Split digraphs and their applications

M. Drew LaMar
The College of William and Mary
Quantitative Biology Laboratory
Department of Biology

National Institute of Standards and Technology
Gaithersburg, MD
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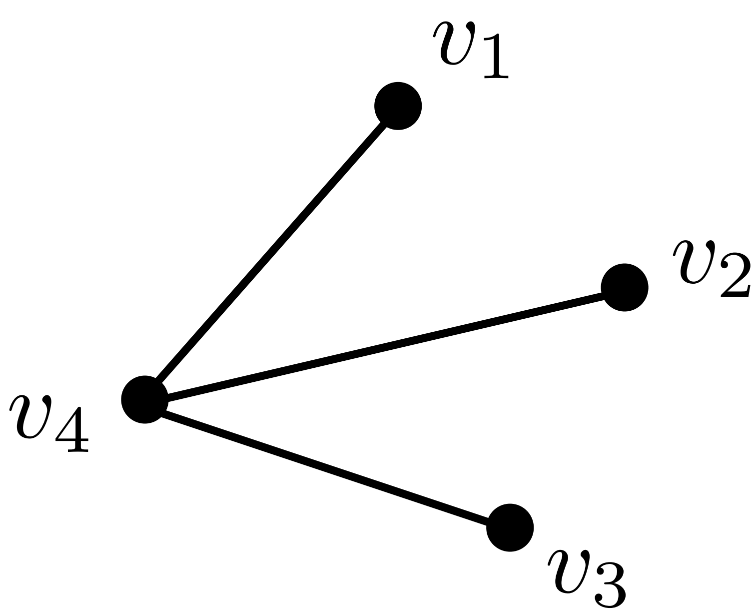
Simple graphs and degree sequences

Undirected

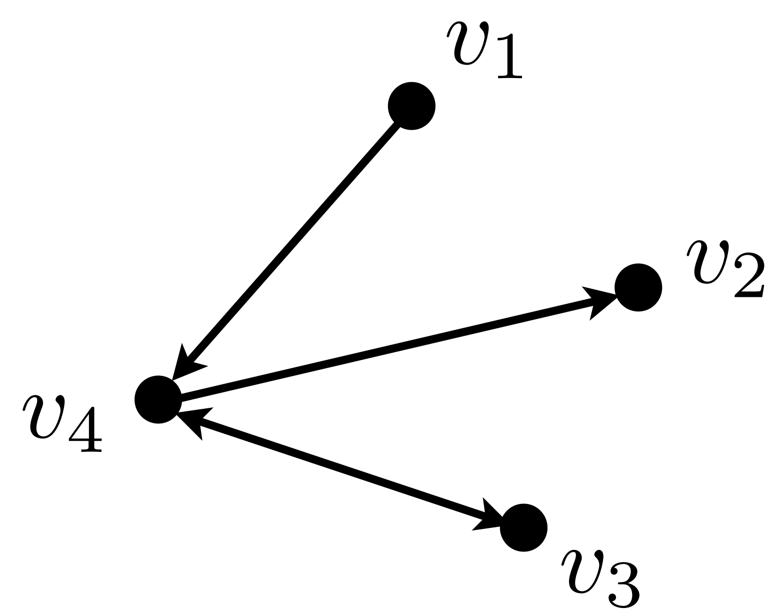
$$G = (V, E)$$



$$d = \begin{pmatrix} 1 & 1 & 1 & 3 \end{pmatrix}$$

Directed

$$\vec{G} = (V, A)$$



$$d = \begin{pmatrix} d^+ \\ d^- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Split graphs

A graph is split if it can be partitioned into a clique and an independent set:

- Subset of perfect graphs
- Superset of threshold graphs
- Only chordal graphs whose complements are also chordal
- $(2K_2, C_4, C_5)$-free
- A graph is split if and only if its degree sequence satisfies a particular Erdős-Gallai inequality with equality

Graphic sequences

Theorem [Erdős-Gallai (1961)]:

Let d be a non-increasing integer sequence. Then d is graphic if and only if $\sum_{i=1}^{|V|} d_i$ is even and for $k = 1, \ldots, |V|$,

$$k(k-1) + \sum_{j=k+1}^{|V|} \min\{k, d_i\} \geq \sum_{i=1}^{k} d_i.$$

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Slack sequence

$$\boxed{s_k \equiv k(k-1) + \sum_{j=k+1}^{|V|} \min\{k, d_i\} - \sum_{i=1}^{k} d_i}$$

Characterizations of split graphs

Definition [Földes and Hammer (1977)]:

A graph G is **split** if and only if $V(G)$ is a disjoint union of two sets C and I such that C is a clique and I is an independent set. In this case, $\mathcal{X} = \{C, I\}$ is called a **split partition**.

$$\begin{array}{cc} & \begin{array}{cc} C & I \end{array} \\ \begin{array}{c} C \\ I \end{array} & \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix} \end{array}$$

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Theorem [Hammer and Simeone (1981)]:

If d is the degree sequence of a graph G, then G is split if and only if $s_m = 0$.

Digraphic sequences

Permutations:

$$\bar{d}_i = d_{a_i}, \qquad \underline{d}_i = d_{b_i}$$

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Example:

$$d = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \qquad \bar{d} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \qquad \underline{d} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

$$a = \begin{pmatrix} 4 & 3 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 4 & 3 & 2 & 1 \end{pmatrix}$$

Digraphic sequences

Theorem [Fulkerson (1960)]:

An integer-pair sequence d is digraphic if and only if $\sum_{i=1}^{N} d_i^+ = \sum_{i=1}^{N} d_i^-$ and for $k = 1, \ldots, N$,

$$\sum_{i=1}^{k} \min[\underline{d}_i^+, k-1] + \sum_{i=k+1}^{N} \min[\underline{d}_i^+, k] \geq \sum_{i=1}^{k} \underline{d}_i^- .$$

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Slack sequences

$$\begin{aligned} \bar{s}_k &= \sum_{i=1}^{k} \min[\bar{d}_i^-, k-1] + \sum_{i=k+1}^{N} \min[\bar{d}_i^-, k] - \sum_{i=1}^{k} \bar{d}_i^+ \\ \underline{s}_k &= \sum_{i=1}^{k} \min[\underline{d}_i^+, k-1] + \sum_{i=k+1}^{N} \min[\underline{d}_i^+, k] - \sum_{i=1}^{k} \underline{d}_i^- \end{aligned}$$

Structural characterization of split digraphs

Definition [Split digraph]:

Given a digraph $\vec{G}$, a vertex partition $\mathcal{X} = \{X^{\pm}, X^{+}, X^{-}, X^{0}\}$ (with possible empty sets) is called a **split partition** of $\vec{G}$ if and only if

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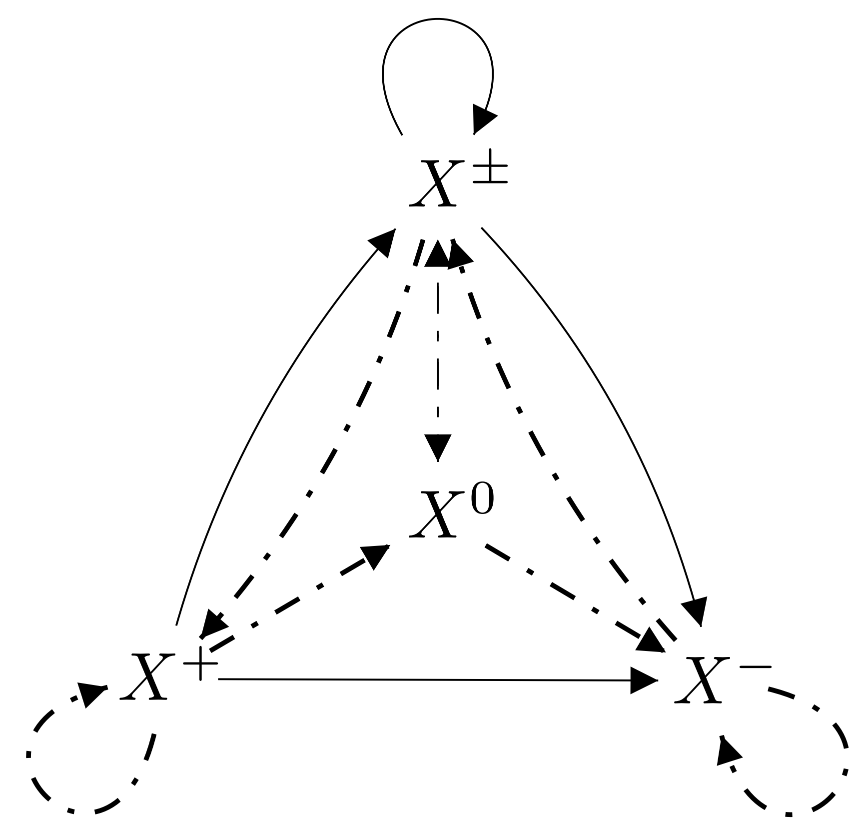
$$
\begin{array}{c c}
 & \begin{array}{cccc} X^{\pm} & X^{+} & X^{-} & X^{0} \end{array} \\
\begin{array}{c} X^{\pm} \\ X^{+} \\ X^{-} \\ X^{0} \end{array} &
\begin{pmatrix} 1 & * & 1 & * \\ 1 & * & 1 & * \\ * & 0 & * & 0 \\ * & 0 & * & 0 \end{pmatrix}
\end{array}
$$



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Theorem [LaMar]:

If d is the degree sequence of a digraph $\vec{G}$, then $\vec{G}$ is split if and only if $\min\{\bar{s}_1, \ldots, \bar{s}_{N-1}, \underline{s}_1, \ldots, \underline{s}_{N-1}\} = 0$.

Splittance of an undirected graph

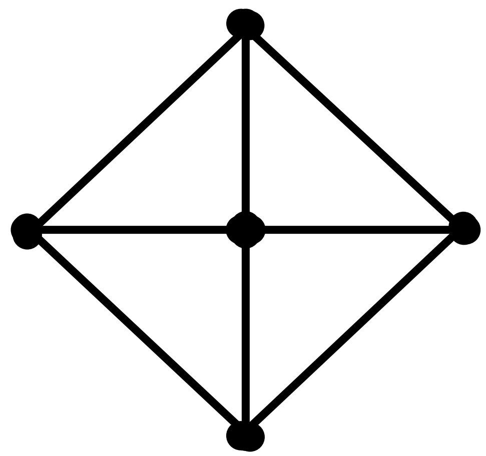
Definition [Graph splittance]:

Define the **splittance** $\sigma(G)$ of G to be the minimum number of edges to add to or remove from G in order to obtain a split graph.

Splittance of an undirected graph

$$d = (4\ 3\ 3\ 3\ 3)$$

1 2 3 4 5



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$$\sigma(d) = (8\ 4\ 2\ 1\ 1\ 2)$$

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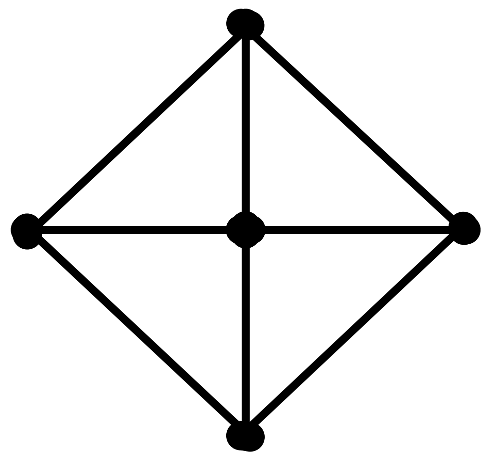

$$\sigma_k(d) = \frac{1}{2}\left\{k(k-1) - \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{N} d_i\right\}$$

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$m(d)$

Corrected Durfee number:

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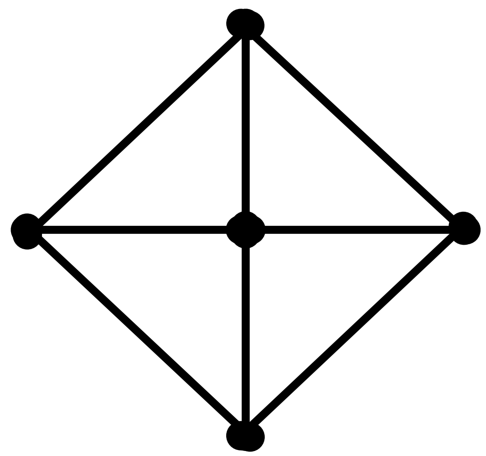

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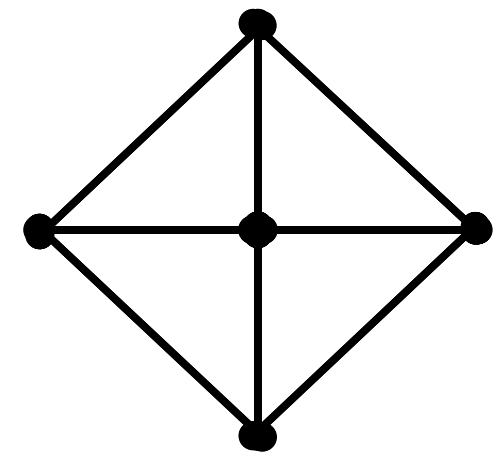

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$$s = (0\ 0\ 1\ 2\ 2\ 0)$$

0 1 2 3 4 5

Theorem [Hammer and Simeone (1981)]:

$$\sigma(G) = \min_k \sigma_k(d) = \sigma_m(d) = \frac{1}{2} s_m(d)$$

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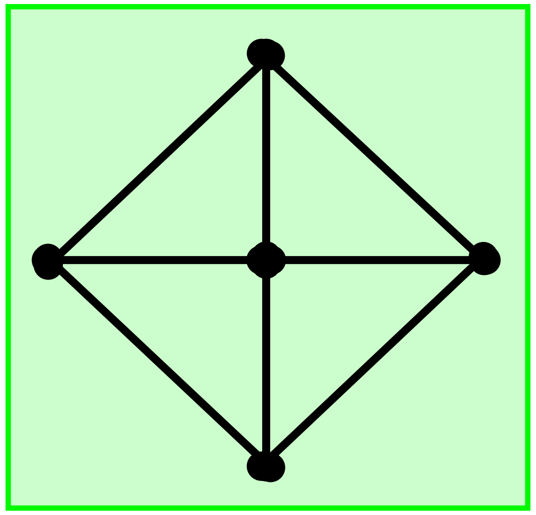

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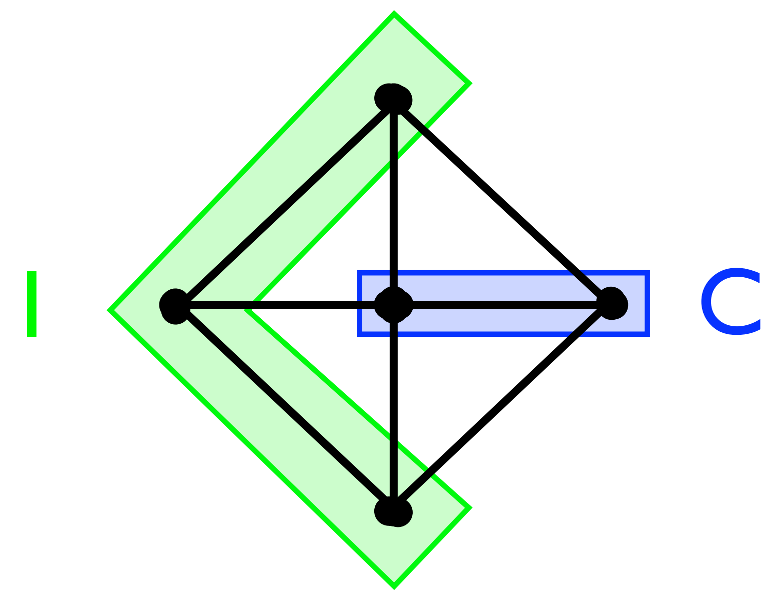

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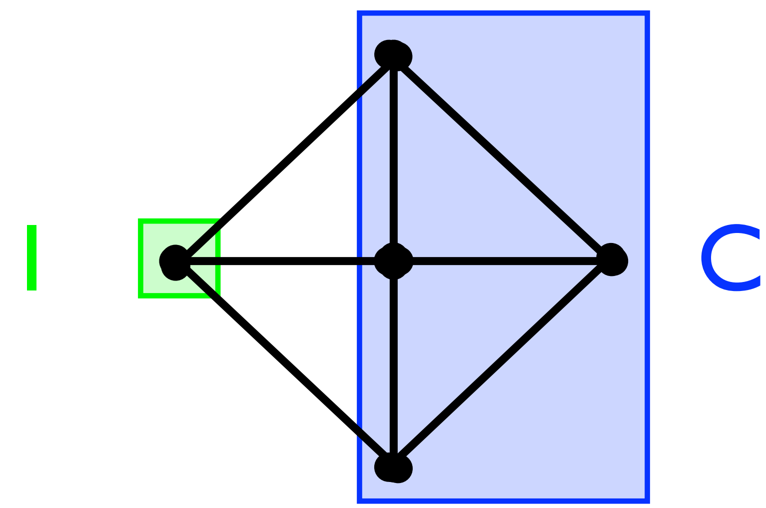

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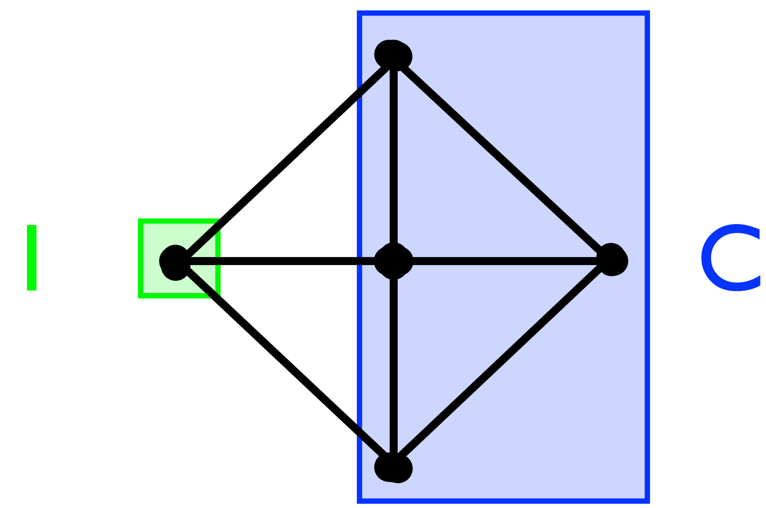

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NOT SPLIT!

