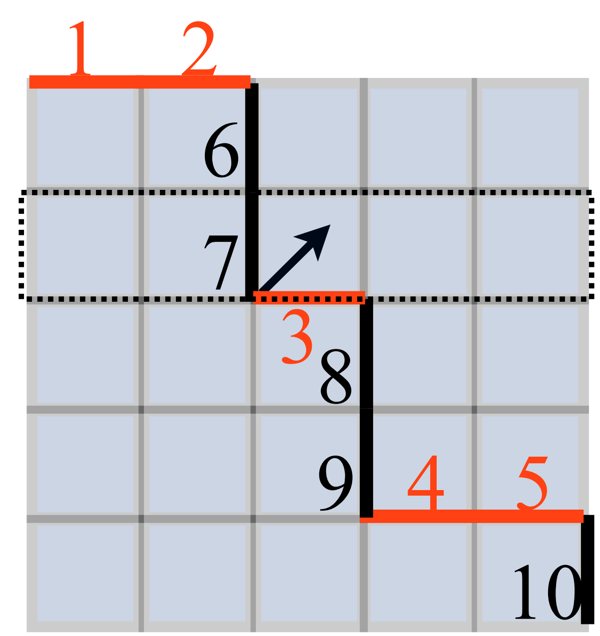

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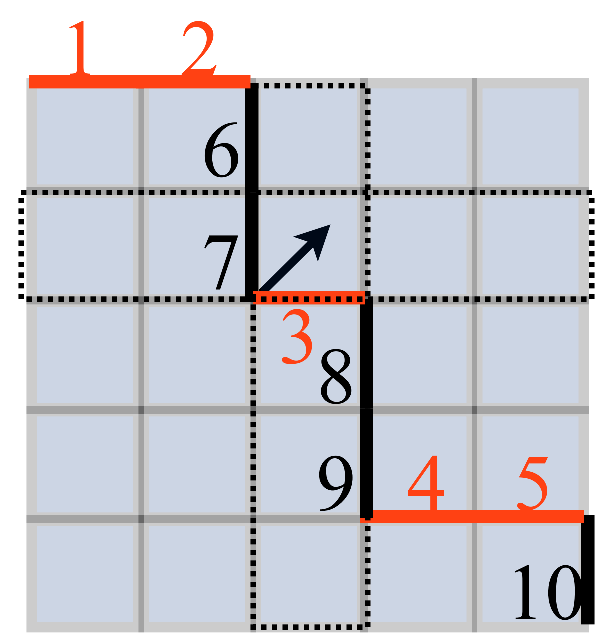

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Slow Mixing Results

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Special Case: Bijection with staircase walks

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1 2 6 7 3 8 9 4 5 10

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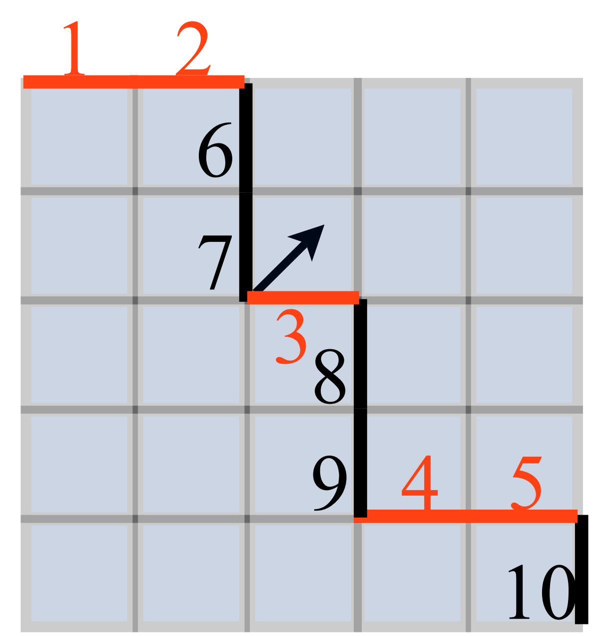
p_{37}

→



$\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's

Slow Mixing Results

- **Thm 2: *M* is not always rapidly mixing.**

Special Case: Bijection with staircase walks

So each choice of p_{ij} where $i \le \frac{n}{2} < j$ determines the bias on square (i,n-j+1)

$$p_{ij} = \begin{cases} 1 & \text{if } i < j \le \frac{n}{2} \text{ OR } \frac{n}{2} < i < j \\ ?? & \text{else} \end{cases}$$

