Signal Adaptive Frame Theory

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Abstract

Adaptive frequency band (AFB) and ultra-wide-band (UWB) systems require either rapidly changing or very high sampling rates. Conventional analog-to-digital devices have non-adaptive and limited dynamic range. We investigate AFB and UWB signal processing via a basis projection method.

The method first windows the signal and then decomposes the signal into a basis via a continuous-time inner product operation, computing the basis coefficients in parallel. The windowing systems are key, and we develop systems that have variable partitioning length, variable roll-off and variable smoothness. These include smooth bounded adaptive partitions of unity (BAPU systems) created using B-splines, systems developed to preserve orthogonality of any orthonormal systems between adjacent blocks, and *almost orthogonal* windowing systems that are more computable than the orthogonality preserving systems. We construct the basis projection method for all three types of windows, analyze various methods for signal segmentation and create systems designed for binary signals.

The projection method is, in effect, an adaptive Gabor system for signal analysis. The natural language in which to express this system is frame theory. We finish our talk by developing projection as signal adaptive frame theory.

Acknowledgments

- Research partially supported by U.S. Army Research Office Scientific Services program, administered by Battelle (TCN 06150, Contract DAAD19-02-D-0001).

Definition (Absolutlely and Square Integrable)

A function f is called *absolutely integrable*, i.e. $f \in L^1(\mathbb{R})$, if

$$\int_{\mathbb{R}} |f(x)| dt \equiv \|f\|_1 < \infty.$$

If f is in L^1, we say that $\|f\|_1$ is the L^1 norm of f. Similarly, a function is called *square integrable*, i.e. $f \in L^2(\mathbb{R})$, if

$$\int_{\mathbb{R}} |f(x)|^2 dt \equiv \|f\|_2 < \infty.$$

If f is in L^2, we say that $\|f\|_2$ is the L^2 norm of f.

Definition (Fourier Series)

Let f be a periodic, integrable function on $\mathbb{R}$, with period 2Φ, i.e., $f \in L^1(\mathbb{T}_{2\Phi})$. The Fourier coefficients of f, $\widehat{f}[n]$, are defined by

$$\widehat{f}[n] = \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} f(t) \exp(-i\pi nt/\Phi)\, dt\,.$$

If $\{\widehat{f}[n]\}$ is absolutely summable ($\sum |\widehat{f}[n]| < \infty$), then the Fourier series of f is

$$f(t) = \sum_{n\in\mathbb{Z}} \widehat{f}[n] \exp(i\pi nt/\Phi)\,.$$

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Definition

Let $T > 0$ and let $g(t)$ be a function such that $\operatorname{supp} g \subseteq [0, T]$. The *$T$-periodization* of g is $[g]^\circ(t) = \sum_{n=-\infty}^{\infty} g(t - nT)\,.$

Definition (Fourier Transform and Inversion Formulae)

Let f be a function in L^1. The Fourier transform of f is defined as

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i t\omega} dt$$

for $t \in \mathbb{R}$ (time), $\omega \in \widehat{\mathbb{R}}$ (frequency). The inversion formula, for $\widehat{f} \in L^1(\widehat{\mathbb{R}})$, is

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- Parseval's equality –

$$\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\widehat{\mathbb{R}})} .$$

C-W-W-K-S-R-... Sampling

- $\mathbb{PW}(\Omega) = \{f : f, \widehat{f} \in L^2, \mathrm{supp}(\widehat{f}) \subset [-\Omega, \Omega]\}$.

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Theorem (C-W-W-K-S-R-... Sampling Theorem)

Let $f \in \mathbb{PW}(\Omega)$, $\delta_{n\sigma}(t) = \delta(t - n\sigma)$ *and* $\mathrm{sinc}_\sigma(t) = \frac{\sin(\frac{2\pi}{\sigma}t)}{\pi t}$.

a.) *If* $\sigma \leq 1/2\Omega$, *then for all* $t \in \mathbb{R}$,

$$f(t) = \sigma \sum_{n=-\infty}^{\infty} f(n\sigma) \frac{\sin(\frac{2\pi}{\sigma}(t - n\sigma))}{\pi(t - n\sigma)} = \sigma \left(\left[\sum_{n=-\infty}^{\infty} \delta_{n\sigma} \right] f \right) * \underset{\sigma}{\mathrm{sinc}}.$$

b.) *If* $\sigma \leq 1/2\Omega$ *and* $f(n\sigma) = 0$ *for all* $n \in \mathbb{Z}$, *then* $f \equiv 0$.

Proof of W-K-S Sampling

Proof : Let $f \in \mathbb{PW}(\Omega)$ and let $\sigma \leq 1/2\Omega$.

$$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{-2i\pi n\omega\sigma} \cdot \chi_{[-1/\sigma,1/\sigma]}(\omega)$$

where the Fourier coefficients are given by

$$c_n = \sigma \int_{-1/\sigma}^{1/\sigma} [\widehat{f}(\omega)]^\circ e^{2i\pi n\omega\sigma} d\omega = \sigma \int_{-\infty}^{\infty} [\widehat{f}(\omega)]^\circ e^{2i\pi(n\sigma)\omega} d\omega = \sigma f(n\sigma)$$

Proof of W-K-S Sampling, Cont'd

Substituting back and solving for f using the inverse Fourier transform, we have that $f(t) =$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{2\pi i \omega t} d\omega \\
= \quad & \int_{-\infty}^{\infty} \sigma \sum_{n=-\infty}^{\infty} f(n\sigma) e^{-2i\pi n\omega\sigma} \cdot \chi_{[-1/\sigma, 1/\sigma]} e^{2\pi i \omega t} d\omega \\
= \quad & \sigma \sum_{n=-\infty}^{\infty} f(n\sigma) \int_{-1/\sigma}^{1/\sigma} e^{(2\pi i (t - n\sigma)\omega)} d\omega \\
= \quad & \sigma \sum_{n=-\infty}^{\infty} f(n\sigma) \frac{\sin(\frac{2\pi}{\sigma}(t - n\sigma))}{\pi(t - n\sigma)} .
\end{aligned}$$