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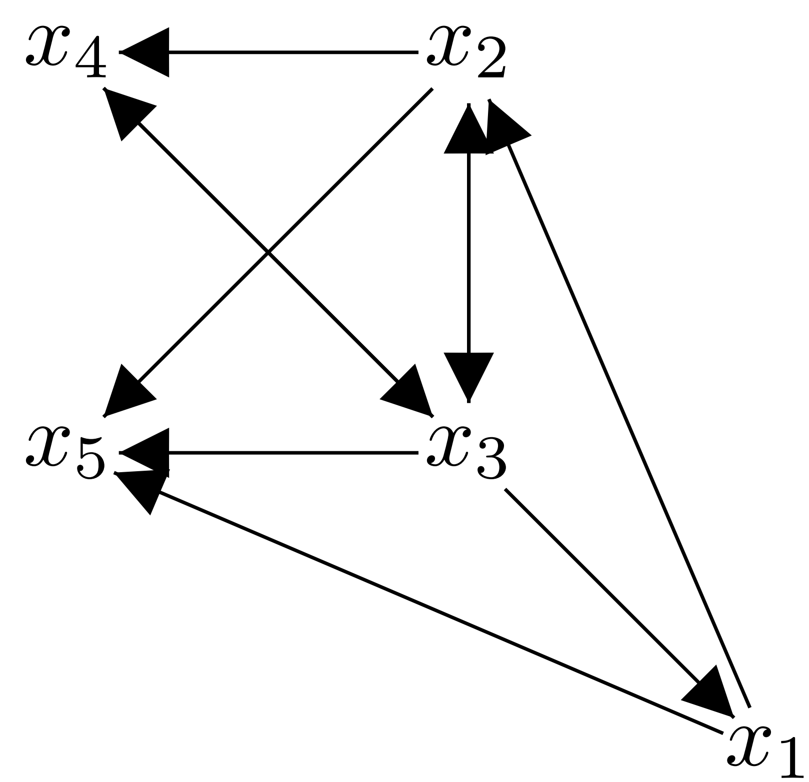
Splittance of a directed graph

Definition [Digraph splittance]:

Define the **splittance** $\sigma(\vec{G})$ of $\vec{G}$ to be the minimum number of arcs to add to or remove from $\vec{G}$ in order to obtain a split digraph.

Splittance of a directed graph

$$\hat{d} = \begin{array}{c} \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$



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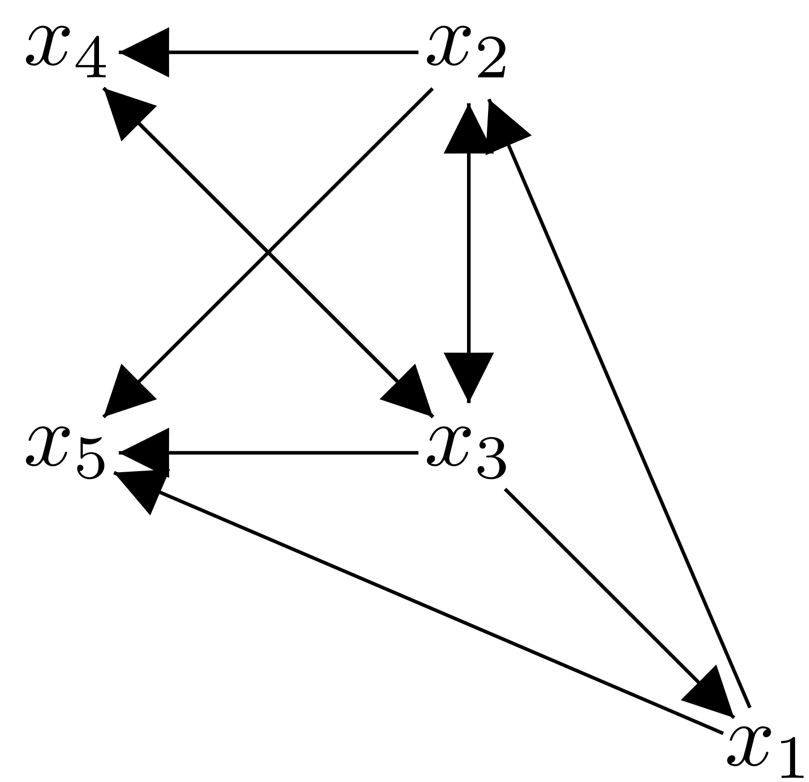
$$S(\hat{d}) = \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix}$$



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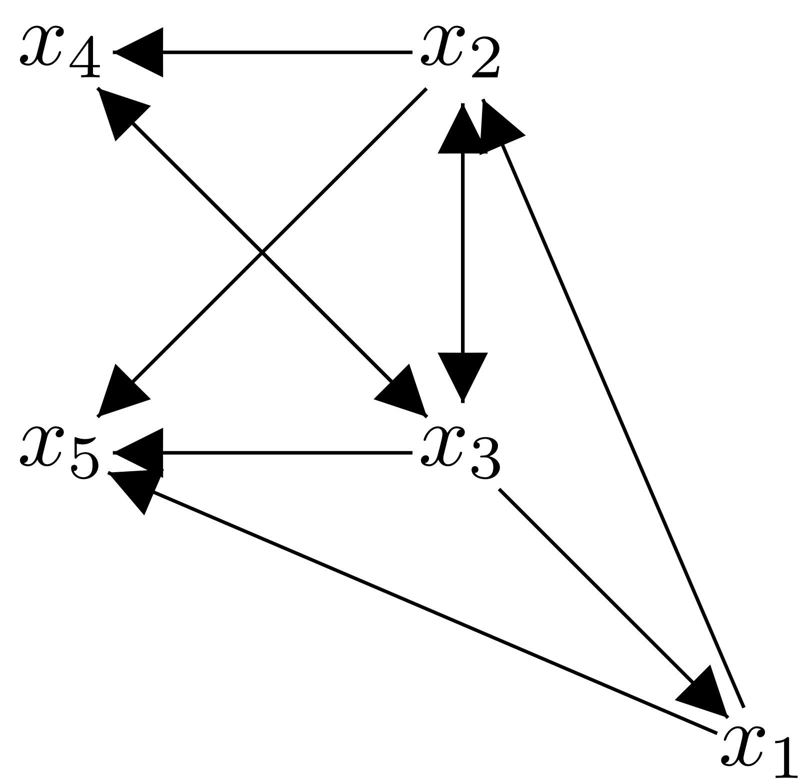

Theorem [LaMar]:

$$\sigma(\vec{G}) = \min_{(k,l) \notin \{(N,0),(0,N)\}} S_{kl}$$

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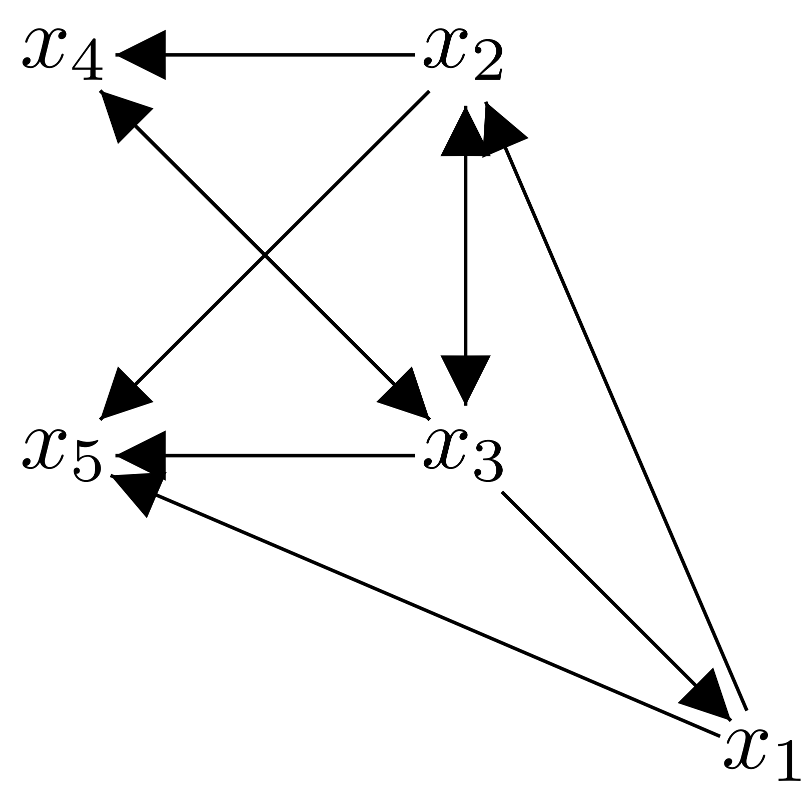
$$a = (3\ 2\ 1\ 4\ 5)$$

$$b = (5\ 3\ 2\ 4\ 1)$$

$$S(\hat{d}) = \begin{array}{cc} \begin{array}{cccccc} & & & & l=4 & \end{array} & \\ \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & \boxed{0} & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix} & \begin{array}{c} \\ \\ k=2 \\ \\ \\ \\ \end{array} \end{array}$$



Splittance of a directed graph

$$\begin{array}{c} \hat{d} = \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \end{array}$$

$$a = (\boxed{3\ 2}\ 1\ 4\ 5) \quad \mathcal{A}_2$$

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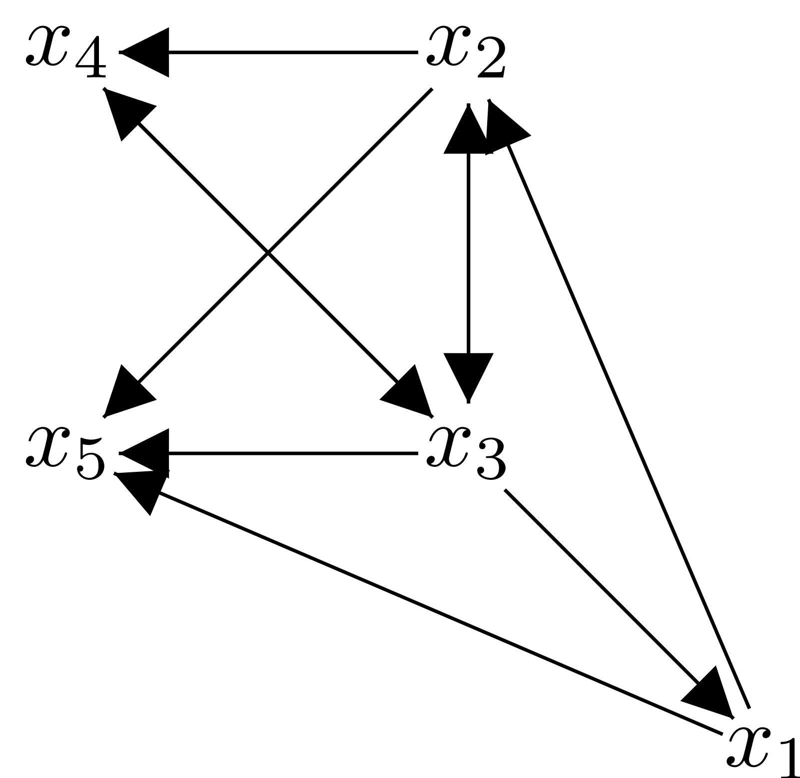

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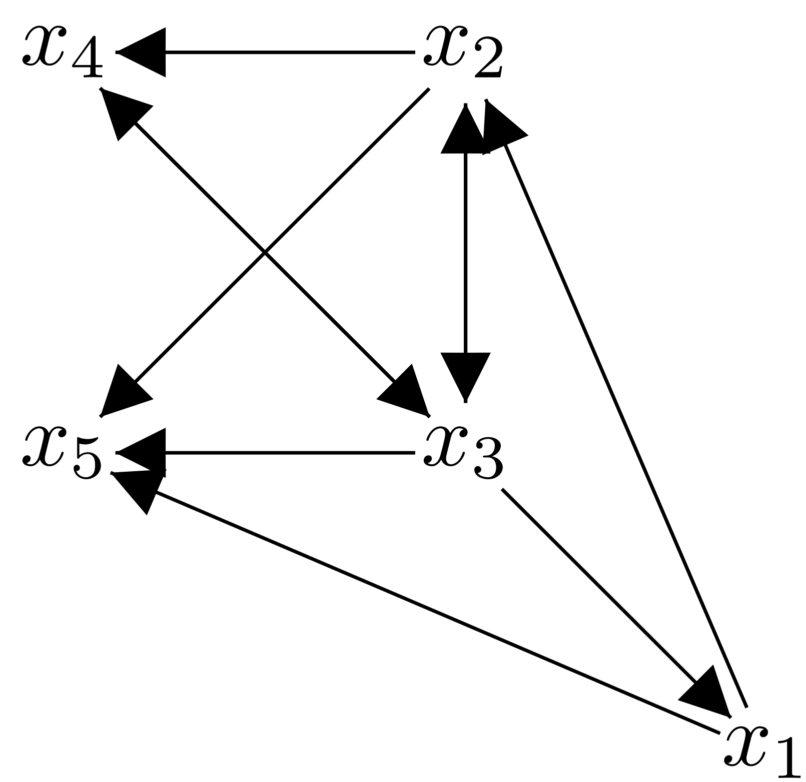
$$S(\hat{d}) = \begin{array}{cc} \begin{array}{cccccc} & & & & \color{red}{l=4} & \end{array} & \\ \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & \boxed{0} & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix} & \begin{array}{c} \\ \\ \color{red}{k=2} \\ \\ \\ \\ \end{array} \end{array}$$



$$X^{\pm} = V_{\mathcal{A}_2 \cap \mathcal{B}_4} = \{x_2, x_3\} \qquad X^{-} = V_{\mathcal{B}_4 \setminus \mathcal{A}_2} = \{x_4, x_5\}$$

$$X^{+} = V_{\mathcal{A}_2 \setminus \mathcal{B}_4} = \emptyset \qquad X^{0} = V_{\{1,\ldots,N\} \setminus \mathcal{A}_2 \cup \mathcal{B}_4} = \{x_1\}$$

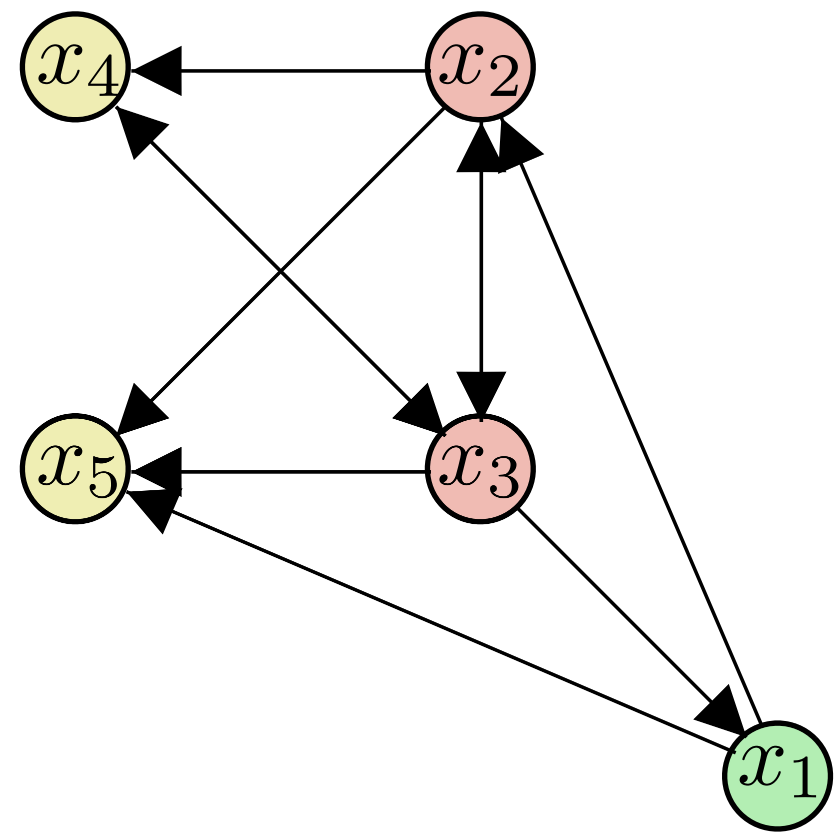
Splittance of a directed graph

$$\begin{array}{cc} \hat{d} = \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \quad\;\; \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \end{array}$$

$$a = (\boxed{3\ 2}\ 1\ 4\ 5)\quad \mathcal{A}_2$$

$$b = (\boxed{5\ 3\ 2\ 4}\ 1)\quad \mathcal{B}_4$$

$$S(\hat{d}) = \begin{array}{c} \begin{matrix} & & & & l = 4 & \end{matrix} \\ \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & \boxed{0} & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix} \end{array} \; k = 2$$



$$X^{\pm} = V_{\mathcal{A}_2 \cap \mathcal{B}_4} = \{x_2, x_3\}$$

$$X^{+} = V_{\mathcal{A}_2 \setminus \mathcal{B}_4} = \emptyset$$

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Splittance of a directed graph

$$\begin{array}{c} \hat{d} = \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \qquad\;\; \begin{matrix} 1 & \;2 & \;3 & \;4 & \;5 \end{matrix} \end{array}$$

$$a = (\boxed{3\ 2}\ 1\ 4\ 5) \quad \mathcal{A}_2$$

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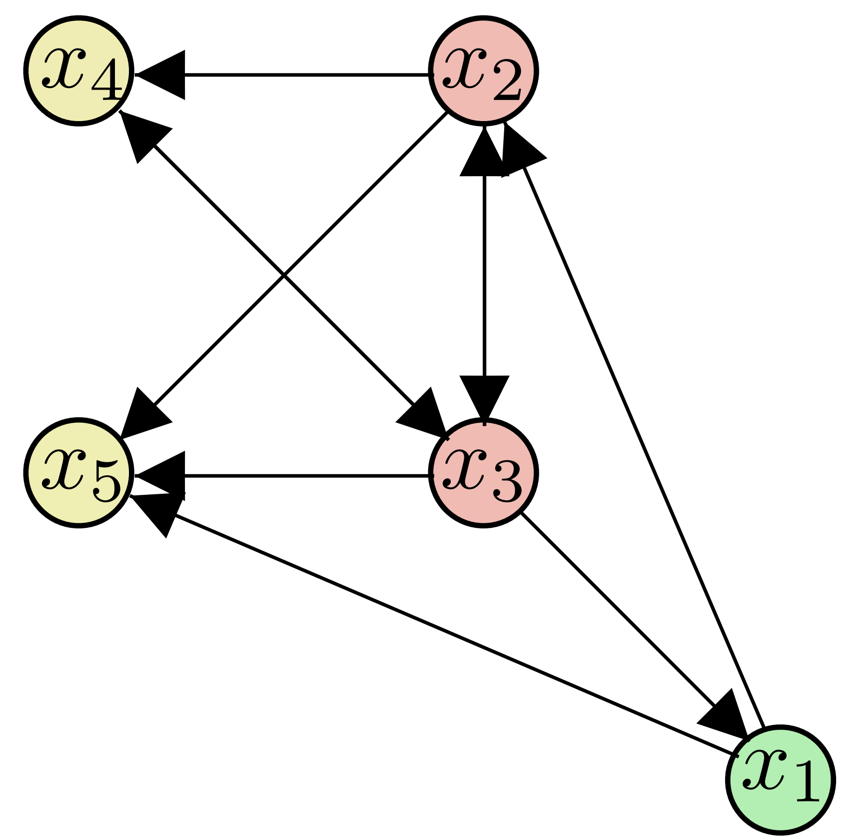