Permutation σ:

1 2 6 7 3 8 9 4 5 10

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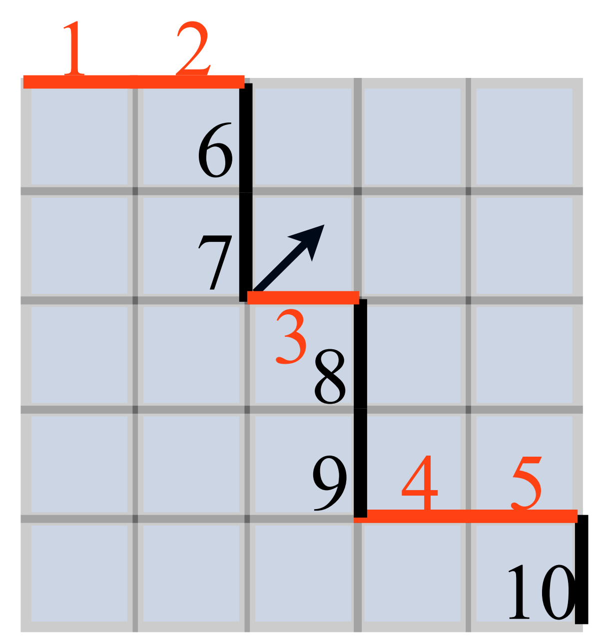

$\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's

Staircase Walks

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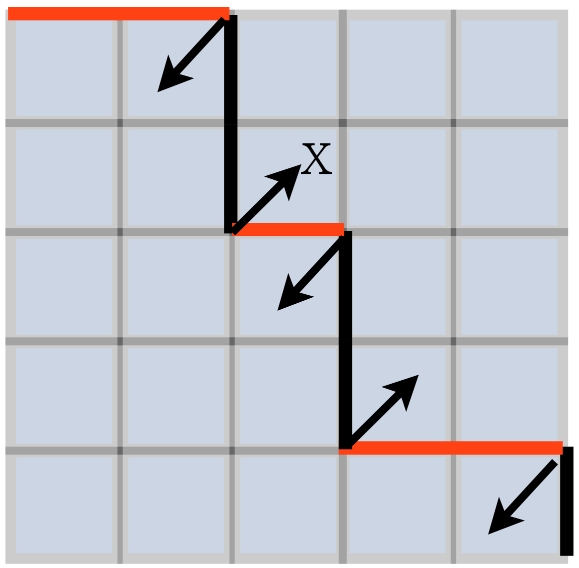

- each box has a different bias p_x

- *M* can add a box or remove a box according to p_x

Staircase Walks

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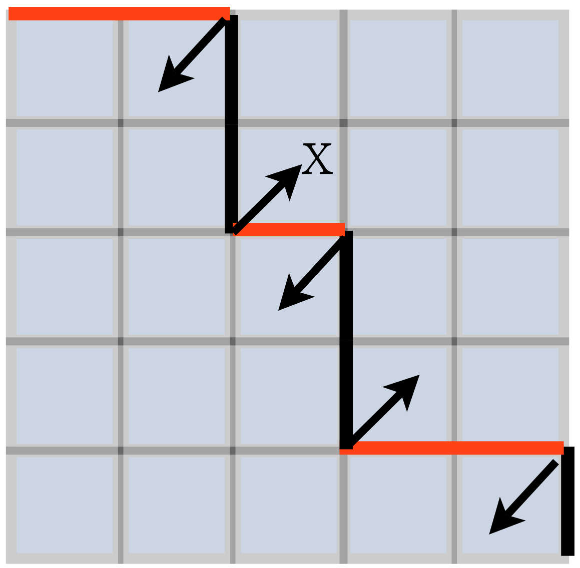


- each box has a different bias p_x

- *M* can add a box or remove a box according to p_x

Tile-based Self-Assembly:

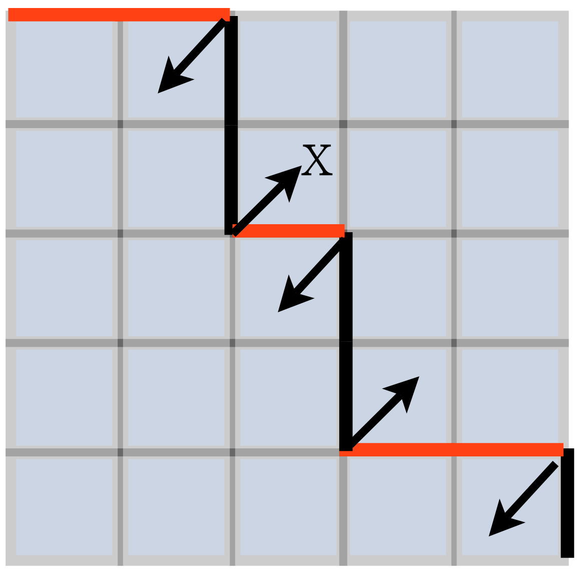
tiles can attach or detach at corners

- attach w.p. p_x
- detach w.p. 1- p_x

rapidly mixing <=> self-assembles efficiently

Staircase Walks

Fluctuating Bias:

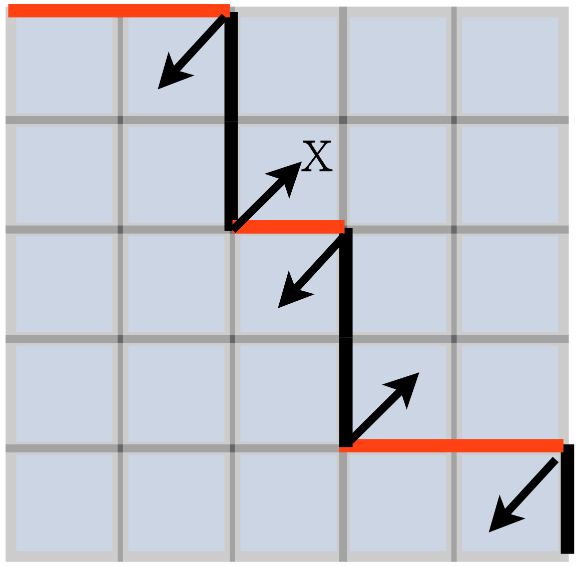


Thm 4: If $p_x \geq p$ (p const. > 1/2) then rapidly mixing.

Thm 5: There exists $\{p_x\}$ s.t $p_x > 1/2$ for all x but mixing time is exponential in n.

Staircase Walks

Fluctuating Bias:



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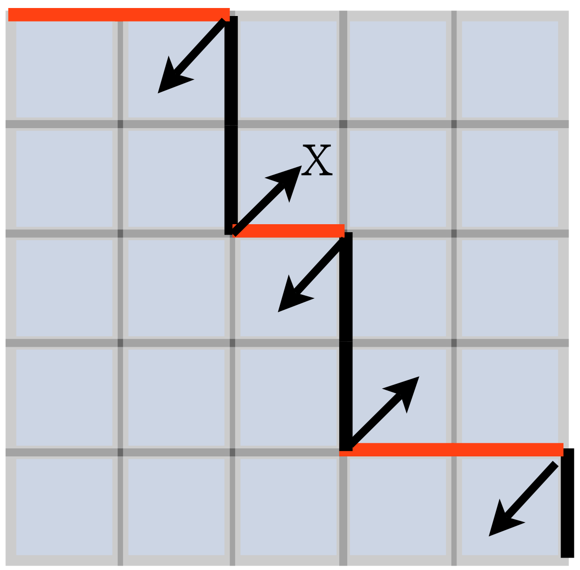