Proof of W-K-S Sampling, Cont'd

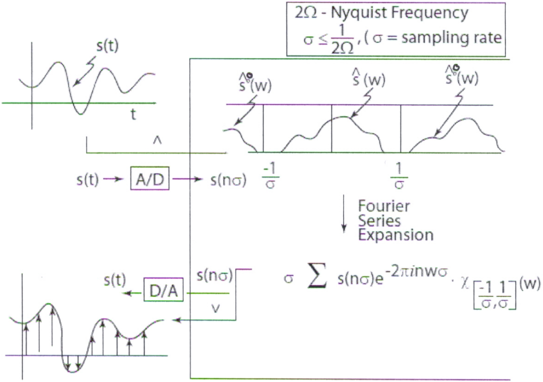

Figure: WKS Sampling

Second Proof of W-K-S Sampling

- Poisson Summation Formula (PSF)

$$\sigma \widehat{\sum_{n\in\mathbb{Z}} \delta_{n\sigma}} = \sum_{n\in\mathbb{Z}} \delta_{n/\sigma} \,.$$

Second Proof of W-K-S Sampling

- Poisson Summation Formula (PSF)

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- **Second Proof :** If $f \in \mathbb{PW}_\Omega$ and $\sigma \leq 1/2\Omega$,

$$\widehat{f}(\omega) = \left(\sum_{n\in\mathbb{Z}} \widehat{f}(\omega - \frac{n}{\sigma}) \right) \cdot \chi_{[-1/\sigma, 1/\sigma)}(\omega) \, .$$

$$\widehat{f}(\omega) = \left(\sum_{n\in\mathbb{Z}} \widehat{f}(\omega - \frac{n}{\sigma}) \right) \cdot \chi_{[-1/\sigma, 1/\sigma)}(\omega) = \left(\sum_{n\in\mathbb{Z}} \left[\delta_{n/\sigma} \right] \widehat{f} \right) \cdot \chi_{[-1/\sigma, 1/\sigma)}(\omega)$$

$$\overset{(PSF)}{\Longleftrightarrow} f(t) = \sigma \left(\left[\sum_{n\in\mathbb{Z}} \delta_{n\sigma} \right] f \right) * \operatorname{sinc}_{\sigma}(t) \, .$$

Errors in W-K-S Sampling

- Truncation Error :

$$f_N(t) = \sigma \sum_{n=-N}^{N} f(n\sigma) \frac{\sin(\frac{2\pi}{\sigma}(t - n\sigma))}{\pi(t - n\sigma)}.$$

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$$E_N = \|f - f_N\|_2^2 = \sigma \sum_{|n|>N} |f(n\sigma)|^2.$$

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- Pointwise error

$$\mathcal{E}_N = \sup |f(t) - f_N(t)| \leq (\sigma E_N)^{1/2}.$$

Errors in W-K-S Sampling, Cont'd

- Aliasing Error - Let $\Omega = 1$, $\sigma \gg 1/2$.

$$\mathcal{E}_A = \sup \left| f(t) - \int_{-1/2}^{1/2} \left(\widehat{f}\right)^{\circ}(\omega) e^{2\pi i t\omega}\, d\omega \right| \leq 2 \int_{|u|\geq 1/2} |\widehat{f}(u)| du.$$

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- $\mathcal{E}_J = \sup \left| f(t) - \sigma \left(\left[\sum_{n=-\infty}^{\infty} \delta_{n\sigma \pm \epsilon_n} \right] f \right) * \mathrm{sinc}_\sigma(t) \right|$. If we assume $|\epsilon_n| \leq J \leq \min\{1/(4\Omega), e^{-1/2}\}$,

$$\mathcal{E}_J \leq KJ \log(1/J),$$

where K is a constant expressed in terms of $\|f\|_\infty$ and $\|f'\|_\infty$.

Projection Method

Adaptive frequency band and ultra-wide-band systems require either rapidly changing or very high sampling rates. These rates stress signal reconstruction in a variety of ways. Clearly, sub-Nyquist sampling creates aliasing error, but error would also show up in truncation, jitter and amplitude, as computation is stressed.

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- Quick and accurate computations of Fourier coefficients, which are computed in parallel.
- Effective adaptive windowing systems.
- The Projection Method is also extremely efficient relative the *Power Game* discussed by Vetterli *et. al.*

Projection Method, Cont'd

- Let $f \in \mathbb{PW}(\Omega)$. For a block of time T, let

$$f(t) = \sum_{k \in \mathbb{Z}} f(t) \chi_{[(k)T,(k+1)T]}(t) \, .$$

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- If we take a given block $f_k(t) = f(t)\chi_{[(k)T,(k+1)T]}(t)$, we can $T-$ periodically continue the function, getting

$$(f_k)^\circ(t) = (f(t)\chi_{[(k)T,(k+1)T]}(t))^\circ \, .$$

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- Expanding $(f_k)^{\circ}(t)$ in a Fourier series, we get

$$(f_k)^{\circ}(t) = \sum_{n \in \mathbb{Z}} \widehat{(f_k)^{\circ}}[n] \exp(2\pi i n t / T) \, .$$

Projection Method, Cont'd

- $$(f_k)^\circ(t) = \sum_{n\in\mathbb{Z}} \widehat{(f_k)^\circ}[n]\exp(2\pi int/T)$$

$$\widehat{(f_k)^\circ}[n] = \frac{1}{T}\int_{(k)_T}^{(k+1)T} f(t)\exp(-2\pi int/T)\,dt\,.$$

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- The original function f is Ω band-limited. However, the truncated block functions f_k are not. Using the original Ω band-limit gives us a lower bound on the number of non-zero Fourier coefficients $\widehat{(f_k)^\circ}[n]$ as follows. We have

$$\frac{n}{T} \leq \Omega\,, \text{i.e. }, n \leq T\cdot\Omega\,.$$

Projection Method, Cont'd

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$$
\begin{aligned}
f(t) &= \sum_{k \in \mathbb{Z}} f(t) \chi_{[(k)T,(k+1)T]}(t) \\
&= \sum_{k \in \mathbb{Z}} \left[(f_k)^{\circ}(t) \right] \chi_{[(k)T,(k+1)T]}(t) \\
&\approx f_{\mathcal{P}} = \sum_{k \in \mathbb{Z}} \left[\sum_{n=-N}^{n=N} \widehat{(f_k)^{\circ}}[n] \exp(2\pi i n t/T) \right] \chi_{[(k)T,(k+1)T]}(t) .
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Projection Method, Cont'd

- This process allows the system to individually evaluate each piece and base its calculation on the needed bandwidth.
- Instead of fixing T, the method allows us to fix any of the three while allowing the other two to fluctuate. From the design point of view, the easiest and most practical parameter to fix is N.
- For situations in which the bandwidth does not need flexibility, it is possible to fix Ω and T by the equation $N = \lceil T \cdot \Omega \rceil$. However, if greater bandwidth Ω is need, choose shorter time blocks T.

