HAZARDOUS BACKWARD IN TIME CONTINUATION IN NONLINEAR PARABOLIC EQUATIONS AND DEBLURRING NONLINEARLY BLURRED IMAGERY.

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Identify sources of groundwater pollution

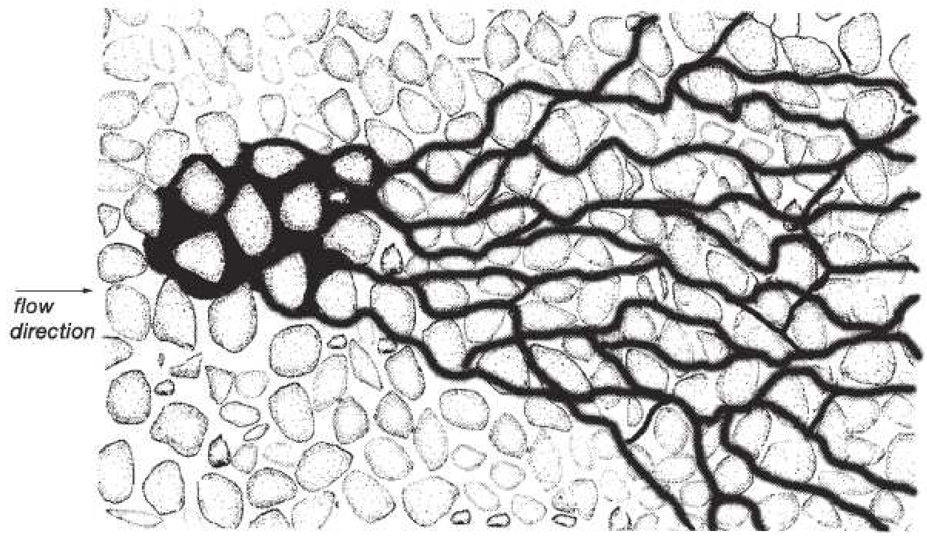


Fig. 1 Contaminant transported in porous media

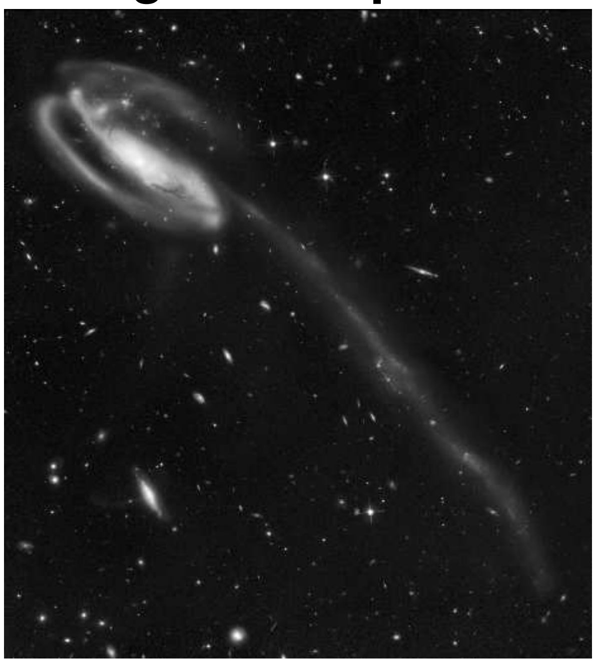
Solve Advection Dispersion Equation backward in time, given present state $g(x, y)$:

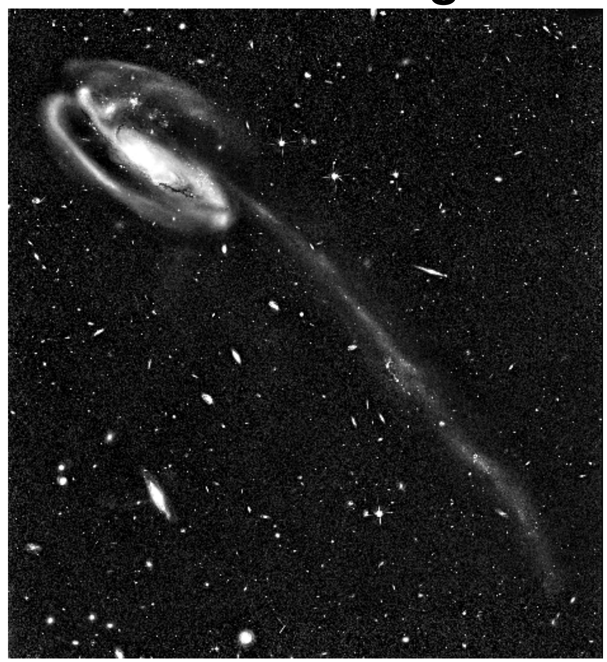
$$C_t = \nabla.\{D\nabla C\} - \nabla.\{vC\}, \qquad 0 < t \leq T,$$

$$C(x, y, T) = g(x, y). \tag{1}$$

DEBLURRING HUBBLE GALAXY IMAGES

Original Tadpole **Deblurred Image**





Solve logarithmic diffusion equation backward in time, given blurred image $g(x,y)$:

$$w_t = -\left[\lambda \log\{1+\gamma(-\Delta)^{\beta}\}\right] w, \qquad 0 < t \leq T,$$
$$w(x,y,T) = g(x,y). \tag{2}$$

BACKWARD PARABOLIC EQNS VERY ILL-POSED

BACKWARD UNIQUENESS AND STABILITY ???

Linear or nonlinear parabolic equn $w_t = \mathcal{L}w,\ t > 0$.

Let $w_1(x,t),\ w_2(x,t)$ be any two solutions.