Definition [Split partition measure]:

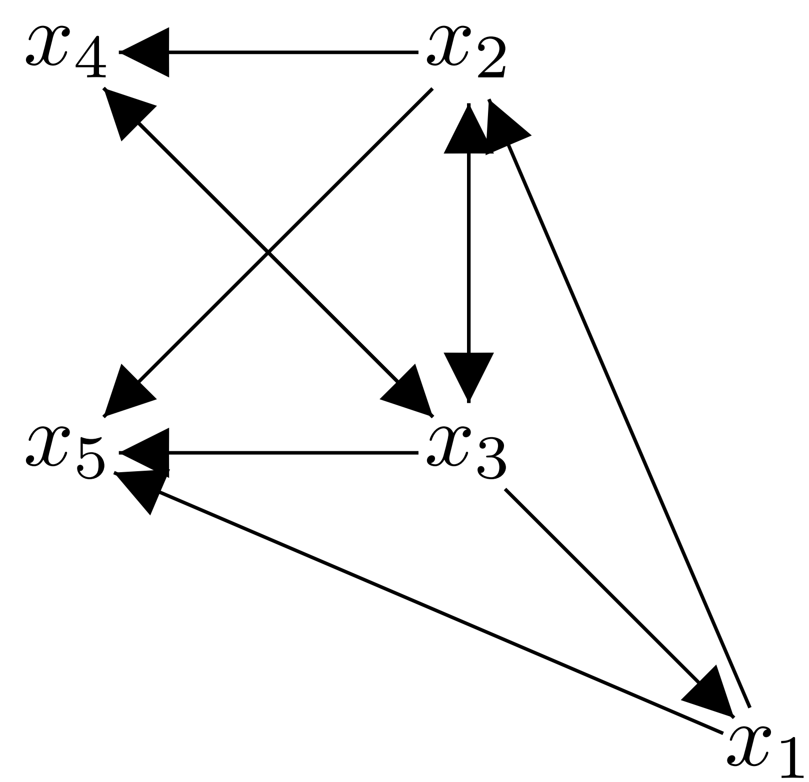
$$\sigma(\mathcal{X}) = |X^{\pm}|\big(k-1\big) + |X^{-}|k + \sum_{X^{+}\cup X^{0}} d_x^{-} - \sum_{X^{\pm}\cup X^{+}} d_x^{+}$$

Splittance of a directed graph

$$\begin{array}{rc} \hat{d} = & \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ & \begin{array}{ccccc} 1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 \end{array} \end{array}$$

$$\boxed{\begin{aligned} \bar{s} &= (0\ 0\ 0\ 1\ 0\ 0) \\ \underline{s} &= (0\ 1\ 1\ 0\ 0\ 0) \end{aligned}}$$

$$S(\hat{d}) = \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix}$$



Splittance of a directed graph

$$\begin{array}{c}\hat{d} = \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}\end{array}$$

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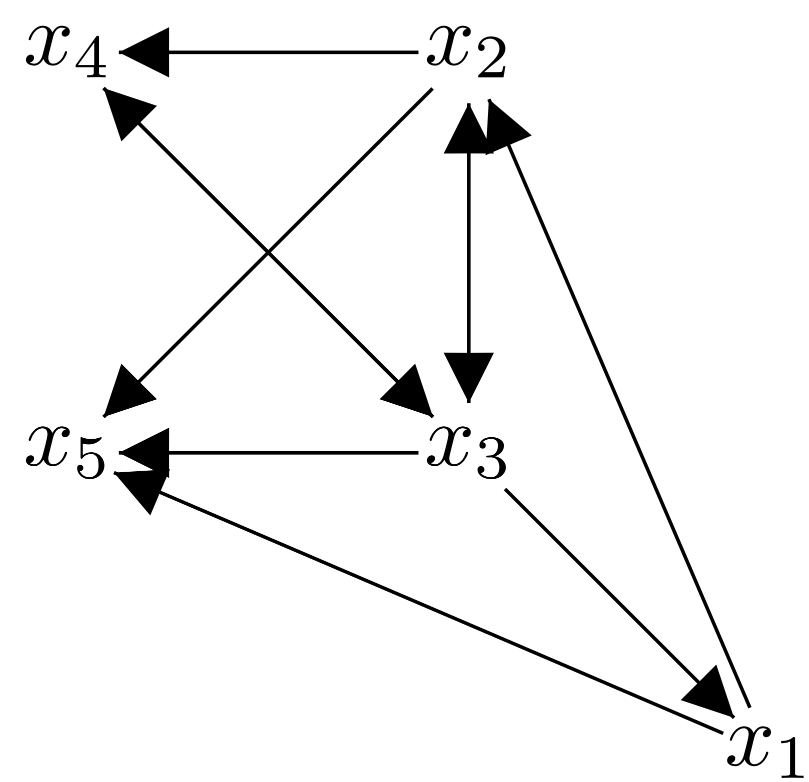
$$S(\hat{d}) = \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix}$$



Theorem [LaMar]:

$$\bar{s} = \min_{l} S_{kl} \qquad \text{and} \qquad \underline{s} = \min_{k} S_{kl}$$

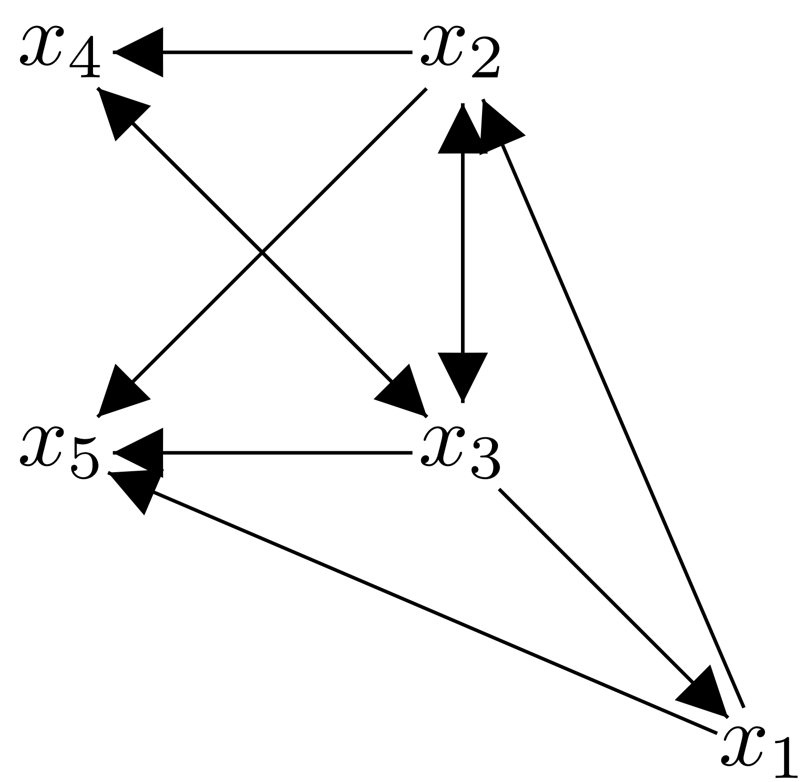
Splittance of a directed graph

$$\begin{array}{c} \hat{d} = \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \quad\quad\; 1 \quad\; 2 \quad\; 3 \quad\; 4 \quad\; 5 \end{array}$$

$$\boxed{\bar{s} = (0\ 0\ 0\ 1\ 0\ 0)}$$

$$\underline{s} = (0\ 1\ 1\ 0\ 0\ 0)$$

$$S(\hat{d}) = \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & \boxed{0} \\ 6 & 4 & 2 & 1 & 0 & \boxed{0} \\ 3 & 2 & 1 & 0 & \boxed{0} & 1 \\ 1 & 1 & 1 & \boxed{1} & 2 & 3 \\ \boxed{0} & 1 & 2 & 3 & 4 & 6 \\ \boxed{0} & 1 & 3 & 5 & 7 & 10 \end{pmatrix} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array}$$



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Splittance of a directed graph

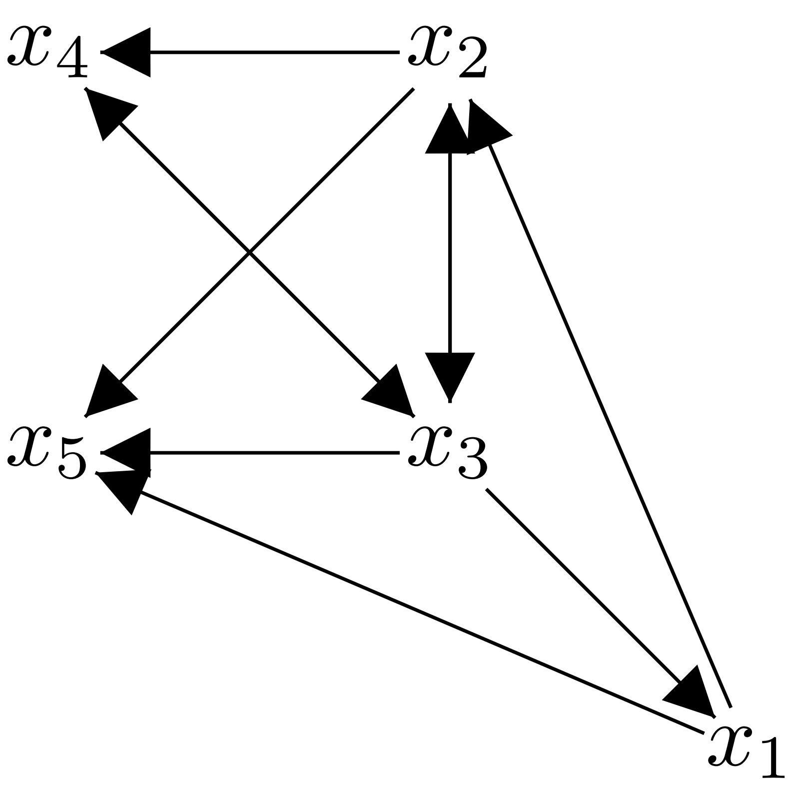
$$\hat{d} = \begin{array}{c} \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \end{array}$$

$$\bar{s} = (0\ 0\ 0\ 1\ 0\ 0)$$

$$\boxed{\underline{s} = (0\ 1\ 1\ 0\ 0\ 0)}$$

$$S(\hat{d}) = \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & \boxed{0} \\ 3 & 2 & 1 & \boxed{0} & \boxed{0} & 1 \\ 1 & 1 & \boxed{1} & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ \boxed{0} & \boxed{1} & 3 & 5 & 7 & 10 \end{pmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$



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Splittance of a directed graph

$$\hat{d} = \begin{array}{c} \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$\bar{s} = (0\ \boxed{0\ 0\ 1\ 0}\ 0)$$

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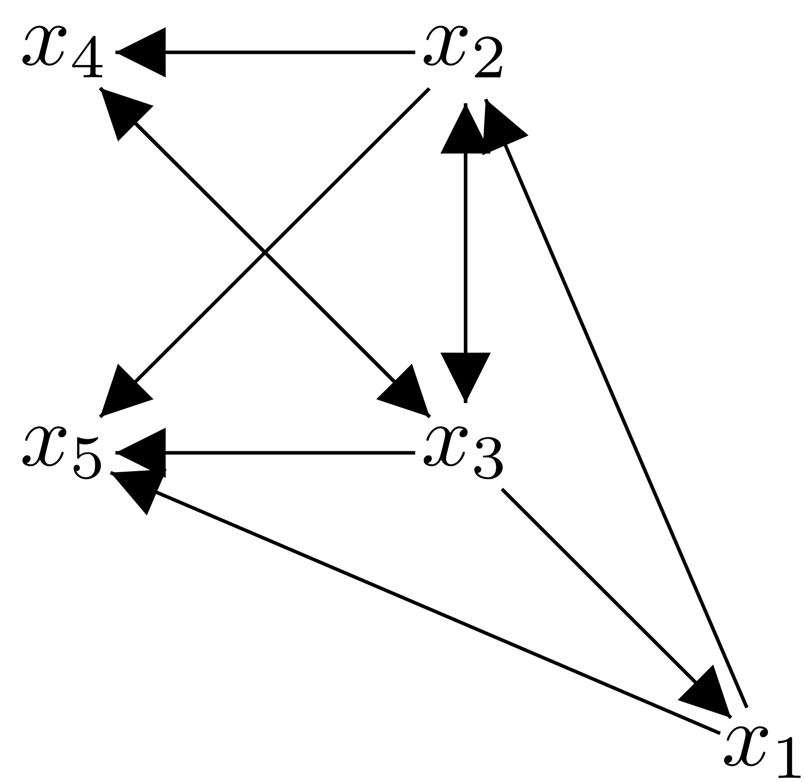
$$S(\hat{d}) = \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix}$$



Theorem [LaMar]:

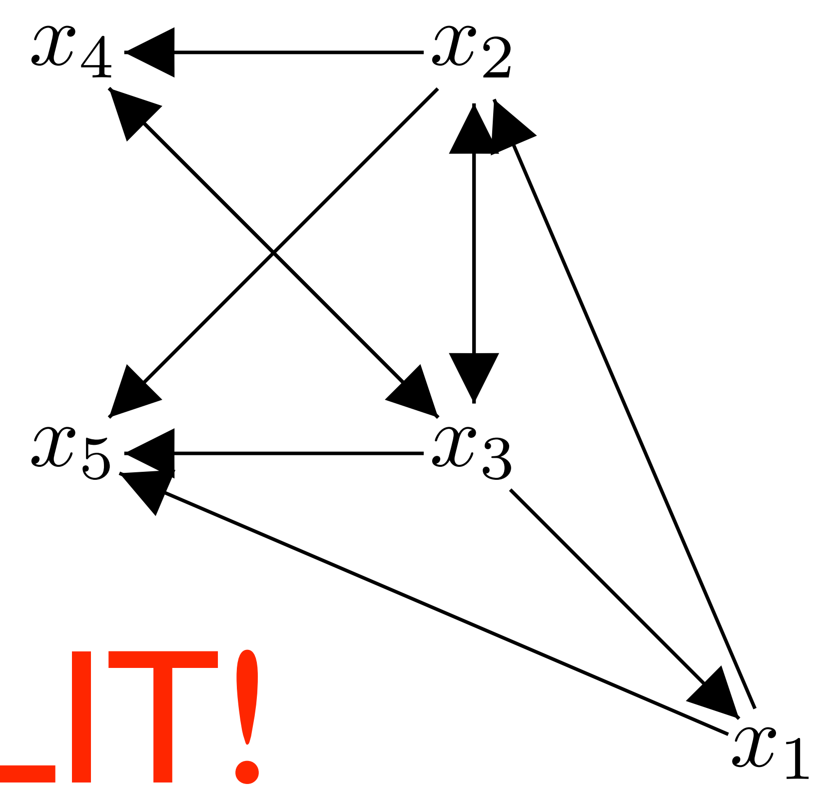
$$\sigma(\vec{G}) = \min\{\bar{s}_1, \ldots, \bar{s}_{N-1}, \underline{s}_1, \ldots, \underline{s}_{N-1}\}$$

Splittance of a directed graph

$$\begin{array}{cc} \hat{d} = & \begin{pmatrix} 2 & 3 & 4 & 1 & 0 \\ 1 & 2 & 2 & 2 & 3 \end{pmatrix} \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

$$\bar{s} = (0 \; \boxed{0 \; 0 \; 1 \; 0} \; 0)$$
$$\underline{s} = (0 \; \boxed{1 \; 1 \; 0 \; 0} \; 0)$$

$$S(\hat{d}) = \begin{pmatrix} 10 & 7 & 5 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 7 & 10 \end{pmatrix}$$



SPLIT!