Projection Method, Cont'd

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Projection Method, Cont'd

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- Change effects the time blocking $\tau(t)$ and the number of basis elements $N(t)$. Let $\overline{\Omega}(t) = \max\{\Omega(t) : t \in \tau(t)\}$. At minimum, $\widehat{(f_k)^\circ}[n]$ is non-zero if

$$\frac{n}{\tau(t)} \leq \overline{\Omega}(t) \text{ or equivalently, } n \leq \tau(t) \cdot \overline{\Omega}(t)\,.$$

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Projection Method, Cont'd

- Let $N(t) = \lceil \tau(t) \cdot \overline{\Omega}(t) \rceil$.
- Let f, $\widehat{f} \in L^2(\mathbb{R})$ and f have a variable but bounded band-limit $\Omega(t)$. Let $\tau(t)$ be an adaptive block of time. Given $\tau(t)$, let $\overline{\Omega}(t) = \max\{\Omega(t) : t \in \tau(t)\}$. Then, for $N(t) = \lceil \tau(t) \cdot \overline{\Omega}(t) \rceil$, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k \in \mathbb{Z}} \left[\sum_{n=-N(t)}^{N(t)} \widehat{(f_k)^{\circ}}[n] e^{(2\pi i n t/\tau)} \right] \chi_{[k\tau,(k+1)\tau]}(t).$$

Projection Method, Cont'd

Problem : Let $f \in \mathbb{PW}(\Omega)$ and let T be a fixed block of time. Then, for $N = \lceil T \cdot \Omega \rceil$,

$$\begin{aligned}\widehat{f_{\mathcal{P}}}(\omega) &= \sum_{k=-\infty}^{\infty} \Bigg[\sum_{n=-N}^{N} \widehat{(f_k)^{\circ}}[n] \exp\left(2\pi i (k - \frac{1}{2}) T\right)(\omega - \frac{n}{T}) \\ &\quad \left(\frac{\sin(\pi(\frac{\omega T}{2} + \frac{n}{2}))}{\pi(\omega + \frac{n}{T})} \right) \Bigg].\end{aligned}$$

Adaptive ON Preserving Windowing Systems

- General method for segmenting Time-Frequency ($\mathbb{R} - \widehat{\mathbb{R}}$) space. The idea is to cut up time into segments of possibly varying length, where the length is determined by signal bandwidth.

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- The techniques developed use the theory of splines, which give control over smoothness in time and corresponding decay in frequency.
- We make our systems so that we have varying degrees of smoothness with cutoffs adaptive to signal bandwidth.
- We also develop our systems so that the orthogonality of bases in adjacent and possible overlapping blocks is preserved.

Adaptive ON Preserving Windowing Systems, Cont'd

Definition (ON Window System)

Let $0 < r \ll T$. An *ON Window System* for adaptive and ultra-wide band sampling is a set of functions $\{\mathbb{W}_k(t)\}$ such that

$$(i.)\quad \mathrm{supp}(\mathbb{W}_k(t)) \subseteq [kT - r, (k+1)T + r] \text{ for all } k\,,$$

$$(ii.)\quad \mathbb{W}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r] \text{ for all } k\,,$$

$$(iii.)\quad \mathbb{W}_k((kT + T/2) - t) = \mathbb{W}_k(t - (kT + T/2)), t \in [0, T/2 + r]\,,$$

$$(iv.)\quad [\mathbb{W}_k(t)]^2 + [\mathbb{W}_{k+1}(t)]^2 = 1\,,$$

$$(v.)\quad \{\widehat{\mathbb{W}_k{}^\circ}[n]\} \in l^1\,.$$

Adaptive ON Preserving Windowing Systems, Cont'd

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- Generate ON Window System by translation of a window $\mathbb{W}_I$ centered at the origin.
- Conditions $(i.)$ and $(ii.)$ are partition properties.
- Conditions $(iii.)$ and $(iv.)$ are needed to preserve orthogonality.
- Conditions $(v.)$ gives the following. Let $f \in \mathbb{PW}(\Omega)$ and let $\{\mathbb{W}_k(t)\}$ be a ON Window System with generating window $\mathbb{W}_I$. Then

$$\frac{1}{T+2r}\int_{-T/2-r}^{T/2+r} [f \cdot \mathbb{W}_I]^\circ(t)\exp(-2\pi int/[T+2r])\,dt$$
$$= \quad \widehat{f} * \widehat{\mathbb{W}_I}[n]\,.$$

Adaptive ON Preserving Windowing Systems, Cont'd

- **Examples :**
  $\{\mathbb{W}_k(t)\} = \bigcup_{k\in\mathbb{Z}} \chi_{[(k)T,(k+1)T]}(t)$
  $\{\mathbb{W}_k(t)\} = \bigcup_{k\in\mathbb{Z}} \mathrm{Cap}_{[(k)T-r,(k+1)T+r]}(t)\,,$
  where

$$\mathrm{Cap}_I(t) =$$

$$\begin{cases} 0 & |t| \geq T/2 + r\,, \\ 1 & |t| \leq T/2 - r\,, \\ \sin(\pi/(4r)(t + (T/2 + r))) & -T/2 - r < t < -T/2 + r\,, \\ \cos(\pi/(4r)(t - (T/2 - r))) & T/2 - r < t < T/2 + r\,. \end{cases}$$

Adaptive ON Preserving Windowing Systems, Cont'd

- Our general window function $\mathbb{W}_I$ is k-times differentiable, has $\operatorname{supp}(\mathbb{W}_I) = [-T/2 - r, T/2 + r]$, and has values

$$\mathbb{W}_I = \begin{cases} 0 & |t| \geq T/2 + r \\ 1 & |t| \leq T/2 - r \\ \rho(\pm t) & T/2 - r < |t| < T/2 + r \end{cases}$$