Let $F(t) = \| w_1(.,t) - w_2(.,t) \|_2$ (L^2 norm)

Logarithmic Convexity techniques lead to

$F(t) \leq \{F(0)\}^{1-\mu(t)}\{F(T)\}^{\mu(t)},\ 0 \leq t \leq T.$

Hölder exponent $\mu(t)$ satisfies $0 \leq \mu(t) \leq 1,\ \mu(T) = 1,$

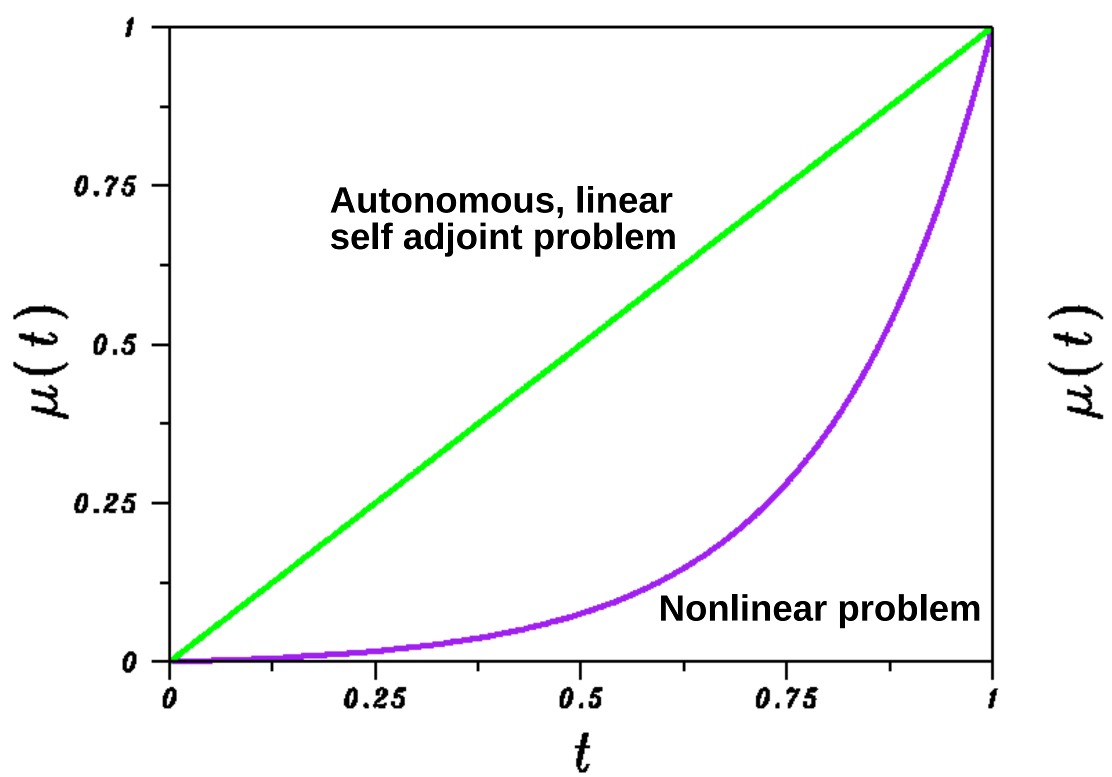
$\mu(0) = 0,\ \mu(t) > 0$ for $t > 0$, and $\mu(t) \downarrow 0$ as $t \downarrow 0$.

If solutions satisfy prescribed bound $\| w(.,0) \|_2 \leq M$, then $F(0) \leq 2M$. Hence $F(T) = 0 \Rightarrow F(t) = 0$ on $[0,T]$. **This implies Backward Uniqueness**.

Fix any small $\delta > 0$. Then $F(T) \leq \delta \Rightarrow F(t) \leq 2M^{1-\mu(t)}\delta^{\mu(t)}$ on $[0,T]$. **This implies Backward Stability.**

Precarious backward stability $F(t) \leq 2M^{1-\mu(t)}\delta^{\mu(t)}$**.**
Autonomous, selfadjoint $\Rightarrow$ $\mu(t) = t/T$. Otherwise, $\mu(t)$ **sublinear** in t, possibly with fast decay to zero.

Behavior of Holder exponent in backward problems



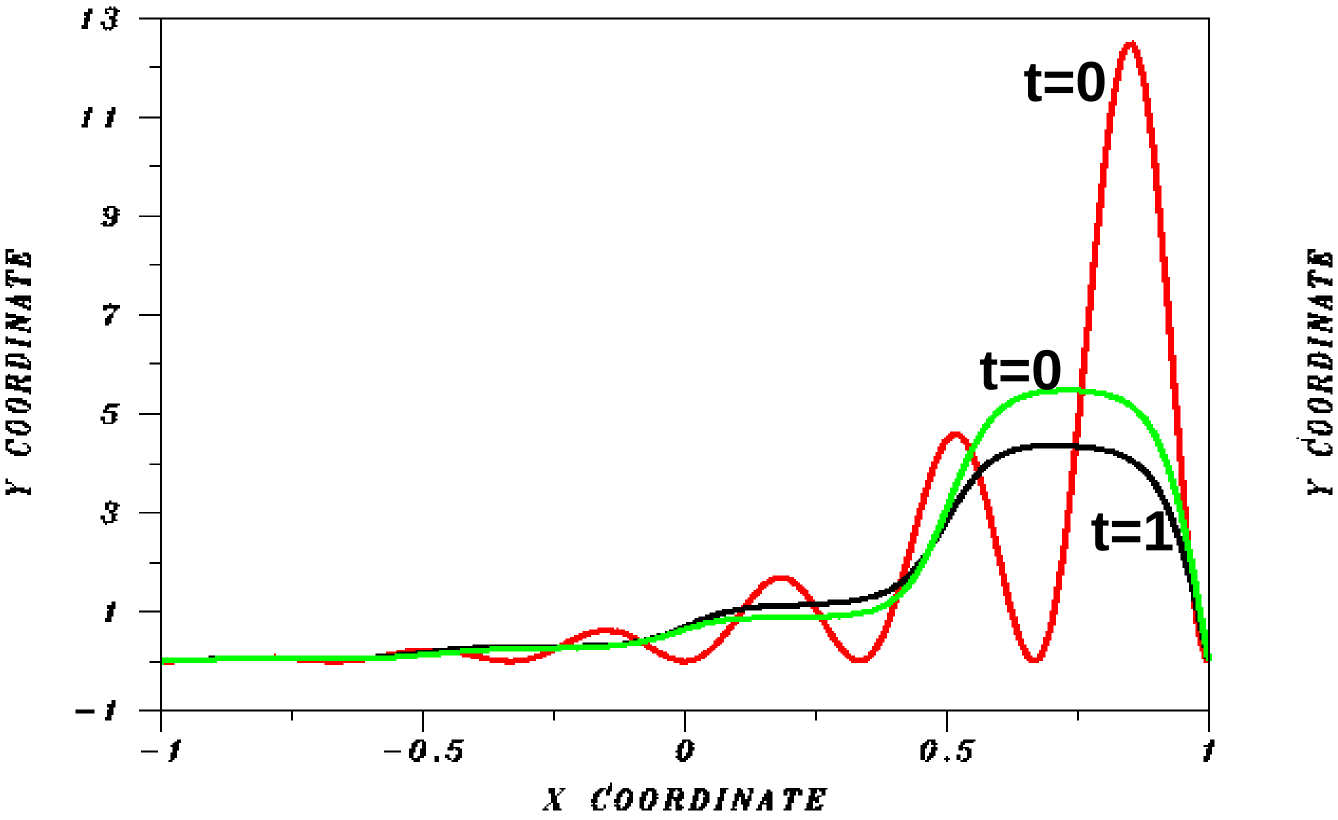
$$w_t = e^{ct} w_{xx},\ \ 0 < x < \pi,\ \ w_x(0,t) = w_x(\pi,t) = 0,\ \ t \geq 0.$$