Theorem [LaMar]:

$$\sigma(\vec{G}) = \min\{\bar{s}_1, \ldots, \bar{s}_{N-1}, \underline{s}_1, \ldots, \underline{s}_{N-1}\}$$

Splittance of a directed graph - directed extensions

Undirected

$$d = (4\ 3\ 3\ 3\ 3)$$

$$\sigma(d) = (8\ 4\ 2\ 1\ 1\ 2)$$

$$s = (0\ 0\ 1\ 2\ 2\ 0)$$

Splittance of a directed graph - directed extensions

Undirected

$$d = (4\ 3\ 3\ 3\ 3)$$

$$\sigma(d) = (8\ 4\ 2\ 1\ 1\ 2)$$

$$s = (0\ 0\ 1\ 2\ 2\ 0)$$

Directed

$$\hat{d} = \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 & 3 \end{pmatrix}$$

$$S(\hat{d}) = \begin{pmatrix} 16 & 12 & 9 & 6 & 3 & 0 \\ 12 & 8 & 6 & 4 & 2 & 0 \\ 9 & 6 & 4 & 3 & 2 & 1 \\ 6 & 4 & 3 & 2 & 2 & 2 \\ 3 & 2 & 2 & 2 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\bar{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

$$\underline{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

Splittance of a directed graph - directed extensions

Undirected

$$d = (4\ 3\ 3\ 3\ 3)$$

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Directed

$$\hat{d} = \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 & 3 \end{pmatrix}$$

$$S(\hat{d}) = \begin{pmatrix} \boxed{16} & 12 & 9 & 6 & 3 & 0 \\ 12 & \boxed{8} & 6 & 4 & 2 & 0 \\ 9 & 6 & \boxed{4} & 3 & 2 & 1 \\ 6 & 4 & 3 & \boxed{2} & 2 & 2 \\ 3 & 2 & 2 & 2 & \boxed{2} & 3 \\ 0 & 0 & 1 & 2 & 3 & \boxed{4} \end{pmatrix}$$

$$\bar{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

$$\underline{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

Splittance of a directed graph - directed extensions

Undirected

$$d = (4\ 3\ 3\ 3\ 3)$$

$$\sigma(d) = (8\ 4\ 2\ 1\ 1\ 2)$$

$$\boxed{\neq 0}$$

$$s = (0\ 0\ 1\ 2\ 2\ 0)$$

Directed

$$\hat{d} = \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 & 3 \end{pmatrix}$$

$$S(\hat{d}) = \begin{pmatrix} \boxed{16} & 12 & 9 & 6 & 3 & 0 \\ 12 & \boxed{8} & 6 & 4 & 2 & 0 \\ 9 & 6 & \boxed{4} & 3 & 2 & 1 \\ 6 & 4 & 3 & \boxed{2} & 2 & 2 \\ 3 & 2 & 2 & 2 & \boxed{2} & 3 \\ 0 & 0 & 1 & 2 & 3 & \boxed{4} \end{pmatrix}$$

$$\boxed{= 0}$$

$$\bar{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

$$\underline{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

Splittance of a directed graph - directed extensions

Undirected

NOT SPLIT!

$$d = (4\ 3\ 3\ 3\ 3)$$

$$\sigma(d) = (8\ 4\ 2\ 1\ 1\ 2)$$

$$s = (0\ 0\ 1\ \underset{\substack{\neq 0\\ \downarrow}}{}2\ 2\ 0)$$

Directed

SPLIT!

$$\hat{d} = \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 & 3 \end{pmatrix}$$

$$S(\hat{d}) = \begin{pmatrix} \boxed{16} & 12 & 9 & 6 & 3 & 0 \\ 12 & \boxed{8} & 6 & 4 & 2 & 0 \\ 9 & 6 & \boxed{4} & 3 & 2 & 1 \\ 6 & 4 & 3 & \boxed{2} & 2 & 2 \\ 3 & 2 & 2 & 2 & \boxed{2} & 3 \\ 0 & 0 & 1 & 2 & 3 & \boxed{4} \end{pmatrix}$$

$= 0$

$$\bar{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

$$\underline{s} = (0\ 0\ 1\ 2\ 2\ 0)$$

Canonical split decomposition

$$\begin{array}{rcl} \Sigma & = & \text{set of splitted graphs} \\ \Gamma & = & \text{set of simple graphs} \end{array}$$

Define the composition operator $\circ : \Sigma \times \Gamma \to \Gamma$ such that if $(G, C, I) \in \Sigma$ and $H \in \Gamma$ with adjacency matrix A, the graph $(G, C, I) \circ H$ has adjacency matrix

$$\begin{array}{cc} & \begin{array}{ccc} C & I & H \end{array} \\ \begin{array}{c} C \\ I \\ H \end{array} & \begin{pmatrix} 1 & * & 1 \\ * & 0 & 0 \\ 1 & 0 & A \end{pmatrix} \end{array}$$

Canonical split decomposition

$$\begin{aligned}\Sigma &= \text{set of splitted graphs}\\ \Gamma &= \text{set of simple graphs}\end{aligned}$$

Define the composition operator $\circ : \Sigma \times \Gamma \to \Gamma$ such that if $(G, C, I) \in \Sigma$ and $H \in \Gamma$ with adjacency matrix A, the graph $(G, C, I) \circ H$ has adjacency matrix

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Canonical Graph Decomposition [Tyshkevich (2000)]:

Every graph G can be represented as a composition

$$G = (G_1, C_1, I_1) \circ \cdots \circ (G_k, C_k, I_k) \circ G_0$$

of indecomposable components. Here (G_i, C_i, I_i) are indecomposable splitted graphs and G_0 is an indecomposable graph.

Canonical split decomposition of digraphs?

Define the composition operator $\circ : \Sigma \times \Gamma \rightarrow \Gamma$ such that if $(\vec{G}, \mathcal{S}) \in \Sigma$ and $\vec{H} \in \Gamma$ with adjacency matrix A, the graph $(\vec{G}, \mathcal{S}) \circ \vec{H}$ has adjacency matrix

$$\begin{array}{c} \\ X^{\pm} \\ X^{+} \\ X^{-} \\ X^{0} \\ H \end{array} \begin{array}{c} \begin{array}{ccccc} X^{\pm} & X^{+} & X^{-} & X^{0} & H \end{array} \\ \begin{pmatrix} 1 & * & 1 & * & 1 \\ 1 & * & 1 & * & 1 \\ * & 0 & * & 0 & 0 \\ * & 0 & * & 0 & 0 \\ 1 & 0 & 1 & 0 & A \end{pmatrix} \end{array}$$

Canonical split decomposition of digraphs?

Define the composition operator $\circ : \Sigma \times \Gamma \to \Gamma$ such that if $(\vec{G}, \mathcal{S}) \in \Sigma$ and $\vec{H} \in \Gamma$ with adjacency matrix A, the graph $(\vec{G}, \mathcal{S}) \circ \vec{H}$ has adjacency matrix

$$\begin{array}{c c}
 & \begin{array}{ccccc} X^{\pm} & X^{+} & X^{-} & X^{0} & H \end{array} \\
\begin{array}{c} X^{\pm} \\ X^{+} \\ X^{-} \\ X^{0} \\ H \end{array} & \begin{pmatrix} 1 & * & 1 & * & 1 \\ 1 & * & 1 & * & 1 \\ * & 0 & * & 0 & 0 \\ * & 0 & * & 0 & 0 \\ 1 & 0 & 1 & 0 & A \end{pmatrix}
\end{array}$$

Conjecture [Canonical Digraph Decomposition]:

Every digraph $\vec{G}$ can be represented as a composition

$$\vec{G} = (\vec{G}_1, \mathcal{S}_1) \circ \cdots \circ (\vec{G}_k, \mathcal{S}_k) \circ \vec{G}_0$$

of indecomposable components. Here $(\vec{G}_i, \mathcal{S}_i)$ are indecomposable splitted digraphs and $\vec{G}_0$ is an indecomposable digraph.

Realizing graphic and digraphic sequences

Theorem [Havel-Hakimi (1962)]:

Let d be a non-increasing integer sequence. Then d is graphic if and only if $\hat{d}$ is graphic, where for some k we have

$$\hat{d}_i = \begin{cases} 0 & \text{for } i = k \\ d_i - 1 & \begin{cases} \text{for } i = 1, \ldots, d_k & \text{if } k > d_k \\ \text{for } i = 1, \ldots, k-1, k+1, \ldots, d_k + 1 & \text{if } k \leq d_k \end{cases} \\ d_i & \text{otherwise.} \end{cases}$$

Realizing graphic and digraphic sequences

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Theorem [Kleitman-Wang (1972)]:

Let $d = (d^+, d^-)$ be an integer-pair sequence that is non-increasing relative to the lexicographical ordering, giving preference to the out-degree. Then d is digraphic if and only if $\hat{d}$ is digraphic, where for some k we have

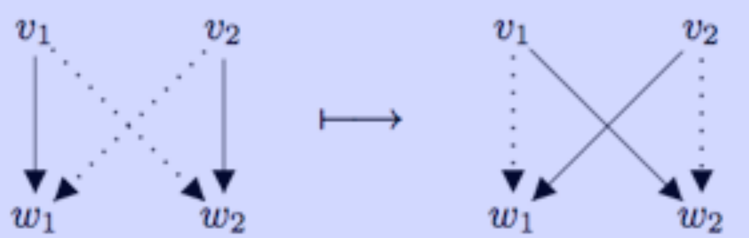
$$\hat{d}_i^+ = \begin{cases} d_i^+ - 1 & \begin{cases} \text{for } i = 1, \ldots, d_k^- & \text{if } k > d_k^- \\ \text{for } i = 1, \ldots, k-1, k+1, \ldots, d_k^- + 1 & \text{if } k \leq d_k^- \end{cases} \\ d_i^+ & \text{otherwise.} \end{cases}$$

$$\hat{d}_i^- = \begin{cases} 0 & \text{for } i = k \\ d_i^- & \text{otherwise.} \end{cases}$$

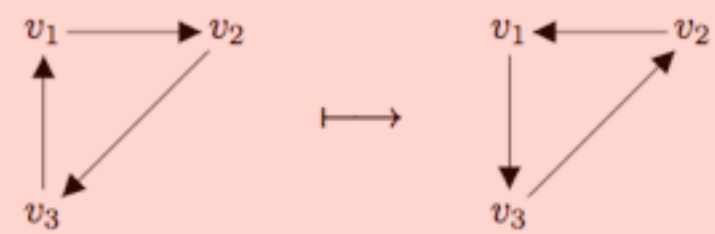
Uniform sampling algorithms ▶ Random walks

Random walk on set of realizations:

2-switch



3-cycle reorientation



Uniform sampling algorithms ▶ Random walks

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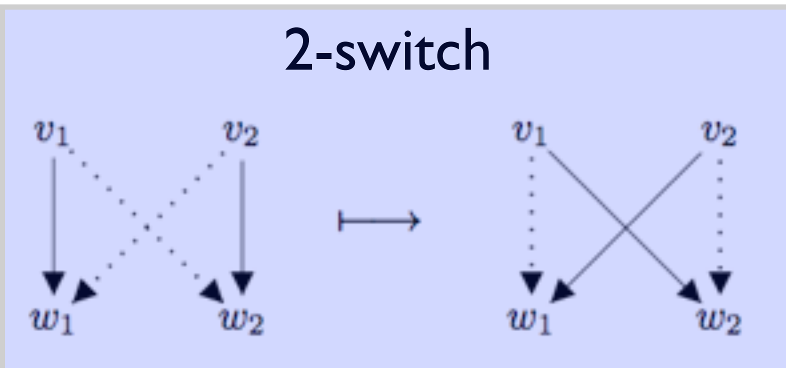
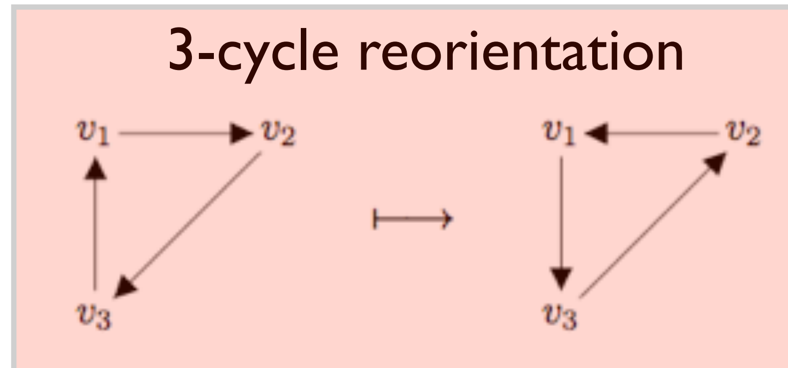
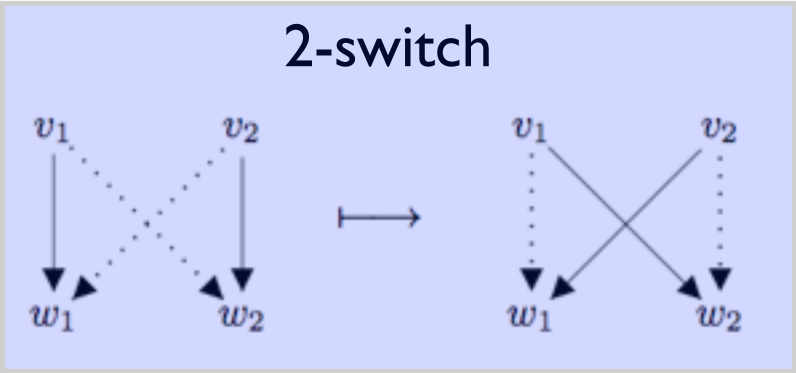


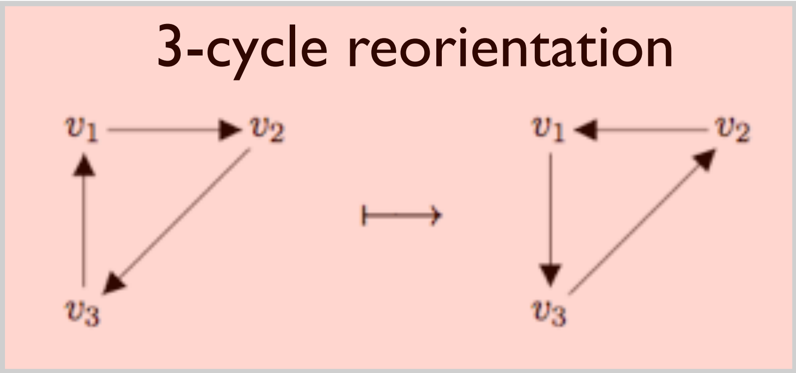


Example:

$$d = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

Random walk on set of realizations:





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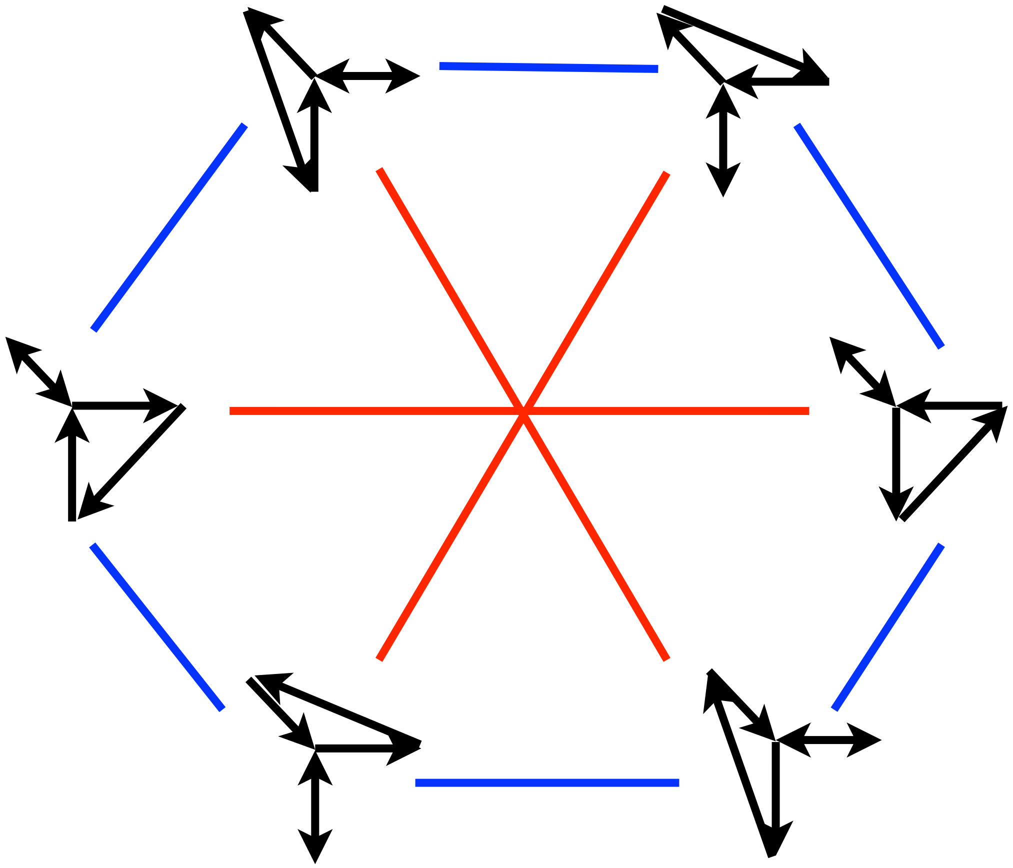

Uniform sampling algorithms ▶ Random walks

Random walk on set of realizations:

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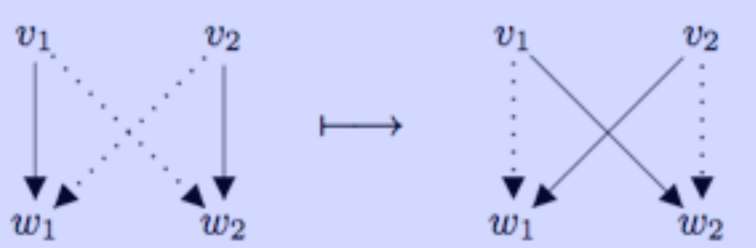
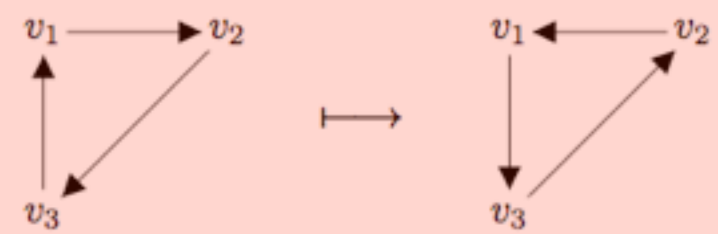