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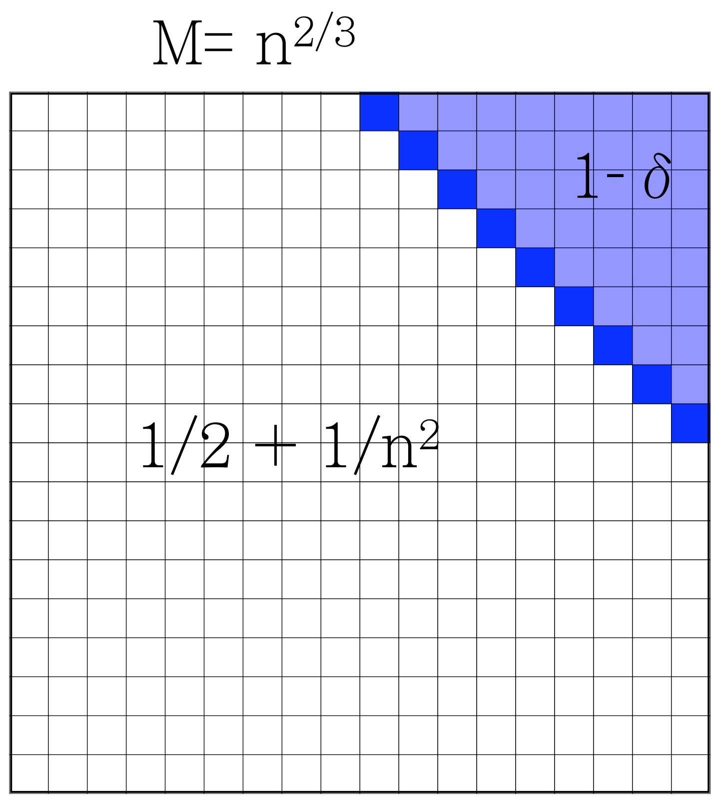
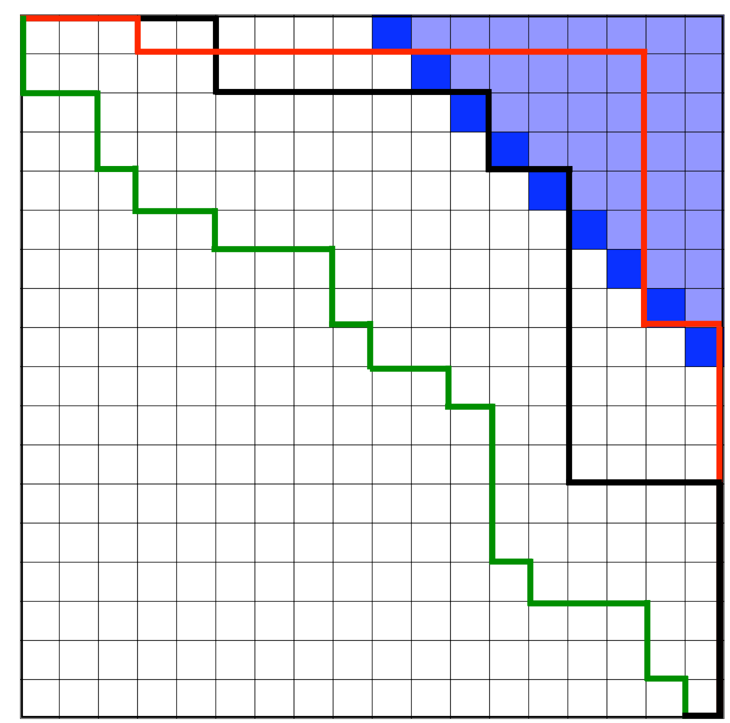
Thm 5: There exists $\{p_x\}$ s.t p_x >1/2 for all x but mixing time is exponential in n.

provides slow mixing example for biased permutations!

Slow Mixing

Thm 5: There exists $\{p_x\}$ s.t $p_x > 1/2$ for all x but mixing time is exponential in n.





Card Shuffling

Applications

Previous Work

New Results

Slow Mixing Results

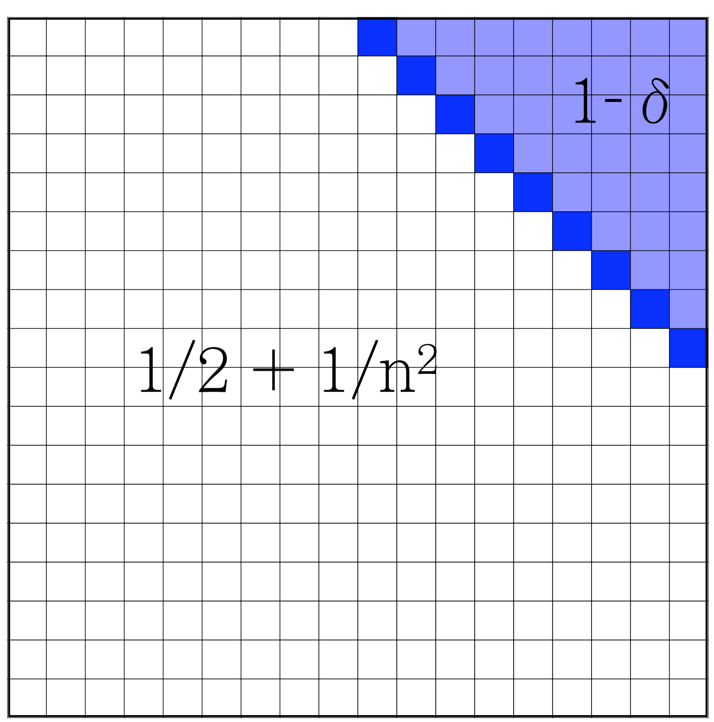
- **Thm 2:** ***M*** **is not always rapidly mixing.**