$$\mu(t) = \{1 - \exp(ct)\}/\{1 - \exp(cT)\},\ \ 0 \leq t \leq T.$$

$$w_t = 0.05\left\{e^{(0.025x+0.05t)}w_x\right\}_x + \{\sin(4\pi x)\}\,w_x,\ \ t \geq 0,$$

$$w^{red}(x,0) = e^{3x}\sin^2(3\pi x),\ \ w(-1,t) = w(1,t) = 0.$$

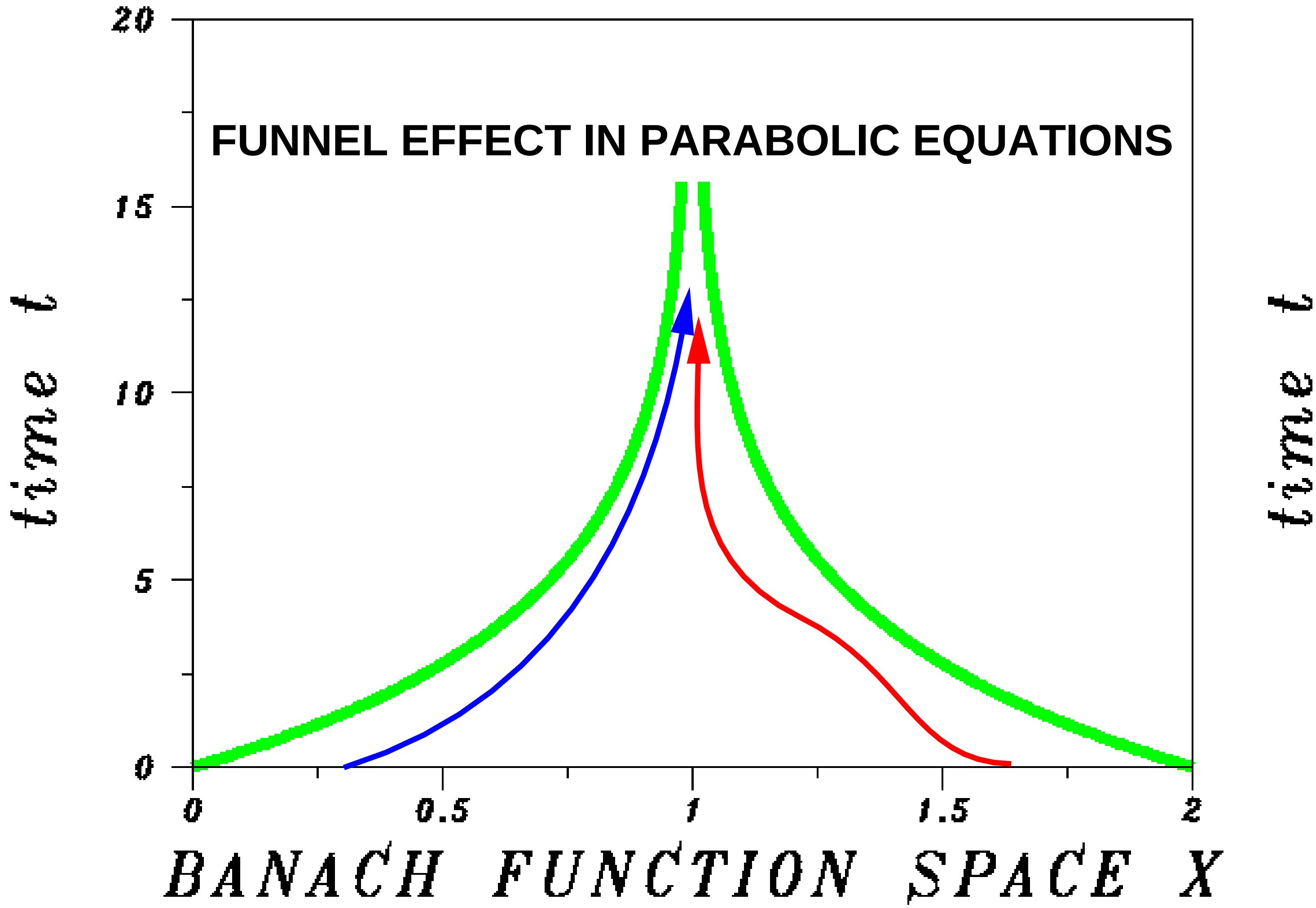
Effective non uniqueness in non self adjoint case



Either red or green initial values at t=0, terminate on black curve at t=1 to within 1.4E−3 pointwise, and L_2 relative error = 2.3E−4.

THE FUNNEL CONJECTURE

FORWARD EVOLUTION IN PARABOLIC EQUATIONS



EXPLORE 2D BACKWARD CONTINUATION

Nonlinear parabolic initial value problem on $\Omega \times (0, T)$

$$w_t = \gamma r(w) \nabla . \{q(x, y, t) \nabla w\} + a w w_x, + b(w cos^2 w) w_y,$$

$$w(x, y, 0) = g(x, y).$$

Ω square $0 < x, y < 1$, $T > 0$, Zero Neumann on $\partial\Omega$.