3-cycle reorientation

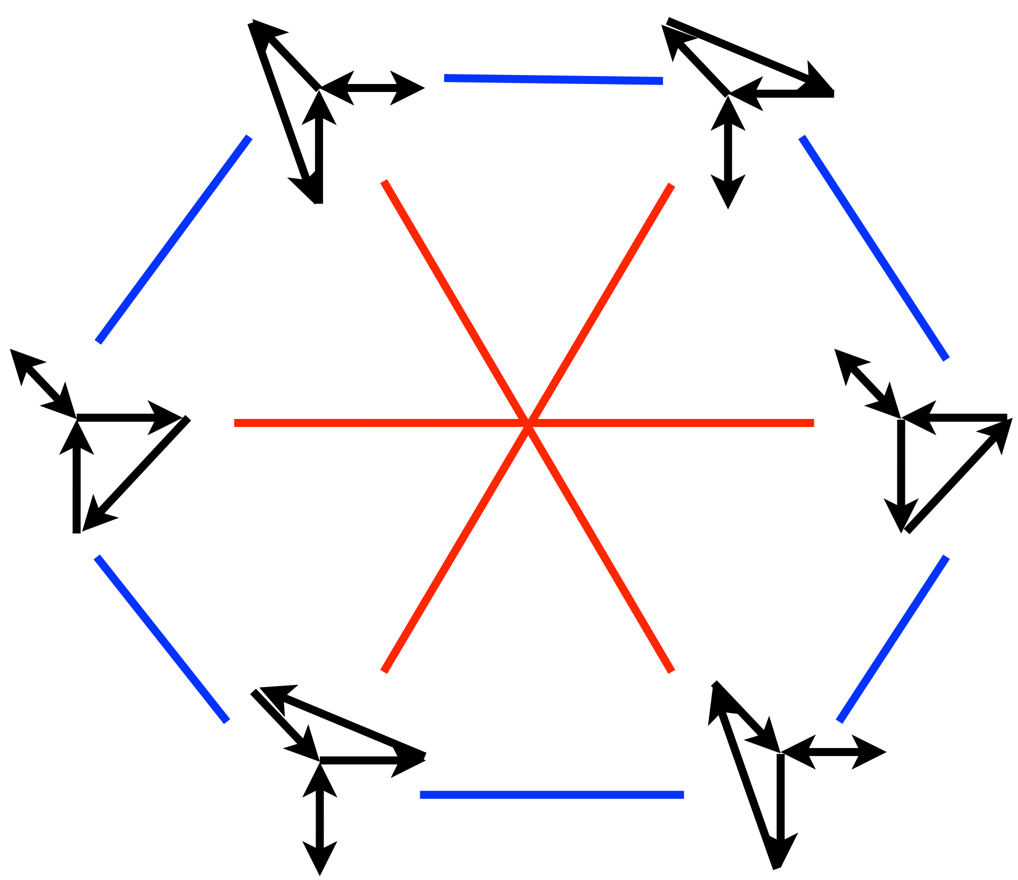


Example:

$$d = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

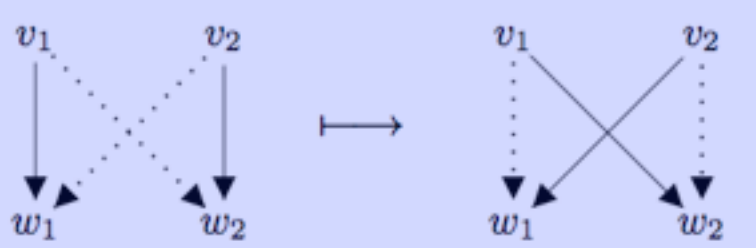
Theorem [Rao et al. (1996)]:

The meta-graph Ω_d is connected.

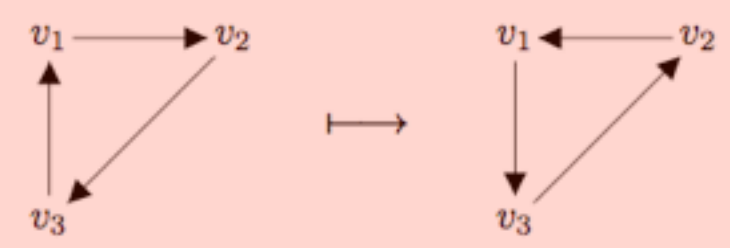


Random walk on set of realizations:

2-switch



3-cycle reorientation



Example:

$$d = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

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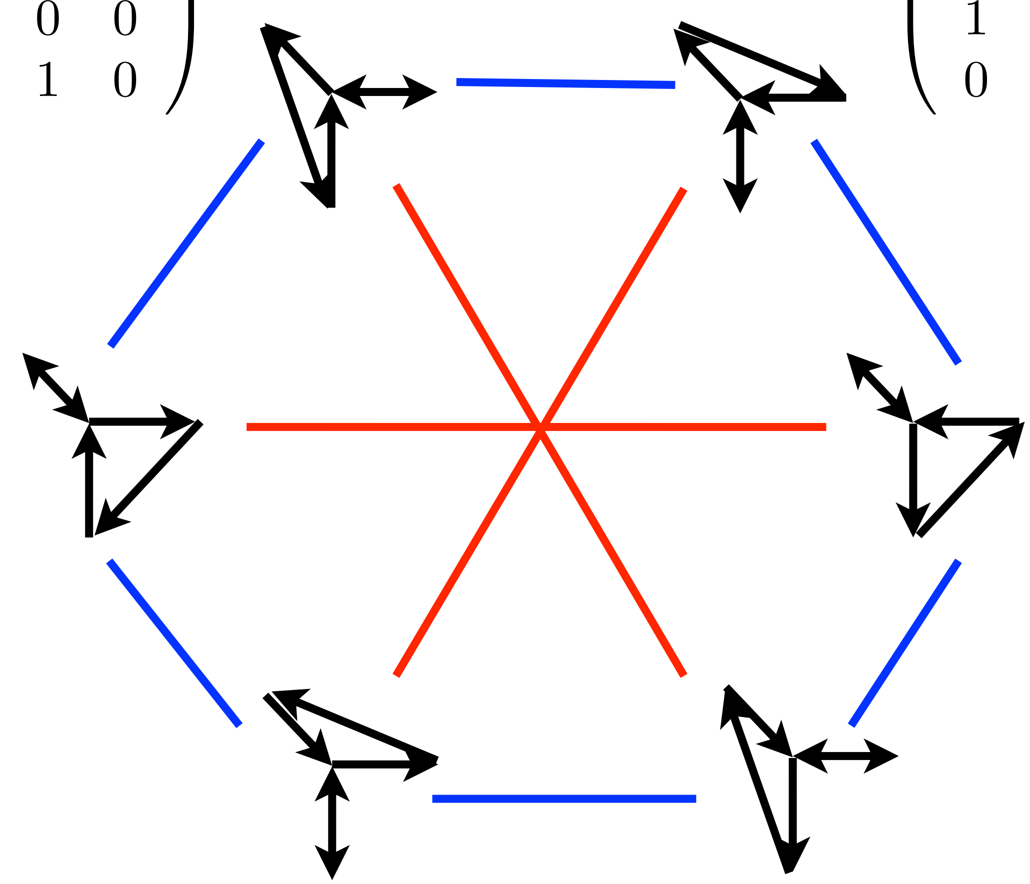


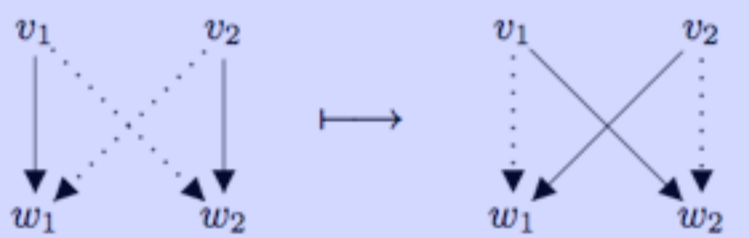
Theorem [Rao et al. (1996)]:

The meta-graph Ω_d is connected.

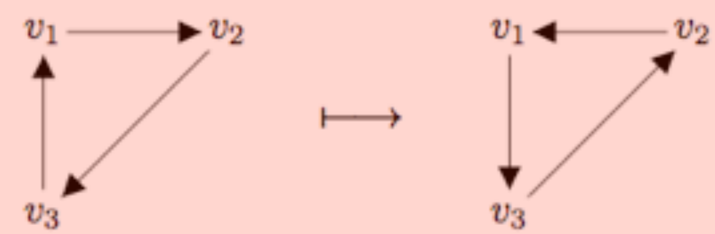
Uniform sampling algorithms ▶ Random walks

Random walk on set of realizations:

2-switch



3-cycle reorientation



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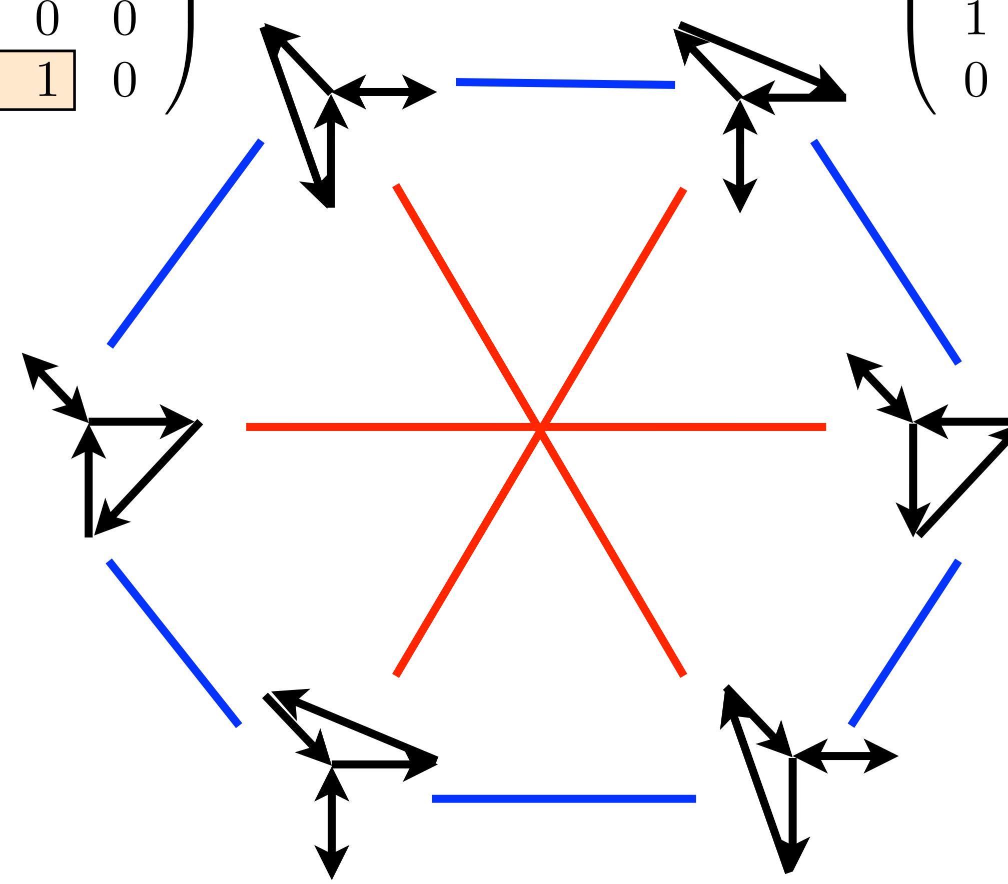

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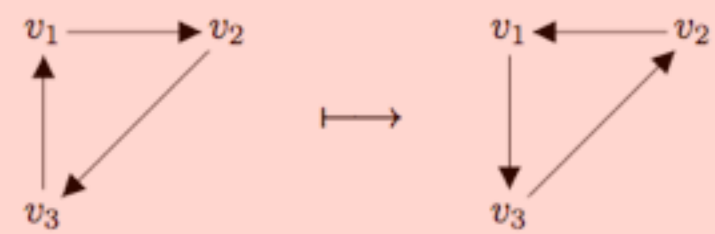

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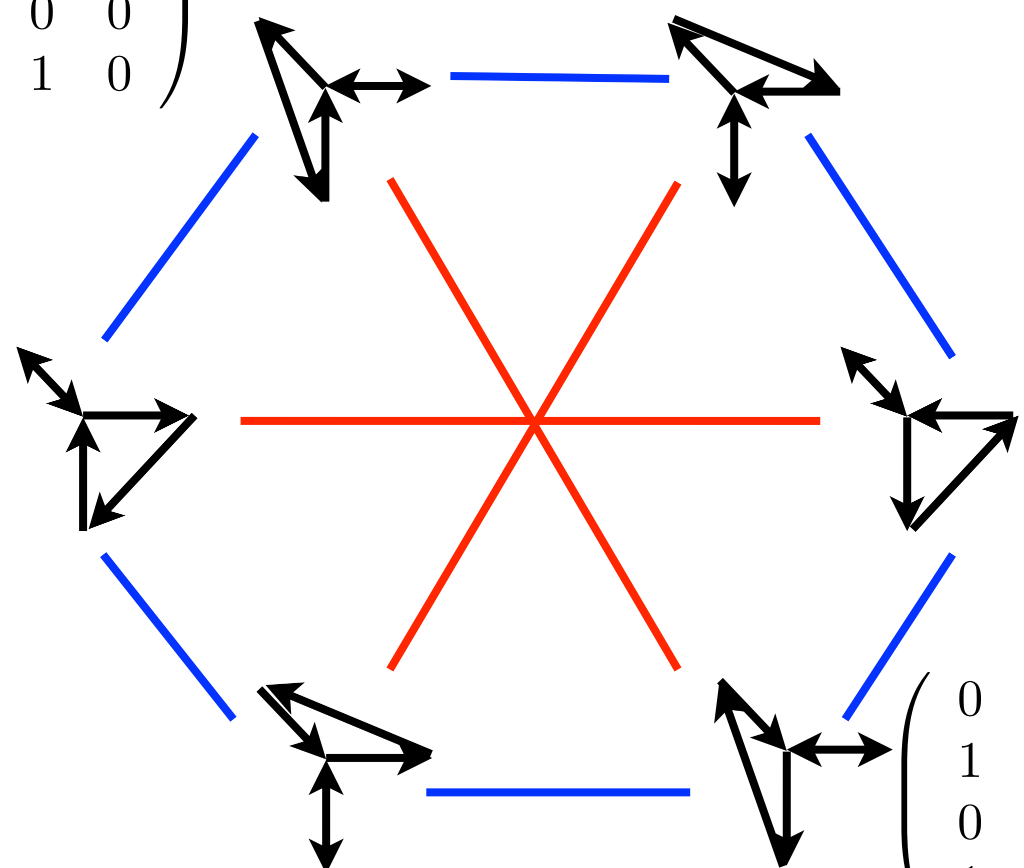

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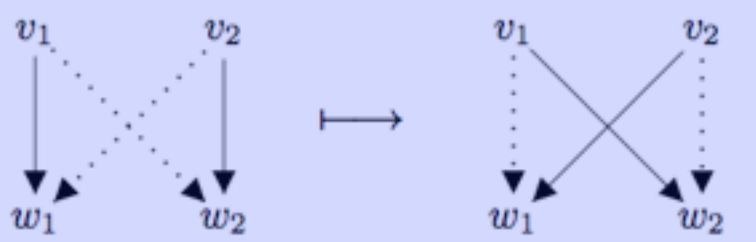

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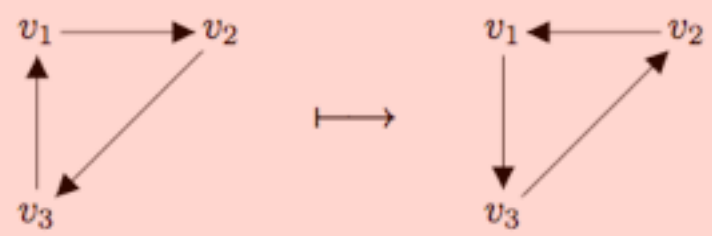

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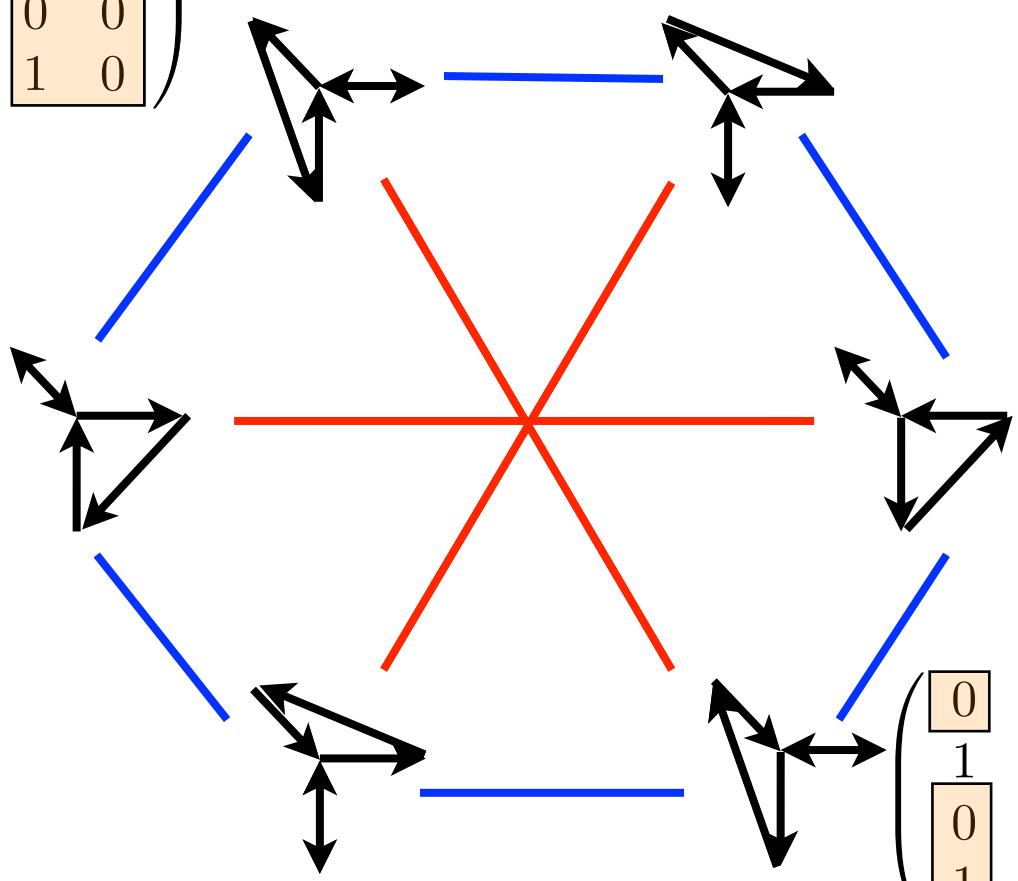