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- We solve for $\rho(t)$ by solving the Hermite interpolation problem

$$\begin{cases} (a.) & \rho(T/2 - r) = 1 \\ (b.) & \rho^{(n)}(T/2 - r) = 0\,,\ n = 1, 2, \ldots, k \\ (c.) & \rho^{(n)}(T/2 + r) = 0\,,\ n = 0, 2, \ldots, k\,, \end{cases},$$

$$[\rho(t)]^2 + [\rho(-t)]^2 = 1 \text{ for } t \in [\pm(T/2 - r), \pm(T/2 + r)]$$

Adaptive ON Preserving Windowing Systems, Cont'd

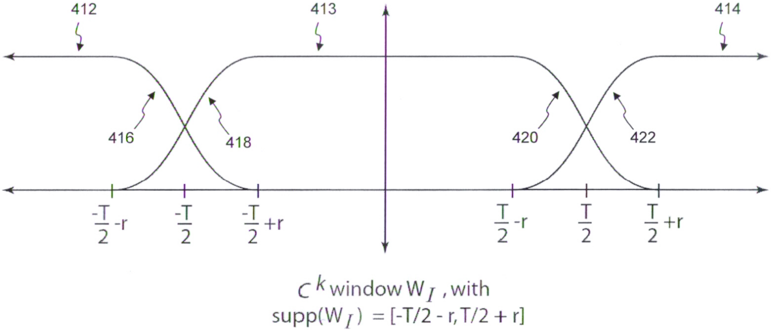

Figure: Window $\mathbb{W}_I$

Adaptive ON Preserving Windowing Systems, Cont'd

- Solving for ρ so that the window in C^1, we get $\rho(t) =$

$$\begin{cases} \frac{1}{\sqrt{2}}\left[1 - \sin(\frac{\pi}{2r}(t + (T/2 + r)))\right] & -T/2 - r < t < -T/2\,, \\ \sqrt{\left[1 - \frac{1}{2}\left[\sin(\frac{\pi}{2r}(t + (T/2 + r)))\right]^2\right]} & -T/2 < t < -T/2 + r\,. \end{cases}$$

Adaptive ON Preserving Windowing Systems, Cont'd

- Solving for ρ so that the window in C^1, we get $\rho(t) =$

$$
\begin{cases}
\dfrac{\frac{1}{\sqrt{2}}\left[1 - \sin\left(\frac{\pi}{2r}(t + (T/2 + r))\right)\right]}{\sqrt{\left[1 - \frac{1}{2}\left[\sin\left(\frac{\pi}{2r}(t + (T/2 + r))\right)\right]^2\right]}} & -T/2 - r < t < -T/2\,, \\
 & -T/2 < t < -T/2 + r\,.
\end{cases}
$$

- With each degree of smoothness, we get an additional degree of decay in frequency.

$\mathbb{W}_k$ Preserve Orthogonality

Let $\{\varphi_j(t)\}$ be an orthonormal basis for $L^2[-T/2, T/2]$. Define

$$\widetilde{\varphi}_j(t) = \begin{cases} 0 & |t| \geq T/2 + r \\ \varphi_j(t) & |t| \leq T/2 - r \\ -\varphi_j(-T - t) & -T/2 - r < t < -T/2 \\ \varphi_j(T - t) & T/2 < t < T/2 + r \end{cases}$$

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

Theorem (The Orthogonality of Overlapping Blocks)

$\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi}_j(t)\}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

Theorem (The Orthogonality of Overlapping Blocks)

$\{\Psi_{k,j}\} = \{\mathbb{W}_k \widetilde{\varphi}_j(t)\}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Sketch of Proof : We want to show that $\langle \Psi_{k,j}, \Psi_{m,n} \rangle = \delta_{k,m} \cdot \delta_{j,n}$. The partitioning properties of the windows give that we need only check overlapping and adjacent windows. Moreover, we need only check window centered at origin.

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

$$
\begin{aligned}
\langle \mathbb{W}_l \widetilde{\varphi}_i, \mathbb{W}_l \widetilde{\varphi}_j \rangle &= \int_{-T/2-r}^{-T/2} (\mathbb{W}_l(t))^2 \varphi_i(-T-t)\varphi_j(-T-t)\, dt \\
&+ \int_{-T/2}^{-T/2+r} ((\mathbb{W}_l(t))^2 - 1)\varphi_i(t)\varphi_j(t)\, dt \\
&+ \int_{-T/2}^{T/2} \varphi_i(t)\varphi_j(t)\, dt \\
&+ \int_{T/2-r}^{T/2} ((\mathbb{W}_l(t))^2 - 1)\varphi_i(t)\varphi_j(t)\, dt \\
&+ \int_{T/2}^{T/2+r} (\mathbb{W}_l(t))^2 \varphi_i(T-t)\varphi_j(T-t)\, dt \,.
\end{aligned}
$$

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- Since $\{\varphi_j\}$ is an ON basis, the third integral equals 1 when $i = j$.

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- Since $\{\varphi_j\}$ is an ON basis, the third integral equals 1 when $i = j$.
- We apply the linear change of variables $t = -T/2 - \tau$ to the first integral and $t = -T/2 + \tau$ to the second integral. We then add these two integrals together to get

$$\int_0^r [(\mathbb{W}_I(T/2-\tau))^2+(\mathbb{W}_I(\tau-T/2))^2-1]\varphi_i(-T/2+\tau)\varphi_j(-T/2+\tau)\, d\tau\,.$$

Conditions (*iii.*) and (*iv.*) give
$[(\mathbb{W}_I(T/2-\tau))^2 + (\mathbb{W}_I(\tau - T/2))^2 - 1] = 0.$

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- Since $\{\varphi_j\}$ is an ON basis, the third integral equals 1 when $i = j$.
- We apply the linear change of variables $t = -T/2 - \tau$ to the first integral and $t = -T/2 + \tau$ to the second integral. We then add these two integrals together to get

$$\int_0^r [(\mathbb{W}_l(T/2-\tau))^2+(\mathbb{W}_l(\tau-T/2))^2-1]\varphi_i(-T/2+\tau)\varphi_j(-T/2+\tau)\,d\tau\,.$$

  Conditions (*iii.*) and (*iv.*) give $[(\mathbb{W}_l(T/2-\tau))^2 + (\mathbb{W}_l(\tau - T/2))^2 - 1] = 0$.
- Applying the linear change of variables $t = T/2 - \tau$ to the fourth integral and $t = T/2 + \tau$ to the fifth integral gives that these two integrals also sum to zero.