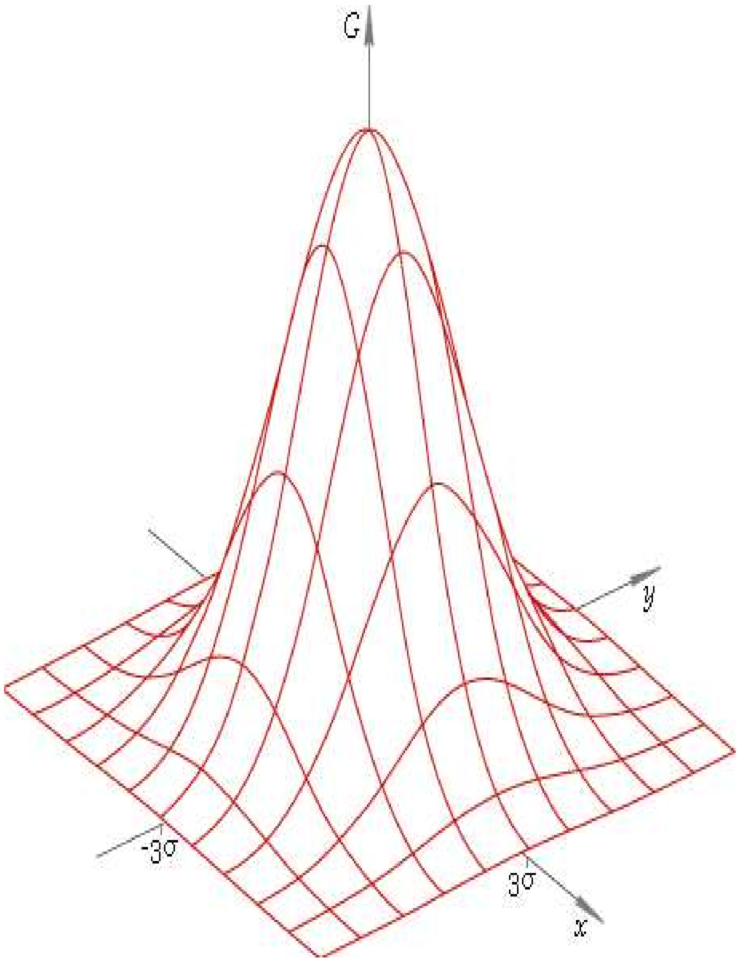
$r(w) = \exp(0.025w)$, $\quad q(x, y, t) = e^{10t}(1 + 5e^{2y} sin \pi x)$.

$\gamma = 8.5 \times 10^{-4}$, a, b constants ≥ 0, yet to be prescribed.

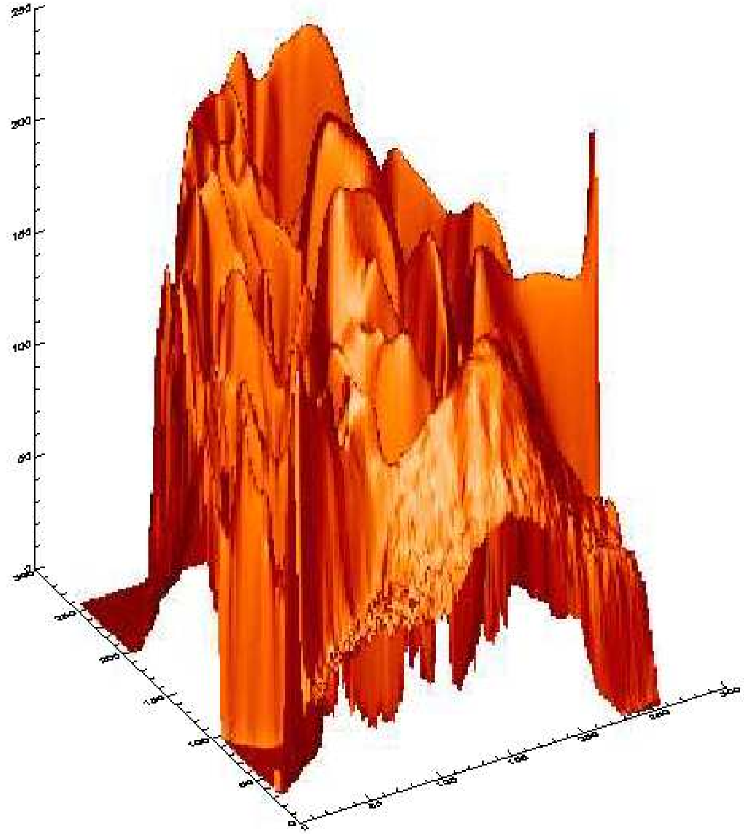
FIND INTERESTING INITIAL VALUES $g(x, y)$????