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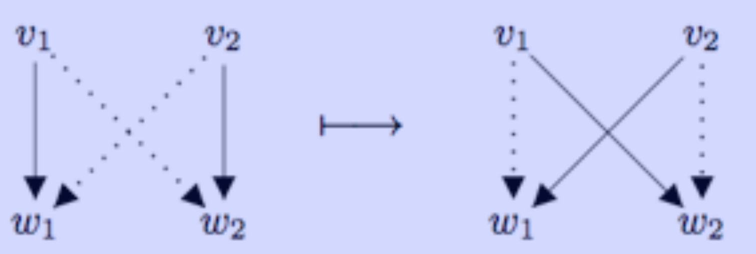

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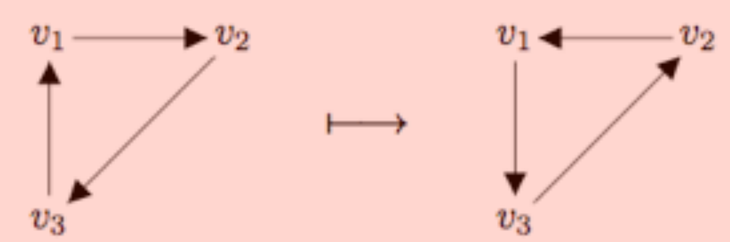

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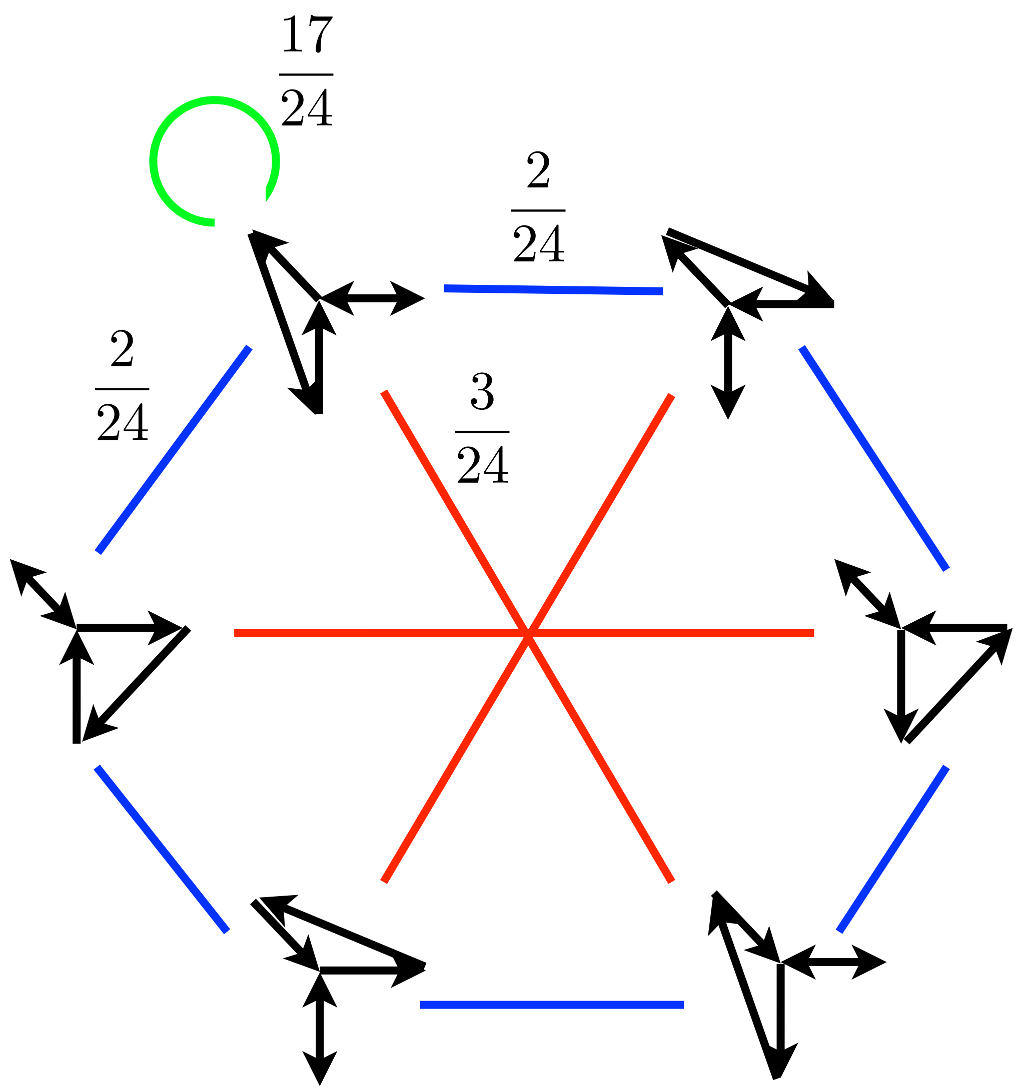

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Uniform sampling algorithms ▶ Importance sampling

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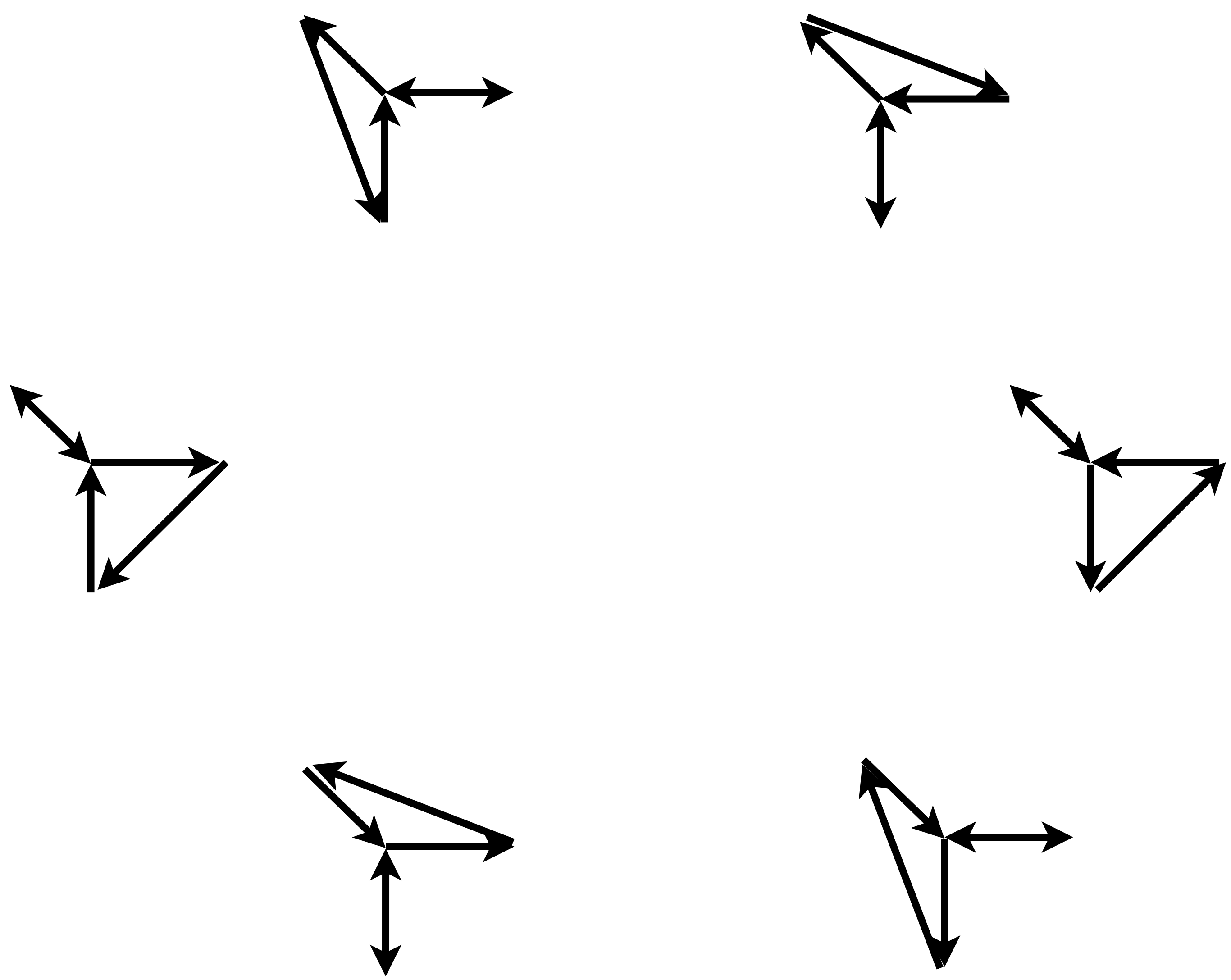

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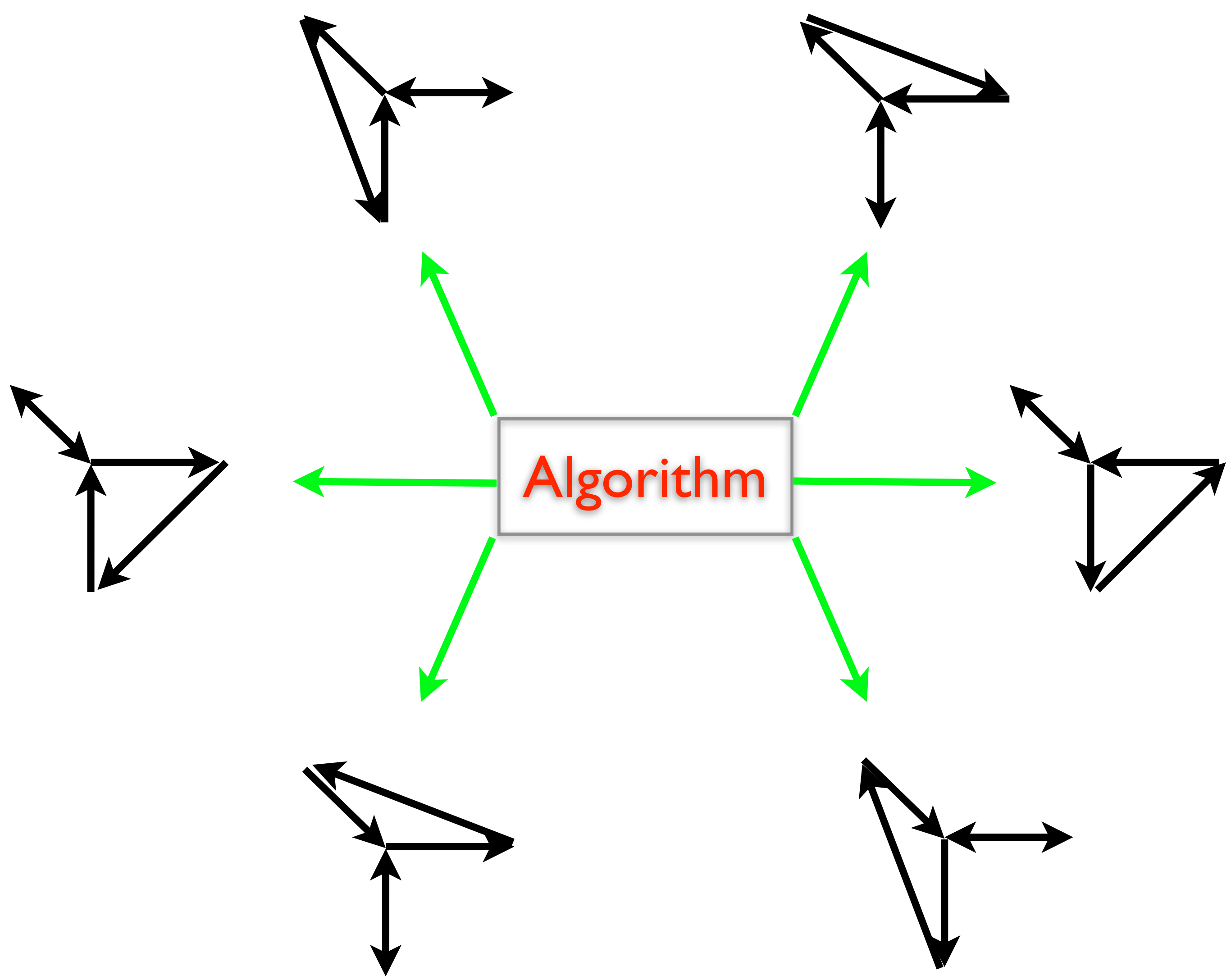

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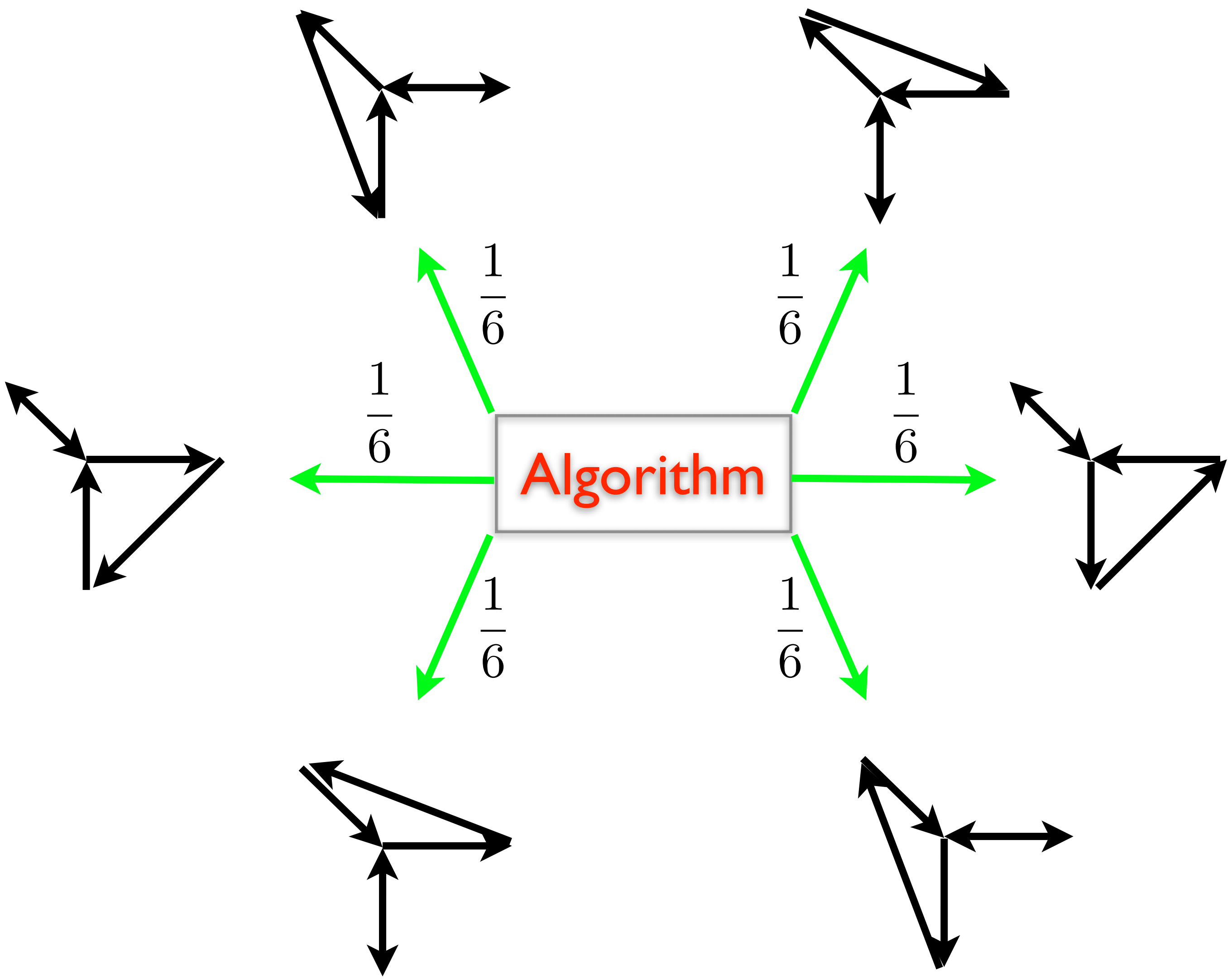