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- A similar computation gives that

$$\langle \mathbb{W}_k \widetilde{\varphi}_i, \mathbb{W}_{k+1} \widetilde{\varphi}_j \rangle = 0\,.$$

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$$\langle \mathbb{W}_k \widetilde{\varphi}_i, \mathbb{W}_l \widetilde{\varphi}_j \rangle = 0\,.$$

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- The partitioning property gives that for $|k-l| \geq 2$,

$$\langle \mathbb{W}_k \widetilde{\varphi}_i, \mathbb{W}_l \widetilde{\varphi}_j \rangle = 0\,.$$

- To finish, we need to show $\{\Psi_{k,j}\}$ spans $L^2(\mathbb{R})$. Given any function $f \in L^2$, consider the windowed element $f_k(t) = \mathbb{W}_k(t) \cdot f(t)$. Let $f_l(t) = \mathbb{W}_l(t) \cdot f(t)$. We have that $\{\varphi_j(t)\}$ is an orthonormal basis for $L^2[-T/2, T/2]$.

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

Let $f_l(t) = \mathbb{W}_l(t) \cdot f(t)$. We have that $\{\varphi_j(t)\}$ is an orthonormal basis for $L^2[-T/2, T/2]$. Given f_l, define

$$\bar{f}_l(t) =$$

$$\begin{cases} 0 & |t| \geq T/2 + r \\ f_l(t) & |t| \leq T/2 - r \\ f_l(t) - f_l(-T - t) & -T/2 - r < t < -T/2 \\ f_l(t) + f_l(T - t) & T/2 < t < T/2 + r \end{cases}$$

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- Since $\bar{f}_l \in L^2[-T/2, T/2]$, we may expand it as

$$\sum_{j=1}^{\infty} \langle \bar{f}_l, \varphi_j \rangle \varphi_j(t) .$$

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- Since $\bar{f}_I \in L^2[-T/2, T/2]$, we may expand it as

$$\sum_{j=1}^{\infty} \left\langle \bar{f}_I, \varphi_j \right\rangle \varphi_j(t)\,.$$

- To extend this to $L^2[-T/2 - r, T/2 + r]$, we expand using $\{\widetilde{\varphi}_j(t)\}$, getting

$$\widetilde{\bar{f}}_I = \sum_{j=1}^{\infty} \left\langle \bar{f}_I, \varphi_j \right\rangle \widetilde{\varphi}_j(t)\,.$$

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- Then

$$\widetilde{\overline{f}}_I = \sum_{j=1}^{\infty} \langle \overline{f}_I, \varphi_j \rangle \, \widetilde{\varphi}_j(t) \, .$$

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- Then

$$\widetilde{\overline{f}}_I = \sum_{j=1}^{\infty} \langle \overline{f}_I, \varphi_j \rangle \, \widetilde{\varphi}_j(t)\,.$$

- $\widetilde{\overline{f}}_I(t) =$

$$\begin{cases} 0 & |t| \geq T/2 + r \\ f_I(t) & |t| \leq T/2 - r \\ f_I(t) - f_I(-T - t) & -T/2 - r < t < -T/2 + r \\ f_I(t) + f_I(T - t) & T/2 - r < t < T/2 + r \end{cases}$$

This construction preserves orthogonality between adjacent blocks.

$\mathbb{W}_k$ Preserve Orthogonality, Cont'd

- To finish, let f be any function in L^2. Consider the windowed element $f_k(t) = \mathbb{W}_k(t) \cdot f(t)$. Repeat the construction above for this window. This shows that, for fixed k, $\{\Psi_{k,j}\}$ spans $L^2([kT - r, (k+1)T + r])$ and preserves orthogonality between adjacent blocks on either side. Summing over all $k \in \mathbb{Z}$ gives that $\{\Psi_{k,j}\}$ is an ON basis for $L^2(\mathbb{R})$. $\square$

Partition of Unity Systems

- Similar construction techniques give us partition of unity functions. The theory of B-splines gives us the tools to create these systems.

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- If we replace condition ($iv.$) with

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  we get a bounded adaptive partition of unity.
- The systems can be built using B-splines, and have Fourier transforms of the form

$$\left[\frac{\sin(2\pi T\omega)}{\pi\omega}\right]^n.$$

Partition of Unity Systems, Cont'd

Definition (Bounded Adaptive Partition of Unity)

A *Bounded Adaptive Partition of Unity* is a set of functions $\{\mathbb{B}_k(t)\}$ such that

$$
\begin{array}{ll}
(i.) & \mathrm{supp}(\mathbb{B}_k(t)) \subseteq [kT - r, (k+1)T + r]\,, \\
(ii.) & \mathbb{B}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r]\,, \\
(iii.) & \mathbb{B}_k((kT + T/2) - t) = \mathbb{B}_k(t - (kT + T/2)), t \in [0, T/2 + r]\,, \\
(iv.) & \displaystyle\sum_k \mathbb{B}_k(t) \equiv 1\,, \\
(v.) & \{\widehat{\mathbb{B}_k{}^{\circ}}[n]\} \in l^1\,.
\end{array}
$$

Partition of Unity Systems, Cont'd

- Conditions $(i.), (ii.)$ and $(iv.)$ make $\{\mathbb{B}_k(t)\}$ a bounded partition of unity.

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- Conditions $(i.), (ii.)$ and $(iv.)$ make $\{\mathbb{B}_k(t)\}$ a bounded partition of unity.
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- We will again generate our systems by translations and dilations of a given window $\mathbb{B}_I$, where $\mathrm{supp}(\mathbb{B}_I) = [(-T/2 - r), (T/2 + r)]$.