EXAMPLES OF DATA

Gaussian inital data



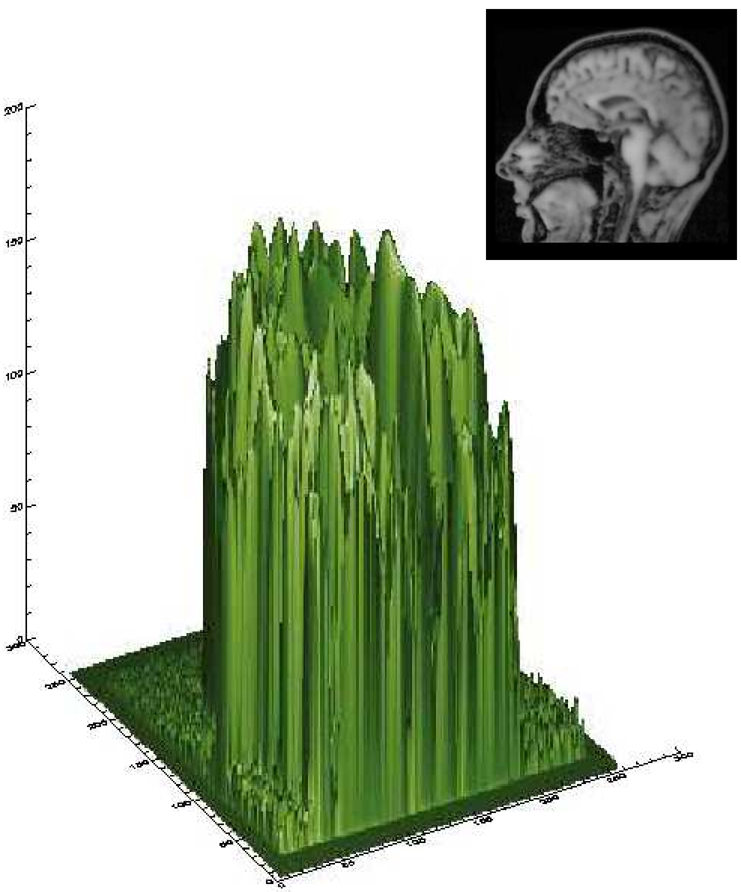
More interesting data



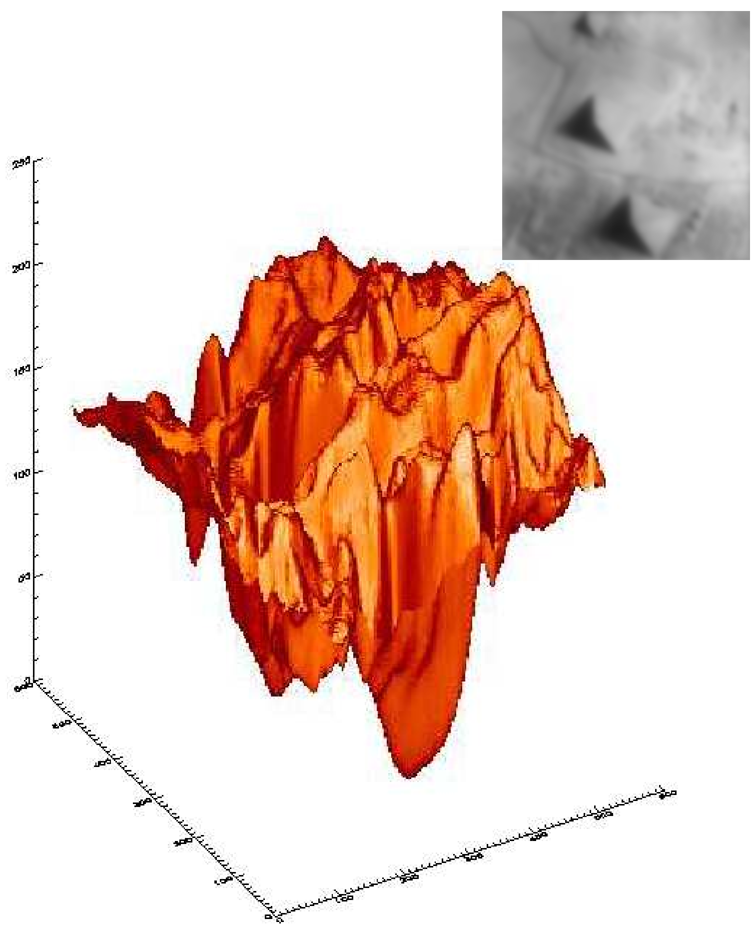
MORE EXAMPLES OF DATA

8 bit images with values between 0 and 255.

MRI Brain image



Pyramids image



NONLINEAR PARABOLIC IMAGE BLURRING

$w_t = \gamma r(w)\nabla.\{q(x,y,t)\nabla w\} + aww_x + b(wcos^2 w)w_y$

Forward problem well-posed in $L^2(\Omega)$. For any initial value $w(x,y,0) = g(x,y)$, unique solution $w(x,y,T)$ exists at time T. **Nonlinear solution operator Λ^T well-defined on $L^2(\Omega)$ by formula $\Lambda^T g(x,y) = w(x,y,T)$.**

$w(x,y,T) \in$ **very restricted class** of smooth functions.

Restrict attention to 256×256 images. Using centered space differencing, with $\Delta x = \Delta y = 1/256$, $\Delta t = 3.0E{-}7$, march forward 400 time steps to $T = 1.2E{-}4$, using following **explicit finite difference scheme**

$$\begin{aligned} W^{n+1} &= W^n + \Delta t\gamma R(W^n)\nabla.\{Q^n\nabla W^n\} + aW^nW^n_x \\ &+ b(W^n cos^2 W^n)W^n_y, \qquad n = 0,399, \\ W^0 &= g(x,y). \end{aligned} \tag{3}$$

Discrete nonlinear solution operator Λ^T_d well-defined on 256×256 images by formula $\Lambda^T_d W^0 = W^{400}$.

NONLINEAR BACKWARD CONTINUATION

Very little known about **nonlinear multidimensional** backward in time parabolic equations. Computation of such problems lies in **uncharted waters**.

$$w_t = \gamma r(w)\nabla.\{q(x,y,t)\nabla w\} + aww_x + b(wcos^2 w)w_y$$

Solution operator $\Lambda^T g(x,y) = w(x,y,T)$, defined on L^2.

VanCittert iterative procedure
Originated in Spectroscopy in 1930's. 1D Linear convolution integral equations, with explicitly known kernels, assumed to have **positive** Fourier transforms.

With $f(x,y) \approx w(x,y,T)$, fixed $0 < \lambda < 1$, consider

$$h^{m+1}(x,y) = h^m(x,y) + \lambda \left\{ f(x,y) - \Lambda^T h^m(x,y) \right\},\ m \geq 1,$$

where $h^1(x,y) = \lambda f(x,y)$.

Convergence $\Rightarrow \Lambda^T h^\infty(x,y) = f(x,y)$**.**

Mathematically impossible with noisy data $f(x,y)$**.**

Nevertheless, Van Cittert is valuable tool in large variety of 2D nonlinear backward equations. In many cases, L^{∞} norm of residual, $\| f - \Lambda^T h^m \|_{\infty}$, decays quasi-monotonically to small value after a finite number N of iterations, and $h^N(x,y)$ is useful approximation to $w(x,y,0)$. Cases also exist where **method fails !!!**.