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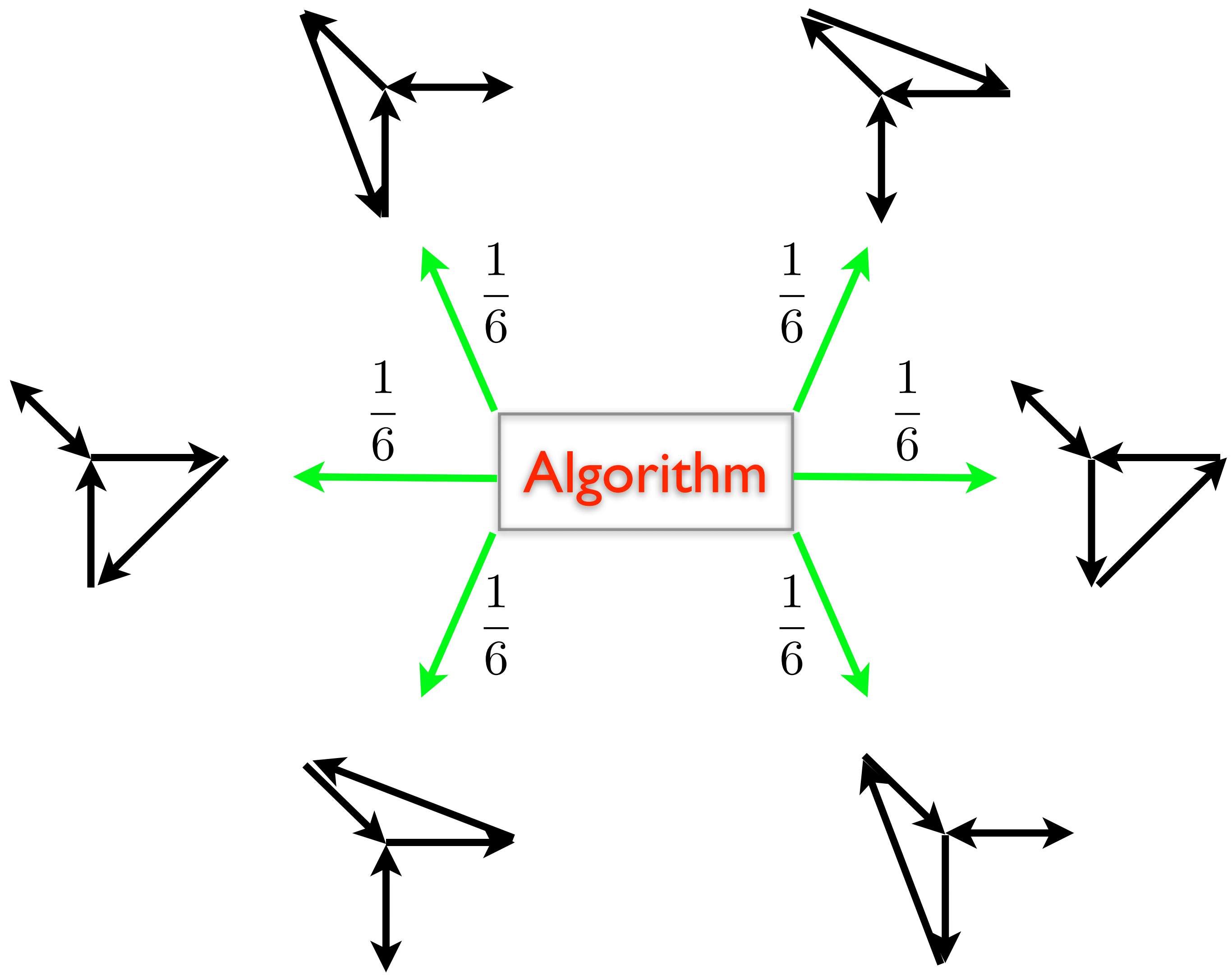
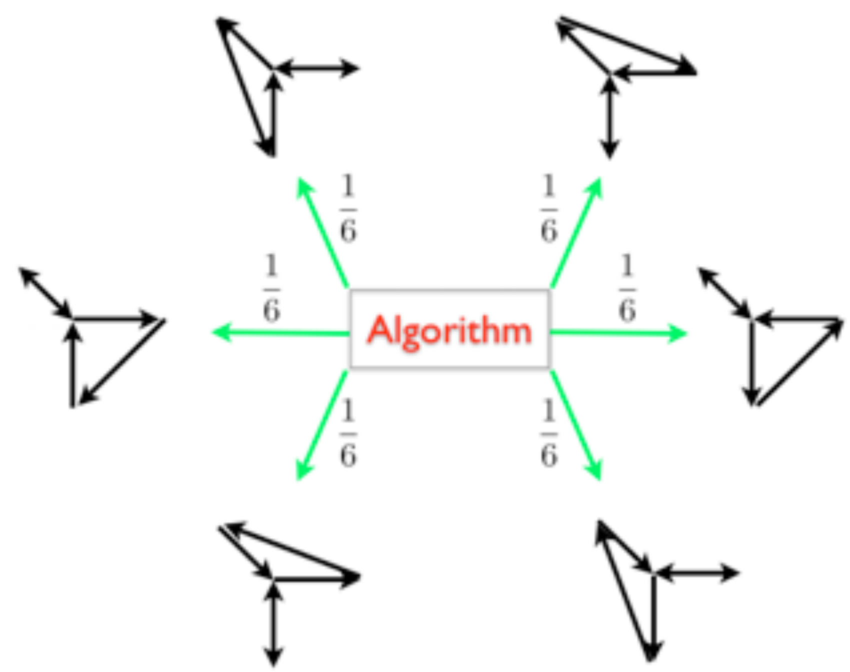


* Blitzstein and Diaconis, "Sequential Importance Sampling Algorithm for Generating Random Graphs with Prescribed Degrees." Internet Mathematics. 2011 Mar. 9;6(4):489–522. (remained unpublished for 6 years)

* del Genio et al, "Efficient and exact sampling of simple graphs with given arbitrary degree sequence." PLoS ONE. 2010 Mar. 31;5(4):1–7.



In general, if we create n networks $\vec{G}_i$, each with creation probability p_i, and we're interesting in some network measure $X_i \equiv f(\vec{G}_i)$, then the sample mean is given by

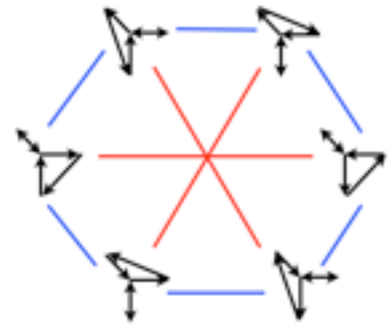
$$\bar{X} = \frac{\sum_{i=1}^{n} \omega_i X_i}{\sum_{i=1}^{n} \omega_i},$$

where $\omega_i = \frac{1}{p_i}$.

Uniform sampling of realizations

Theorem [Rao et al. (1996)]:

The meta-graph Ω_d is connected.



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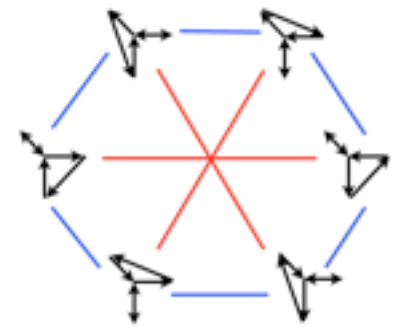

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$\Omega'_d = (\mathcal{V}, \mathcal{E}_2)$ is disconnected if and only if d is $\vec{C}_3^*$-anchored.

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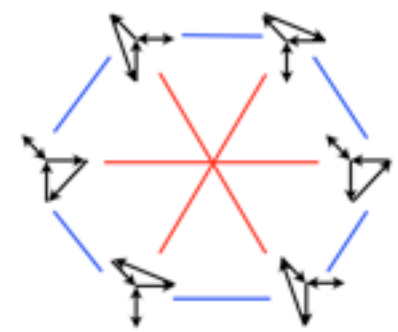

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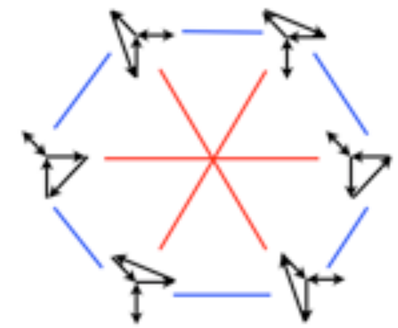

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Corollary:

$$\Omega_d \simeq \Omega_d[\mathcal{V}(G_2)] \times \left(\times_{i=1}^{k} K_2\right),$$

where G_2 is one connected component of Ω'_d and k denotes the number of anchored 3-cycles.

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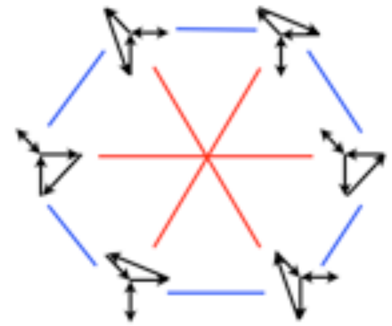

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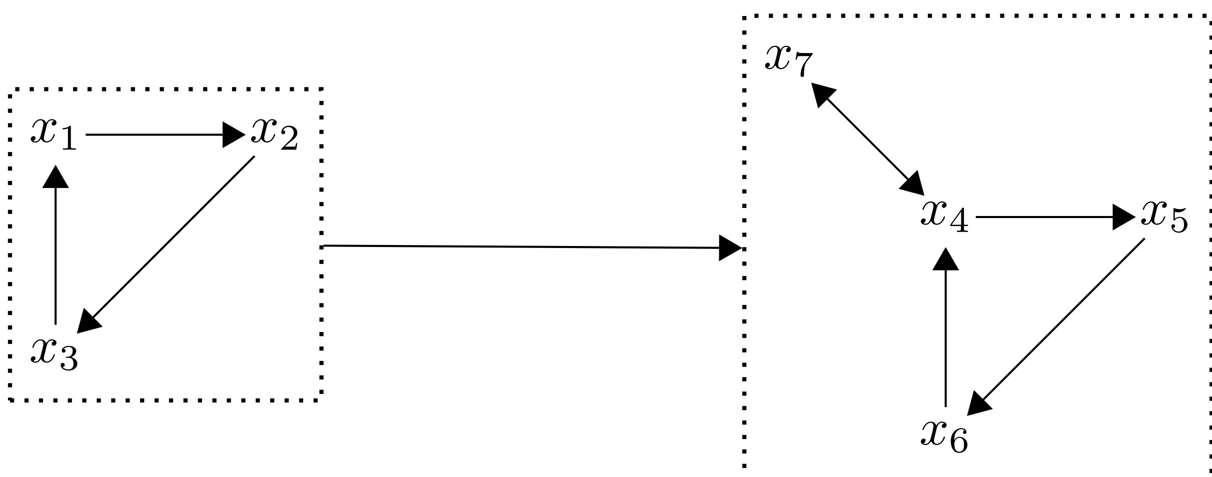
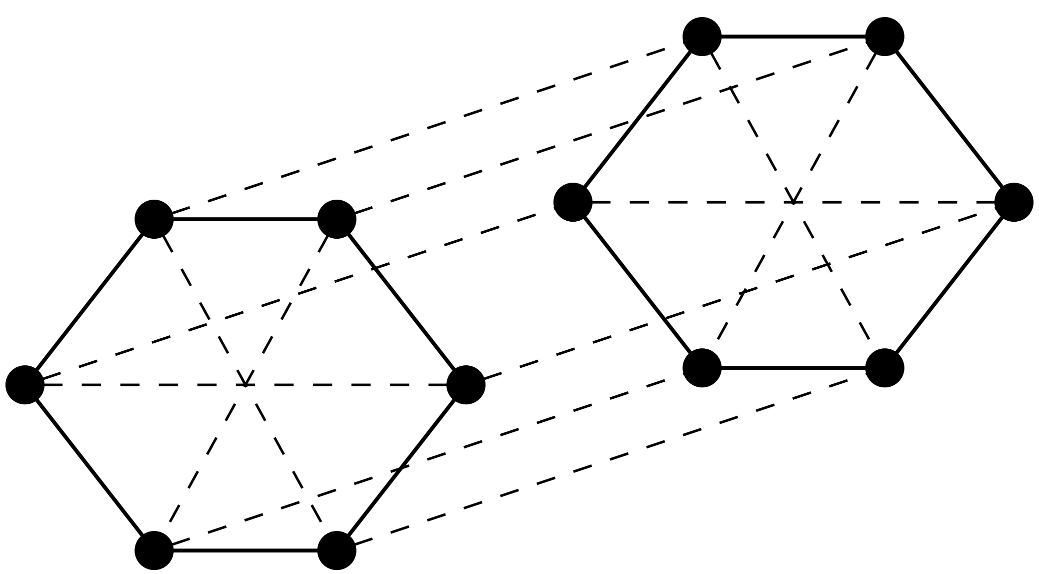


$$\begin{pmatrix} 5 & 5 & 5 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 & 4 & 4 & 4 \end{pmatrix}$$



$\vec{C}_3$-anchored degree sequences

Forcibly $\vec{H}$-digraphic:

Given a degree sequence d and digraph $\vec{H}$, we say d is **forcibly $\vec{H}$-digraphic** if and only if for all $\vec{G} \in R(d)$, there is a subgraph $\vec{H}' \subset \vec{G}$ such that $\vec{H}' \cong \vec{H}$.

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$\vec{H}$-anchored:

Given a digraph $\vec{H}$, we will call a degree sequence d **$\vec{H}$-anchored** if it is forcibly $\vec{H}$-digraphic and there exists a nonempty set of coordinates $\mathcal{J}(\vec{H})$, called an **$\vec{H}$-anchor set**, such that for every coordinate $i \in \mathcal{J}(\vec{H})$ and every $\vec{G} \in R(d)$, there is an induced subgraph $\vec{H}' \subseteq \vec{G}$ with $\vec{H}' \cong \vec{H}$ and $v_i \in V(\vec{H}')$.

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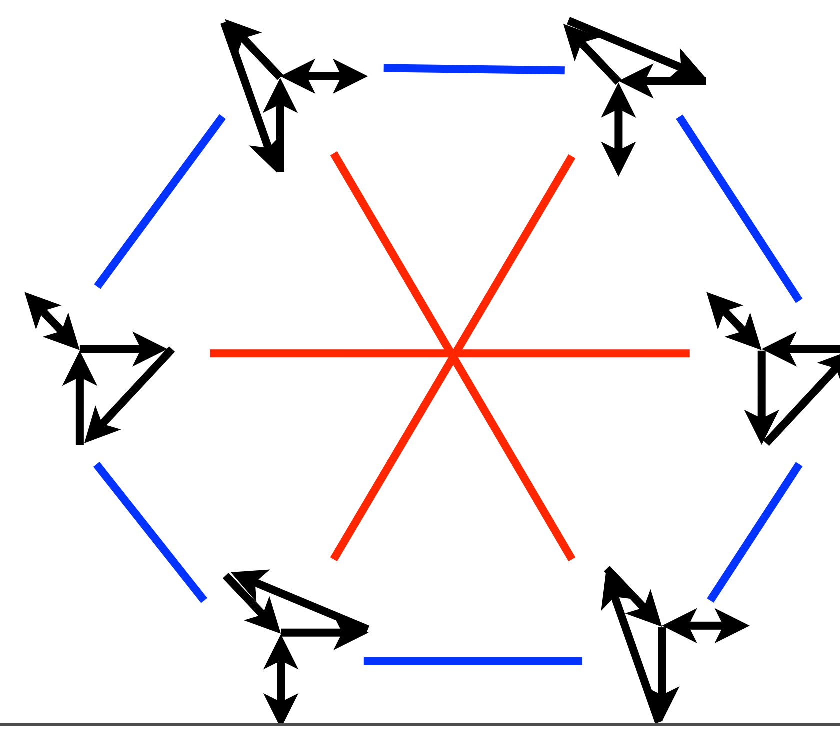
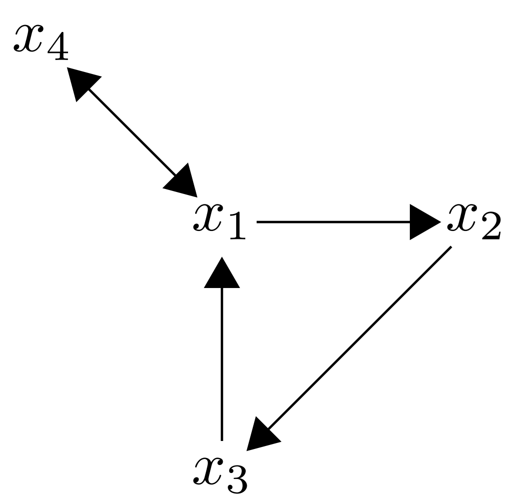
Example ($\vec{C}_3$-anchored):

$$d = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$



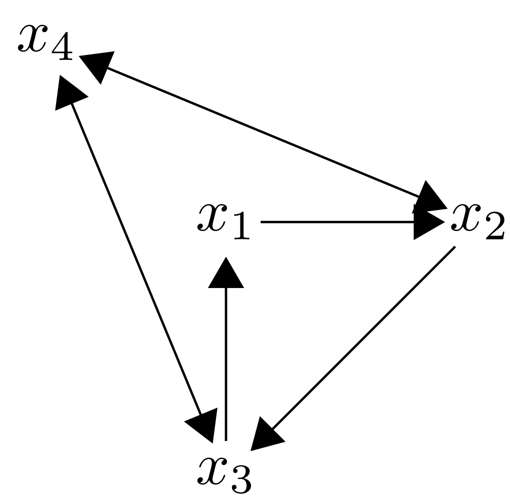
Examples:

(a)



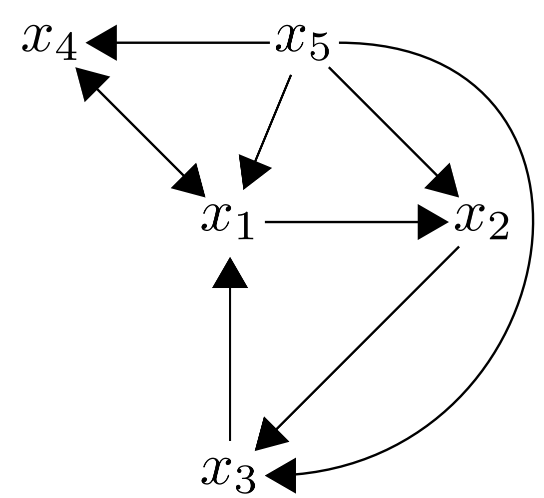
$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

(b)



$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

(c)



$$\begin{pmatrix} 2 & 1 & 1 & 1 & 4 \\ 3 & 2 & 2 & 2 & 0 \end{pmatrix}$$

Structural characterization of $\vec{C}_3$-anchored digraphs

Examples:

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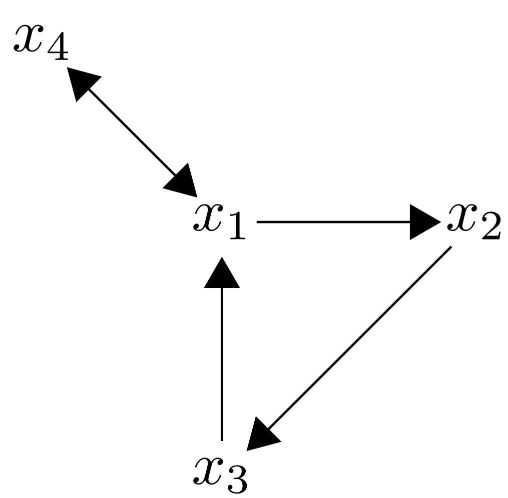

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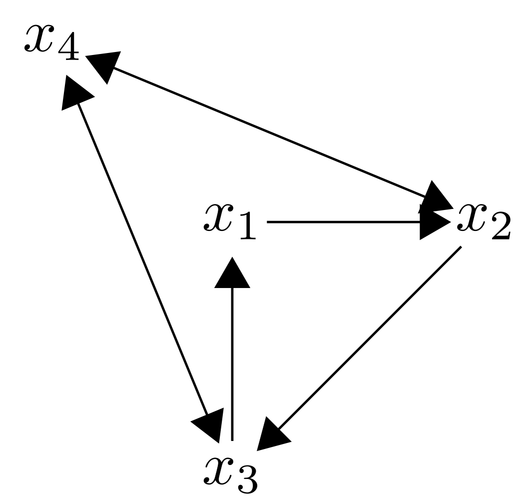