Partition of Unity Systems, Cont'd

- Conditions $(i.), (ii.)$ and $(iv.)$ make $\{\mathbb{B}_k(t)\}$ a bounded partition of unity.
- The change in condition $(iv.)$ means that these systems do not preserve orthogonality between blocks.
- We will again generate our systems by translations and dilations of a given window $\mathbb{B}_I$, where $\mathrm{supp}(\mathbb{B}_I) = [(-T/2 - r), (T/2 + r)]$.
- Our first example was developed by studying the de la Vallée-Poussin kernel used in Fourier series. Let $0 < r \ll T$ and let

$$\mathrm{Tri}_L(t) = \max\{[((2T/(4r)) + r) - |t|/(2r)], 0\}\,,$$
$$\mathrm{Tri}_S(t) = \max\{[((2T/(4r)) + r - 1) - |t|/(2r)], 0\} \text{ and}$$
$$\mathrm{Trap}(t) = \mathrm{Tri}_L(t) - \mathrm{Tri}_S(t)\,.$$

The Trap function has perfect overlay in the time domain and $1/\omega^2$ decay in frequency space.

Partition of Unity Systems, Cont'd

- **Examples :**
  $\{\mathbb{B}_k(t)\} = \bigcup_{k\in\mathbb{Z}} \chi_{[(k)T,(k+1)T]}(t)$
  $\{\mathbb{B}_k(t)\} = \bigcup_{k\in\mathbb{Z}} \mathrm{Trap}_{[(k)T-r,(k+1)T+r]}(t)\,.$

Partition of Unity Systems, Cont'd

- **Examples :**
  $\{\mathbb{B}_k(t)\} = \bigcup_{k\in\mathbb{Z}} \chi_{[(k)T,(k+1)T]}(t)$
  $\{\mathbb{B}_k(t)\} = \bigcup_{k\in\mathbb{Z}} \mathrm{Trap}_{[(k)T-r,(k+1)T+r]}(t)\,.$
- Our general window function $\mathbb{W}_I$ is k-times differentiable, has $\mathrm{supp}(\mathbb{B}_I) = [(-T/2-r),(T/2+r)]$ and has values

$$\mathbb{B}_I = \begin{cases} 0 & |t| \geq T/2+r \\ 1 & |t| \leq T/2-r \\ \rho(\pm t) & T/2-r < |t| < T/2+r \end{cases}$$

Partition of Unity Systems, Cont'd

- We again solve for $\rho(t)$ by solving the Hermite interpolation problem

$$\begin{cases} (a.) & \rho(T/2 - r) = 1 \\ (b.) & \rho^{(n)}(T/2 - r) = 0\,,\ n = 1, 2, \ldots, k \\ (c.) & \rho^{(n)}(T/2 + r) = 0\,,\ n = 0, 1, 2, \ldots, k\,, \end{cases}$$

with the conditions that $\rho \in C^k$ and

$$[\rho(t)] + [\rho(-t)] = 1 \text{ for } t \in [T/2 - r, T/2 + r]\,.$$

Partition of Unity Systems, Cont'd

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with the conditions that $\rho \in C^k$ and

$$[\rho(t)] + [\rho(-t)] = 1 \text{ for } t \in [T/2 - r, T/2 + r]\,.$$

- We use B-splines as our cardinal functions. Let $0 < \alpha \ll \beta$ and consider $\chi_{[-\alpha,\alpha]}$. We want the n-fold convolution of $\chi_{[\alpha,\alpha]}$ to fit in the interval $[-\beta, \beta]$.

Partition of Unity Systems, Cont'd

- Then we choose α so that $0 < n\alpha < \beta$ and let

$$\Psi(t) = \underbrace{\chi_{[-\alpha,\alpha]} * \chi_{[-\alpha,\alpha]} * \cdots * \chi_{[-\alpha,\alpha]}(t)}_{n-times} .$$

Partition of Unity Systems, Cont'd

- Then we choose α so that $0 < n\alpha < \beta$ and let

$$\Psi(t) = \underbrace{\chi_{[-\alpha,\alpha]} * \chi_{[-\alpha,\alpha]} * \cdots * \chi_{[-\alpha,\alpha]}(t)}_{n-times} .$$

- The β-periodic continuation of this function, $\Psi^\circ(t)$ has the Fourier series expansion

$$\sum_{k \neq 0} \frac{\alpha}{n\beta} \left[\frac{\sin(\pi k\alpha/n\beta)}{2\pi k\alpha/n\beta} \right]^n \exp(\pi ikt/\beta) .$$

Partition of Unity Systems, Cont'd

- The C^k solution for ρ is given by a theorem of Schoenberg. Schoenberg solved the Hermite interpolation problem

$$\begin{cases} (a.) & S^{(n)}(-1) = 0\,,\ n = 0, 1, 2, \ldots, k\,, \\ (b.) & S(1) = 1\,, \\ (b.) & S^{(n)}(1) = 0\,,\ n = 1, 2, \ldots, k\,. \end{cases}$$

Partition of Unity Systems, Cont'd

- The C^k solution for ρ is given by a theorem of Schoenberg. Schoenberg solved the Hermite interpolation problem

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- An interpolant that minimizes the Chebyshev norm is called the *perfect spline*. The perfect spline $S(t)$ for Hermite problem above is given by the integral of the function

$$M(x) = (-1)^n \sum_{j=0}^{k} \frac{\Psi(t - t_j)}{\phi'(t_j)}\,,$$

where Ψ is the $(k+1)$ convolution of characteristic functions, the knot points are $t_j = -\cos(\frac{\pi j}{k})$ and $\phi(t) = \prod_{j=0}^{k}(t - t_j)$.

Partition of Unity Systems, Cont'd

- We then have that

$$\rho(t) = S \circ \ell(t)\,, \text{ where } \ell(t) = \frac{1}{r}t - \frac{2T}{2r}\,.$$

Partition of Unity Systems, Cont'd

- We then have that

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- For this ρ, and for

$$\mathbb{B}_I = \begin{cases} 0 & |t| \geq T/2 + r \\ 1 & |t| \leq T/2 - r \\ \rho(\pm t) & T/2 - r < |t| < T/2 + r \end{cases}$$

we have that $\widehat{\mathbb{B}_I}(\omega)$ is given by the antiderivative of a linear combination of functions of the form

$$\left[\frac{\sin(2\pi T\omega)}{\pi\omega}\right]^{k+1},$$

and therefore has decay $1/\omega^{k+2}$ in frequency.

Almost ON Systems

- Cotlar, Knapp and Stein introduced *almost orthogonality* via operator inequalities.

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- We are looking to create windowing systems that are more computable/constructible such as the *Bounded Adaptive Partition of Unity* systems $\{\mathbb{B}_k(t)\}$ with the orthogonality preservation of the *ON Window System* $\{\mathbb{W}_k(t)\}$.