VANCITTERT NONLINEAR DEBLURRING

Given 256×256 blurred image $f(x,y) \approx w(x,y,T)$, use above centered difference scheme to do

$$*** \; H^{m+1} = H^m + \lambda \left\{ f(x,y) - \Lambda_d^T H^m \right\}, \; m \geq 1, \; ***$$

where $H^1 = \lambda f(x,y)$. Here, $\Lambda_d^T H^m$ means marching explicit difference scheme 400 time steps forward, using image $H^m(x,y)$ as initial data W^0.

Self-regularizing property of VanCittert Deblurring

Iterative process recovers **low frequency** information in first several iterations. Many more iterations needed to recover high-frequency information. Interactively stopping iteration after finitely many steps, can recover deblurred image relatively free from noise.
(**Unless method fails for that image !!!**)

Diagnostic image metrics L^1 and TV norms
Blurring and deblurring experiments will use following norms to evaluate results

$$\| f \|_1 = \left\{ (256)^{-2} \sum_{x,y=1}^{256} |f(x,y)| \right\},$$

$$\| \nabla f \|_1 = (256)^{-2} \sum_{x,y=1}^{255} \left(\{f_x(x,y)\}^2 + \{f_y(x,y)\}^2 \right)^{1/2}.$$

Peak signal to noise image quality metric (PSNR)

ASSUMES KNOWN IDEAL IMAGE $k(x,y)$.

Let $h(x,y)$ be degraded version of ideal $k(x,y)$,

$$*** \; PSNR = -20 \log_{10}\{\| h - k \|_2 /255\} \; ***.$$

$PSNR = \infty$ if $h(x,y) = k(x,y)$, and $PSNR$ decreases monotonically as $h(x,y)$ diverges from $k(x,y)$.

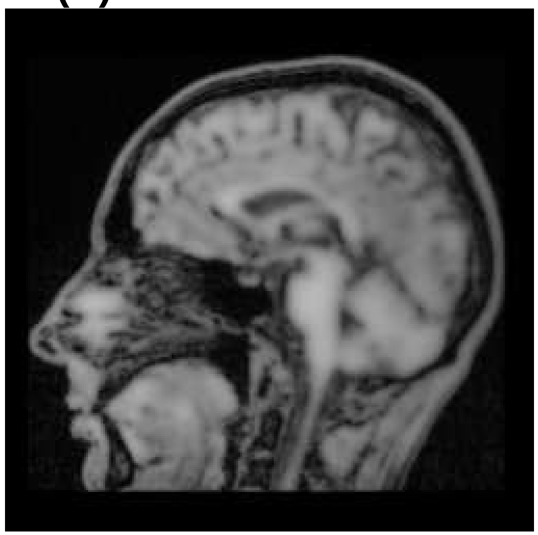
$$w_t = \gamma r(w) \nabla . \{q(x,y,t) \nabla w\} + aww_x + b(wcos^2 w)w_y$$

$$r(w) = \exp(0.025w), \quad q(x,y,t) = e^{10t}(1 + 5e^{2y} sin \pi x).$$

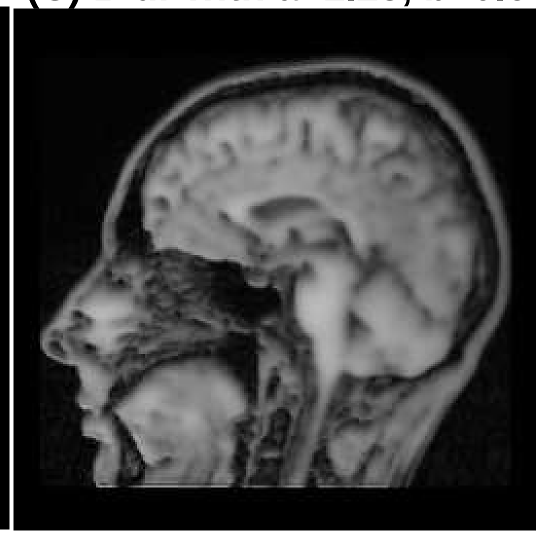
NONLINEAR PARABOLIC BLURRING OF SHARP MRI BRAIN IMAGE

(A) Original sharp image

(B) Blur with a=b=0



(C) Blur with a=1.25, b=0.6

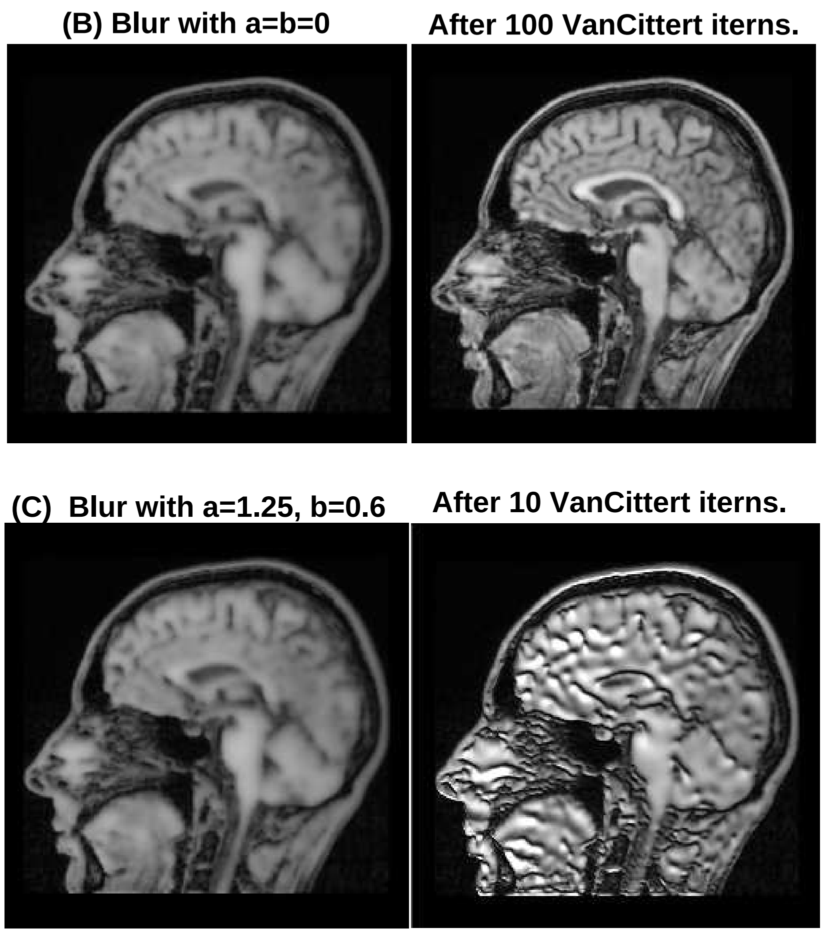


Behavior in above nonlinear blurring

Image	$\parallel f \parallel_1$	$\parallel \nabla f \parallel_1$	*PSNR*
Sharp A	59	3360	∞
Blurred B	55	1740	25
Blurred C	55	1770	20