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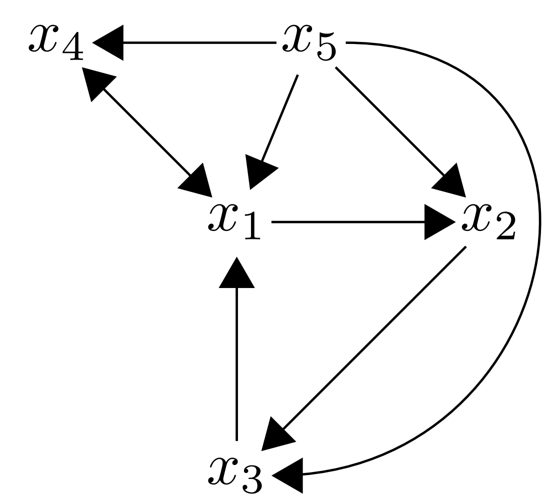

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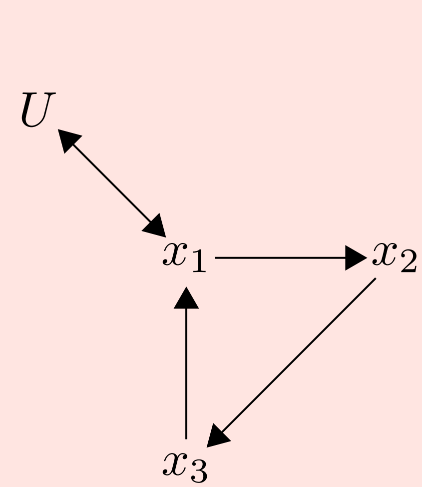
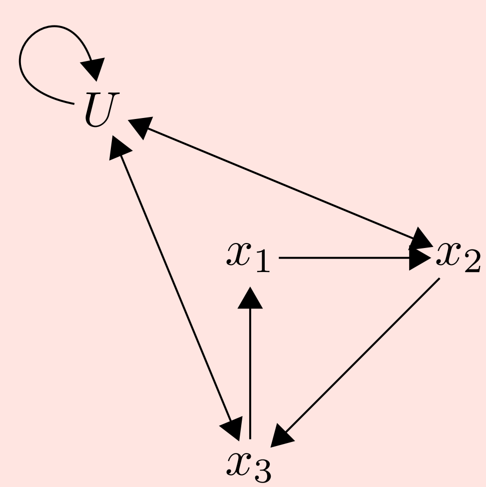
$\vec{F}_1$	$\vec{F}_2$
	

Structural characterization of $\vec{C}_3$-anchored digraphs

Theorem [LaMar]:

$\vec{G}$ is a $\vec{C}_3$-anchored digraph if and only if

$$\vec{G} \simeq (\vec{H}, \mathcal{S}) \circ \vec{F}_i$$

where $(\vec{H}, \mathcal{S})$ is a splitted digraph and $\vec{F}_i$ are the indecomposable digraphs defined previously.

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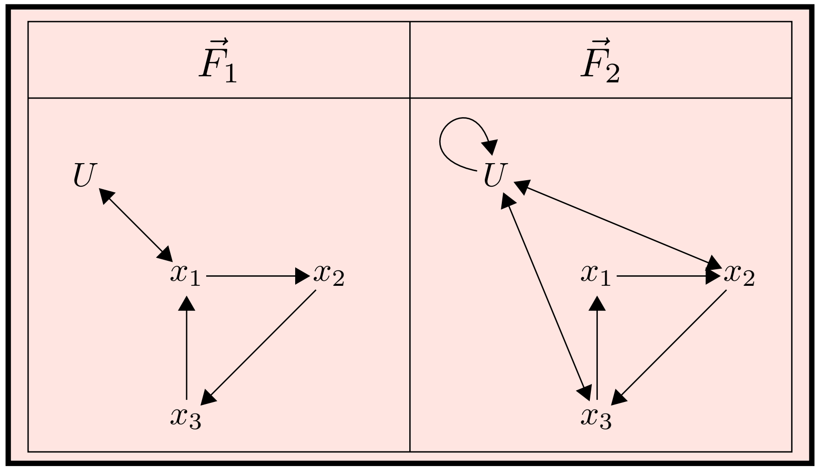

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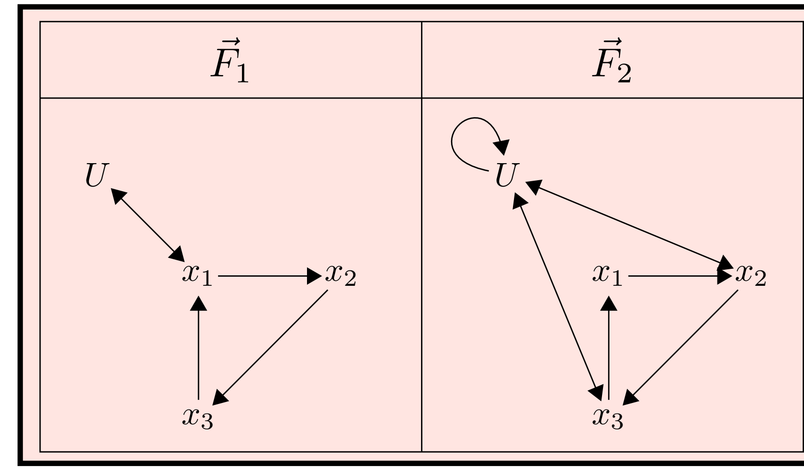

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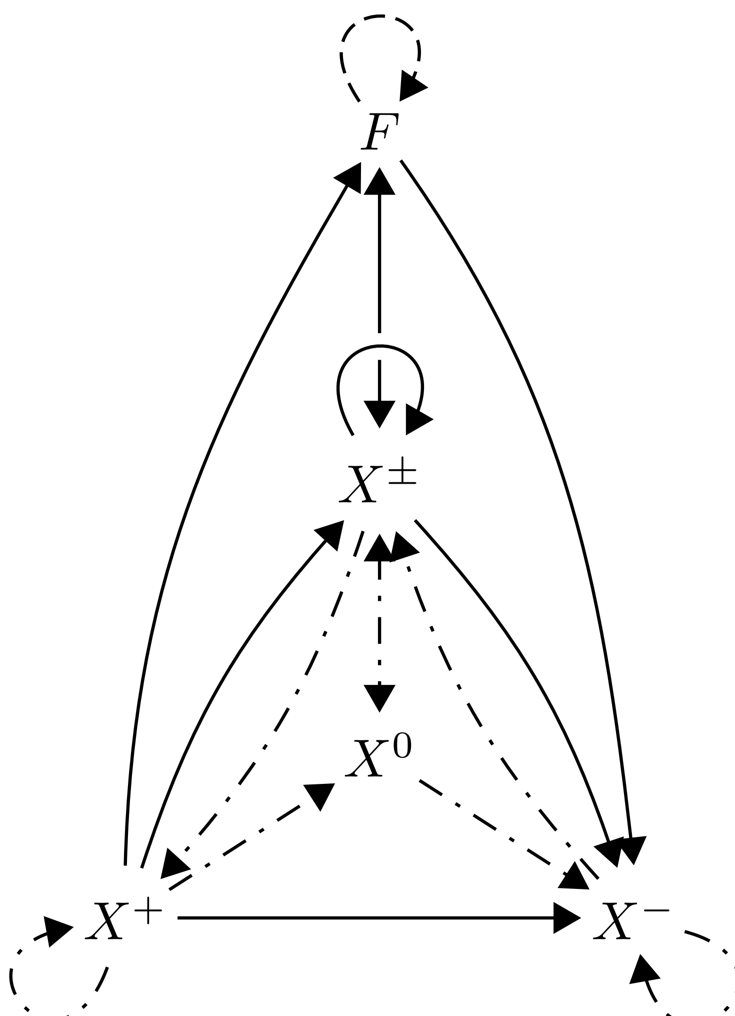

$$\begin{array}{c} \\ X^{\pm} \\ X^{+} \\ X^{-} \\ X^{0} \\ F \end{array} \begin{array}{c} \begin{array}{ccccc} X^{\pm} & X^{+} & X^{-} & X^{0} & F \end{array} \\ \begin{pmatrix} 1 & * & 1 & * & 1 \\ 1 & * & 1 & * & 1 \\ * & 0 & * & 0 & 0 \\ * & 0 & * & 0 & 0 \\ 1 & 0 & 1 & 0 & \bullet \end{pmatrix} \end{array}$$

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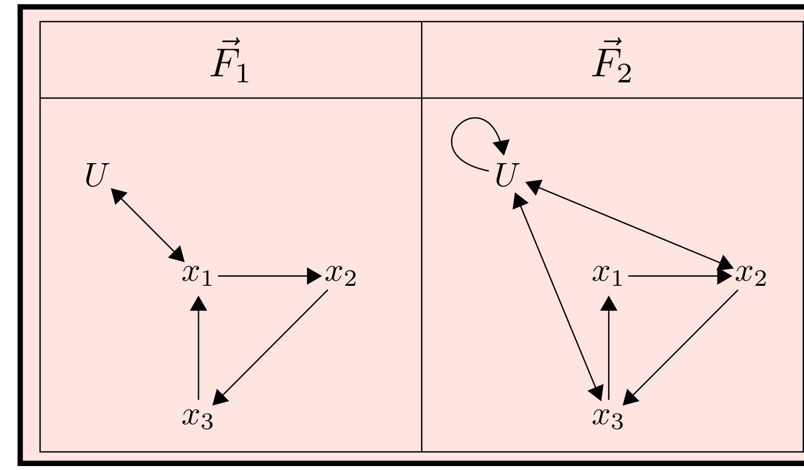

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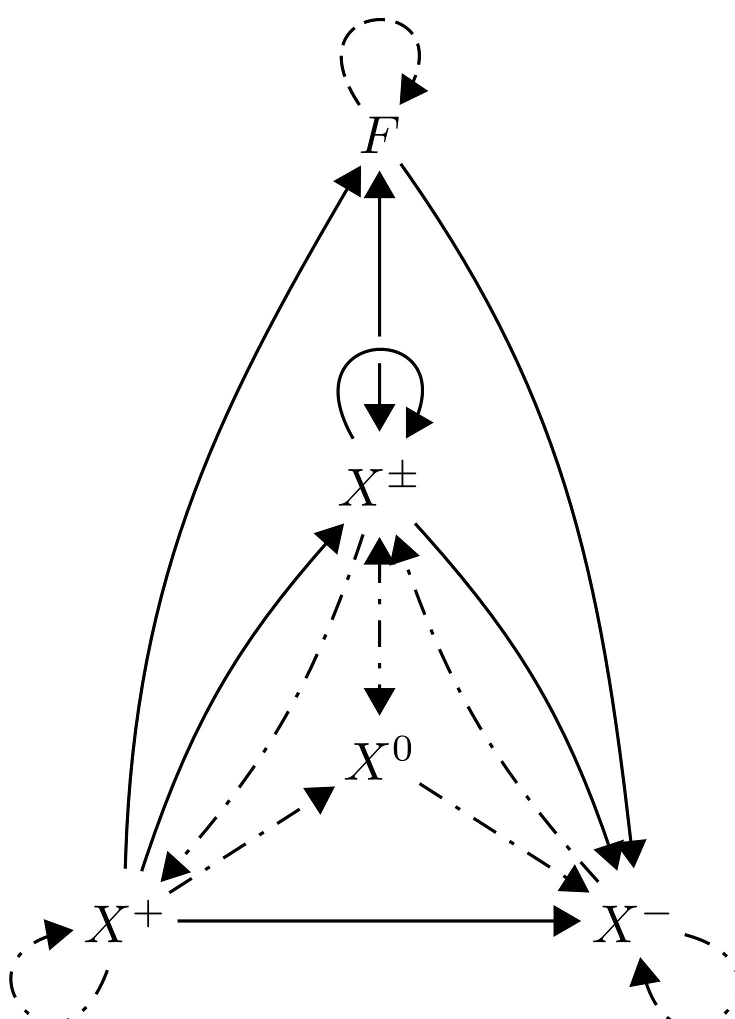

$\vec{G}$ is $\vec{C}_3^*$-anchored $\Longleftrightarrow$ $\vec{F}_i = \vec{C}_3$

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Degree sequence characterization of $\vec{C}_3$-anchored digraphs

Theorem [LaMar]:

The degree sequence d is $\vec{C}_3$-anchored if and only if for $J = \{j_1, \ldots, j_n\}$ a set of indices with $3 \le n \le N-1$ and $(k,l) \ge (0,0)$ an index pair, d satisfies one of the three cases

$$\begin{array}{rl} (i) & n = 3 \text{ and } d_{j_1} = d_{j_2} = d_{j_3} = (l+1, k+1) \\ (ii) & n > 3 \text{ and } d_{j_1} = (l+n-2, k+n-2),\ d_{j_2} = \cdots = d_{j_n} = (l+1, k+1) \\ (iii) & n > 3 \text{ and } d_{j_1} = \cdots = d_{j_{n-1}} = (l+n-2, k+n-2),\ d_{j_n} = (l+1, k+1) \end{array}$$

with

$$(d_{j_1}, \ldots, d_{j_n}) = (\bar{d}_{k+1}, \ldots, \bar{d}_{k+n}) = (\underline{d}_{l+1}, \ldots, \underline{d}_{l+n})$$

and the slack sequences satisfying

$$(0, 1, \ldots, 1, 0) = (\bar{s}_k, \bar{s}_{k+1}, \ldots, \bar{s}_{k+n-1}, \bar{s}_{k+n}) = (\underline{s}_l, \underline{s}_{l+1}, \ldots, \underline{s}_{l+n-1}, \underline{s}_{l+n}).$$

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$\vec{C}_3^*$-anchored

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$$\textit{(iii)} \quad n > 3 \text{ and } d_{j_1} = \cdots = d_{j_{n-1}} = (l+n-2, k+n-2),\ d_{j_n} = (l+1, k+1)$$

with

$$(d_{j_1}, \ldots, d_{j_n}) = (\bar{d}_{k+1}, \ldots, \bar{d}_{k+n}) = (\underline{d}_{l+1}, \ldots, \underline{d}_{l+n})$$

and the slack sequences satisfying

$$(0, 1, \ldots, 1, 0) = (\bar{s}_k, \bar{s}_{k+1}, \ldots, \bar{s}_{k+n-1}, \bar{s}_{k+n}) = (\underline{s}_l, \underline{s}_{l+1}, \ldots, \underline{s}_{l+n-1}, \underline{s}_{l+n}).$$

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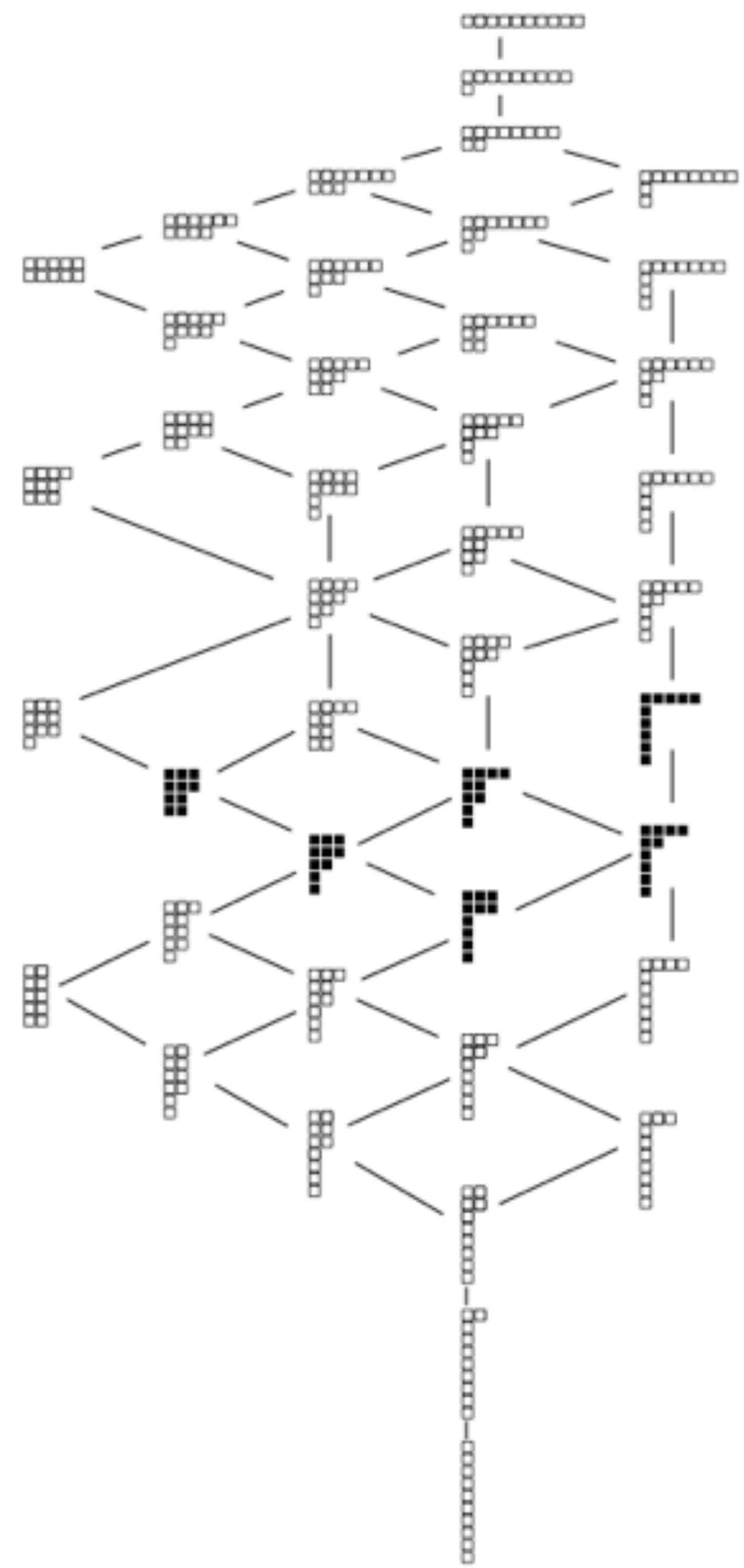

Fig. 2. Hasse diagram for Par(10).

Thank you!