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- Cotlar, Knapp and Stein introduced *almost orthogonality* via operator inequalities.
- We are looking to create windowing systems that are more computable/constructible such as the *Bounded Adaptive Partition of Unity* systems $\{\mathbb{B}_k(t)\}$ with the orthogonality preservation of the *ON Window System* $\{\mathbb{W}_k(t)\}$.
- Consider $\{\mathbb{W}_k(t)\} = \bigcup_{k\in\mathbb{Z}} \mathrm{Cap}_{[(k)T-r,(k+1)T+r]}(t)$, where

$$\mathrm{Cap}_I(t) =$$

$$\begin{cases} 0 & |t| \geq T/2 + r\,, \\ 1 & |t| \leq T/2 - r\,, \\ \sin(\pi/(4r)(t + (T/2 + r))) & -T/2 - r < t < -T/2 + r\,, \\ \cos(\pi/(4r)(t - (T/2 - r))) & T/2 - r < t < T/2 + r\,. \end{cases}$$

Almost ON Systems, Cont'd

Definition (Almost ON System)

Let $0 < r \ll T$. An *Almost ON System* for adaptive and ultra-wide band sampling is a set of functions $\{\mathbb{A}_k(t)\}$ for which there exists δ, $0 \le \delta \le 1/2$, such that

$$
\begin{array}{ll}
(i.) & \operatorname{supp}(\mathbb{A}_k(t)) \subseteq [kT - r, (k+1)T + r] \text{ for all } k\,, \\
(ii.) & \mathbb{A}_k(t) \equiv 1 \text{ for } t \in [kT + r, (k+1)T - r] \text{ for all } k\,, \\
(iii.) & \mathbb{A}_k((kT + T/2) - t) = \mathbb{A}_k(t - (kT + T/2)), t \in [0, T/2 + r]\,, \\
(iv.) & 1 - \delta \le [\mathbb{A}_k(t)]^2 + [\mathbb{A}_{k+1}(t)]^2 \le 1 + \delta\,, \\
(v.) & \{\widehat{\mathbb{A}_k{}^{\circ}}[n]\} \in l^1\,.
\end{array}
$$

Almost ON Systems, Cont'd

- Start with $\bigcup_{k\in\mathbb{Z}} \mathrm{Cap}_{[(k)T-r,(k+1)T+r]}(t)$, where

$$\mathrm{Cap}_I(t) =$$

$$\begin{cases} 0 & |t| \geq T/2 + r\,, \\ 1 & |t| \leq T/2 - r\,, \\ \sin(\pi/(4r)(t + (T/2 + r))) & -T/2 - r < t < -T/2 + r\,, \\ \cos(\pi/(4r)(t - (T/2 - r))) & T/2 - r < t < T/2 + r\,. \end{cases}$$

Almost ON Systems, Cont'd

- Start with $\bigcup_{k\in\mathbb{Z}} \mathrm{Cap}_{[(k)T-r,(k+1)T+r]}(t)$, where

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- Let $\Delta_{(T,r)} = \frac{T+2r}{m}$. By placing equidistant knot points

$$-T/2 - r = x_0, -T/2 - r + \Delta_{(T,r)} = x_1, \ldots, T/2 + r = x_m,$$

we can construct C^m polynomial splines S_{m+1} approximating

$$\mathrm{Cap}(t) \text{ in } [(-T/2 - r), (T/2 + r)]\,.$$

Almost ON Systems, Cont'd

- A theorem of Curry and Schoenberg gives that the set of B-splines

$$\{B^{(m+1)}_{-(m+1)}, \ldots, B^{(m+1)}_{k}\}$$

forms a basis for S_{m+1}.

Almost ON Systems, Cont'd

- A theorem of Curry and Schoenberg gives that the set of B-splines

$$\{B_{-(m+1)}^{(m+1)}, \ldots, B_k^{(m+1)}\}$$

forms a basis for S_{m+1}.

- Therefore,

$$\mathrm{Cap}(t) \approx \sum_{i=-(m+1)}^{k} a_i B_i^{(m+1)}(t)\,.$$

Let

$$\delta = \left\| \sum_{i=-(m+1)}^{k} a_i B_i^{(m+1)}(t) - \mathrm{Cap}(t) \right\|_{\infty}.$$

Then, $\delta < 1/2$, with the largest value for the piecewise linear spline approximation. Moreover, $\delta \longrightarrow 0$ as m and k increase.

Almost ON Systems, Cont'd

- The partition of unity systems do *not* preserve orthogonality between blocks. However, they are easier to compute, being based on spline constructions.

Almost ON Systems, Cont'd

- The partition of unity systems do *not* preserve orthogonality between blocks. However, they are easier to compute, being based on spline constructions.
- Therefore, these systems can be used to approximate the Cap system with B-splines. Here we get windowing systems that nearly preserve orthogonality. Each added degree of smoothness in time adds to the degree of decay in frequency.

Projection Revisited

Theorem (Wideband Sampling via Projection)

Let $\{\mathbb{W}_k(t)\}$ be a ON Window System, and let $\{\Psi_{k,j}\}$ be an orthonormal basis that preserves orthogonality between adjacent windows. Let $f \in \mathbb{PW}(\Omega)$ and $N = N(T, \Omega)$ be such that $\langle f, \Psi_n \rangle = 0$ for all $n > N$. Then, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k=-\infty}^{\infty} \left[\sum_{n=-N}^{N} \langle f \cdot \mathbb{W}_k, \Psi_{k,n} \rangle \Psi_{k,n}(t) \right].$$

Projection Revisited, Cont'd

Theorem (Adaptive Sampling via Projection)

Let $f, \widehat{f} \in L^2(\mathbb{R})$ and f have a variable but bounded band-limit $\Omega(t)$. Let $\tau(t)$ be an adaptive block of time. Let $\{\mathbb{W}_k(t)\}$ be a ON Window System with window size $\tau(t) + 2r$ on the kth block, and let $\{\Psi_{k,n}\}$ be an orthonormal basis that preserves orthogonality between adjacent windows. Let $N(t) = N(\tau(t), \Omega(t))$ be such that $\langle f, \Psi_{k,n} \rangle = 0$ for all $n > N$. Then, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k=-\infty}^{\infty} \left[\sum_{n=-N(t)}^{N(t)} \langle f \cdot \mathbb{W}_k, \Psi_{k,n} \rangle \Psi_{k,n}(t) \right].$$