$$w_t = \gamma r(w)\nabla.\{q(x,y,t)\nabla w\} + aww_x + b(wcos^2 w)w_y$$

NONLINEAR DEBLURRING OF MRI BRAIN IMAGE



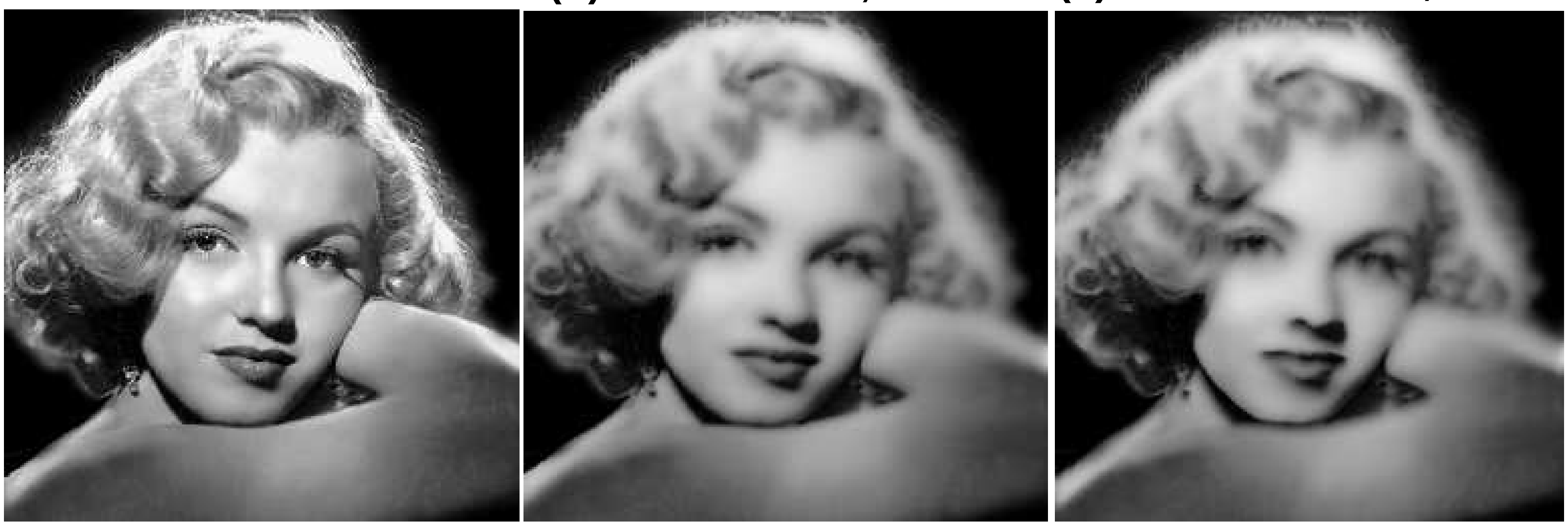
Behavior in above nonlinear deblurring

Image	$\| f \|_1$	$\| \nabla f \|_1$	*PSNR*
Deblurred B	59	2980	34
Deblurred C	63	4780	17

$$w_t = \gamma r(w)\nabla.\{q(x,y,t)\nabla w\} + aww_x + b(wcos^2 w)w_y$$

NONLINEAR PARABOLIC BLURRING OF SHARP FACE IMAGE

(D) Original sharp image **(E) Blur with a=0, b=0.6** **(F) Blur with a=0.83, b=0.6**



Behavior in above nonlinear blurring

Image	$\parallel f \parallel_1$	$\parallel \nabla f \parallel_1$	*PSNR*
Sharp D	107	3100	∞
Blurred E	101	1580	24
Blurred F	101	1550	20

$$w_t = \gamma r(w)\nabla.\{q(x,y,t)\nabla w\} + aww_x + b(wcos^2 w)w_y$$

NONLINEAR DEBLURRING OF FACE IMAGE

(E) Blur with a=0, b=0.6

After 100 VanCittert iterns.

(F) Blur with a=0.83, b=0.6

After 20 VanCittert iterns.

Behavior in above nonlinear deblurring

Image	$\parallel f \parallel_1$	$\parallel \nabla f \parallel_1$	*PSNR*
Deblurred E	106	2580	29
Deblurred F	112	5800	18

$$w_t = \gamma s(w)\nabla.\{q(x,y,t)\nabla w\} + c\sqrt{|w|}w_x + d(w cos^2 w)w_y$$
$$s(w) = 1.0 + 0.00125w^2, \quad q = e^{10t}(1 + 5e^{2y} sin \pi x).$$

NONLINEAR PARABOLIC BLURRING OF SHARP CARRIER IMAGE

(G) Original sharp image

(H) Blur with c=2.5, d=0.3

(I) Blur with c=2.5, d=1.5

Behavior in above nonlinear blurring

Image	$\| f \|_1$	$\| \nabla f \|_1$	*PSNR*
Sharp G	139	4760	∞
Blurred H	134	1720	20
Blurred I	134	1770	20

$$w_t = \gamma s(w)\nabla.\{q(x,y,t)\nabla w\} + c\sqrt{|w|}w_x + d(wcos^2 w)w_y$$

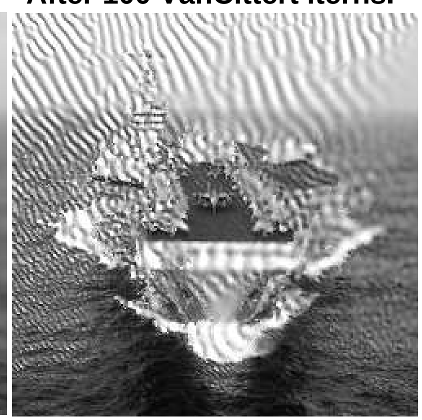
NONLINEAR DEBLURRING OF CARRIER IMAGE

(H) Blur with c=2.5, d=0.3

After 100 VanCittert iterns.