Bijection with staircase walks:

So each choice of p_{ij} where $i \le \frac{n}{2} < j$ determines the bias on square (i,n-j+1)

$$p_{ij} = \begin{cases} 1 & \text{if } i < j \le \frac{n}{2} \text{ OR } \frac{n}{2} < i < j \\ 1/2+1/n^2 & \text{if } i+(n-j+1) < M \\ 1-\delta & \text{otherwise} \end{cases}$$

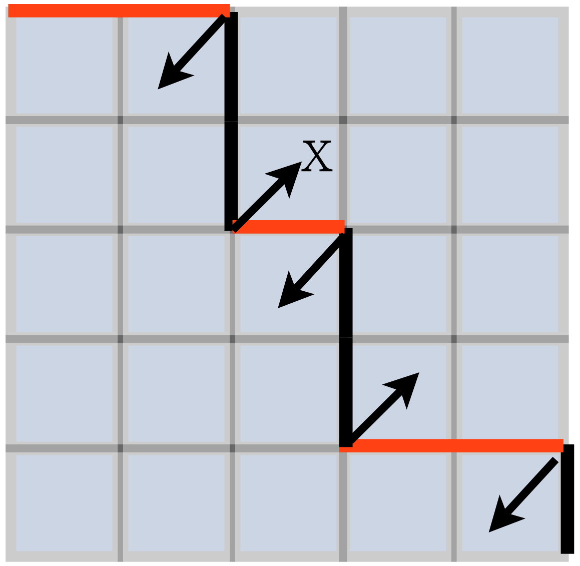
$M = n^{2/3}$



Thank you!

Staircase Walks

Uniform Bias:



Thm [Benjamini, Berger, Hoffman, Mossel]: If $p_x = p$ for all x, then M is rapidly mixing.

Thm 3 [GPR]: If $p_x = p$ for all x, then M is rapidly mixing.

simpler, generalizes easily

proof by coupling with exponential distance metric

new path coupling theorem

Proof of Thm 1

Thm 1: If $p_{i,j} = r_i \geq 1/2 \quad \forall\, i < j$, then M is rapidly mixing.

M' can swap pairs that are not nearest neighbors

- maintain same stationary distribution

Permutation σ:

5	6	3	1	2	7	4

Permutation τ':

5	6	1	2	3	7	4

$$\frac{P'(\sigma,\tau)}{P'(\tau,\sigma)} = \frac{\pi(\tau)}{\pi(\sigma)}$$

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Proof of Thm 1

Introduction

Biased Permutations

Nanoscience

Colloids

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$$\boxed{P'(\sigma,\tau) = p_{2,3}}$$

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Permutation σ:

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$$\frac{P'(\sigma,\tau)}{P'(\tau,\sigma)} = \frac{\pi(\tau)}{\pi(\sigma)} = \left(\frac{\pi(\tau)}{\pi(\tau')}\right)\left(\frac{\pi(\tau')}{\pi(\tau'')}\right)\left(\frac{\pi(\tau'')}{\pi(\sigma)}\right)$$

$$= \left(\frac{\cancel{p_{2,1}}}{\cancel{p_{1,2}}}\right)\left(\frac{p_{2,3}}{p_{3,2}}\right)\left(\frac{\cancel{p_{1,3}}}{\cancel{p_{3,1}}}\right) = \frac{p_{2,3}}{p_{3,2}}$$

Previous work: Uniform sampling

How long does it take to mix?

Permutations:

5	6	3	7	2	1	4
0	0	0	0	0	1	0
0	0	0	0	1	1	0
0	0	1	0	1	1	0
0	0	1	0	1	1	1
1	0	1	0	1	1	1
1	1	1	0	1	1	1
1	1	1	1	1	1	1

1	4	5	3	6	2	7
1	0	0	0	0	0	0
1	0	0	0	0	1	0
1	0	0	1	0	1	0
1	1	0	1	0	1	0
1	1	1	1	0	1	0
1	1	1	1	1	1	0
1	1	1	1	1	1	1

Coupling time (perm) $\leq$ max *{Coupling time(lattice paths)}*