(I) Blur with c=2.5, d=1.5

After 100 VanCittert iterns.



Behavior in above nonlinear deblurring

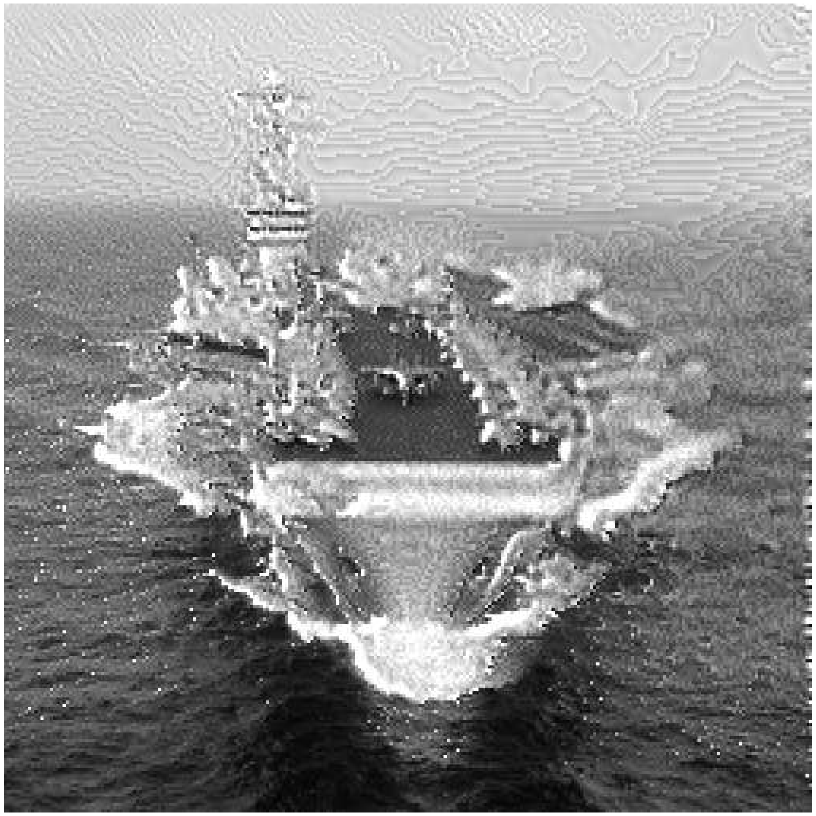
Image	$\parallel f \parallel_1$	$\parallel \nabla f \parallel_1$	*PSNR*
Deblurred H	137	3700	23
Deblurred I	141	7700	18

$$w_t = \gamma s(w)\nabla.\{q(x,y,t)\nabla w\} + c\sqrt{|w|}w_x + d(w cos^2 w)w_y$$

DEBLURRING CARRIER IMAGE WITH FALSE VALUES

Use false c=2.5, d=0.3, to deblur image blurred with c=2.5, d=1.5.

After 100 VanCittert iterations



FAILED BLIND DECONVOLUTION EXPERIMENT

Blind deconvolution of KITT PEAK M51 image

Original M51

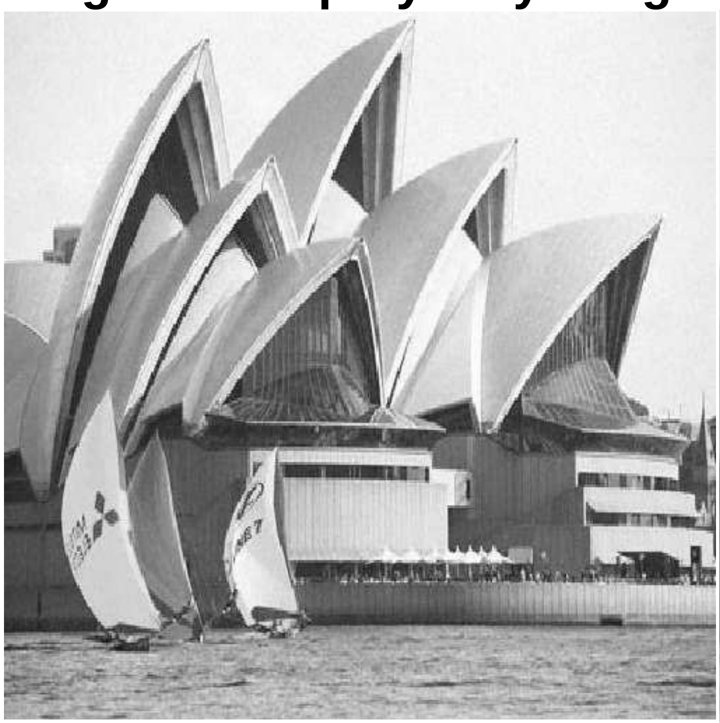
After Linnik deblur

LINEAR NONSELFADJOINT PARABOLIC PDE

$$w_t = \gamma r \nabla . \{q(x, y, t)\nabla w\} + a w_x + b w_y$$
$$r = 30, \quad q = e^{10t}(1 + 5e^{2y} sin \pi x) \quad a = 65, \quad b = 35.$$

LINEAR NONSELFADJOINT PARABOLIC BLURRING

Original sharp Sydney image



Blurred Sydney image

LINEAR NONSELFADJOINT PARABOLIC PDE

$$w_t = \gamma r \nabla . \{q(x, y, t)\nabla w\} + a w_x + b w_y$$
$$r = 30, \quad q = e^{10t}(1 + 5e^{2y} sin \pi x) \quad a = 65, \quad b = 35.$$

LINEAR NONSELFADJOINT PARABOLIC DEBLURRING

Blurred Sydney image

After 20 VanCittert iterations