Projection Revisited, Cont'd

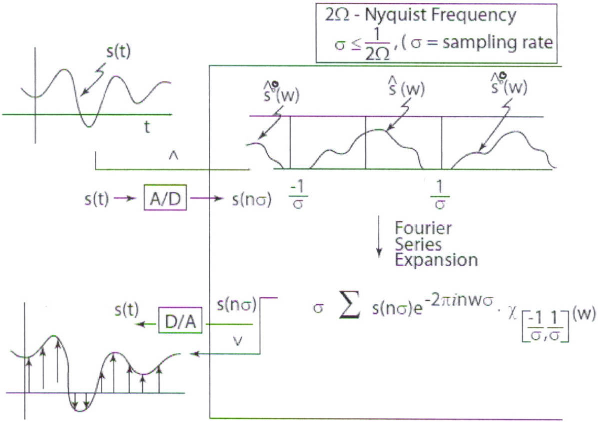


Figure: WKS Sampling – Stationary View of Signal

Projection Revisited, Cont'd

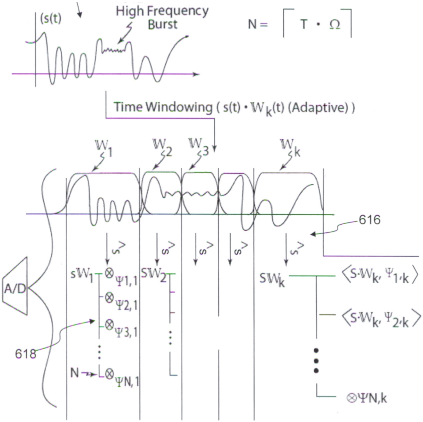


Figure: Projection Part 1 – Windowed Stationarity

Projection Revisited, Cont'd

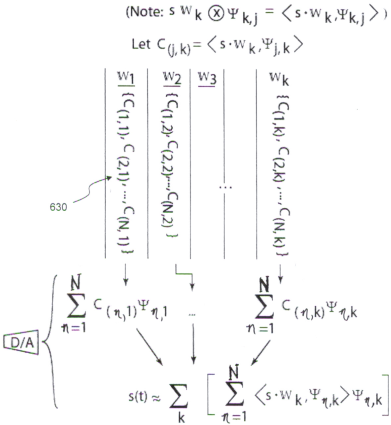
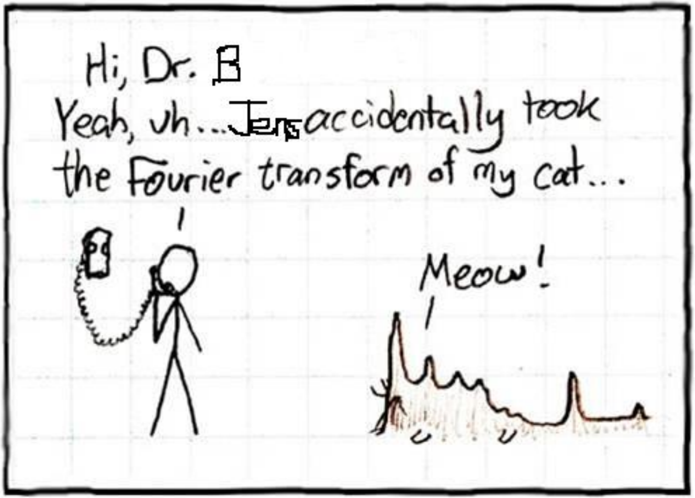


Figure: Projection Part 2 – Windowed Stationarity

Perspective on Bandwidth



Error Analysis

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$$
\begin{aligned}
\mathcal{E}_{k_\mathcal{P}} &= \sup\left|(f(t) \cdot \mathbb{W}_k) - \left[\sum_{n=-N}^{N} \langle f \cdot \mathbb{W}_k, \Psi_{k,n}\rangle \Psi_{k,n}(t)\right]\right| \\
&= \sup\left[\sum_{|n|>N} \langle f \cdot \mathbb{W}_k, \Psi_{k,n}\rangle \Psi_{k,n}(t)\right] \\
&\leq \left[\sum_{|n|>N} \frac{M}{n^{k+2}}\right].
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- Additional projection onto the Gegenbauer polynomials gives error summable over all blocks.

Walsh Functions

- The Walsh functions $\{\Upsilon_n\}$ form an orthonormal basis for $L^2[0,1]$. The basis functions have the range $\{1,-1\}$, with values determined by a dyadic decomposition of the interval. The Walsh functions are of modulus 1 everywhere.

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- The functions are give by the rows of the unnormalized Hadamard matrices, which are generated recursively by

$$H(2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H(2^{(k+1)}) = H(2) \otimes H(2^k) = \begin{bmatrix} H(2^k) & H(2^k) \\ H(2^k) & -H(2^k) \end{bmatrix}.$$

Projection Method and Binary Signals

- Translate and scale the function on this kth interval back to $[0,1]$ by a linear mapping. Denote the resultant mapping as f_{k_T}. The resultant function is an element of $L^2[0,1]$. Given that $f \in \mathbb{PW}(\Omega)$, there exists an $M > 0$ ($M = M(\Omega)$) such that $\langle f_{k_T}, \Upsilon_n \rangle = 0$ for all $n > M$. The decomposition of f_{k_T} into Walsh basis elements is $\sum_{n=0}^{M} \langle f_k, \Upsilon_n \rangle \Upsilon_n$. Translating and summing up gives the Projection representation $f_{\mathcal{P}_T}$

$$f_{\mathcal{P}_T}(t) = \sum_{k \in \mathbb{Z}} \left[\sum_{n=0}^{N} \langle f_{k_T}, \Upsilon_n \rangle \Upsilon_n \right] \mathbb{W}_k(t).$$

Time-Frequency Analysis

- Let $\tau(t)$ be an adaptive block of time. Let $\{\mathbb{W}_k(t)\}$ be a ON Window System with window size $\tau(t) + 2r$ on the kth block, and let $\{\Psi_{k,j}\}$ be an orthonormal basis that preserves orthogonality between adjacent windows. Let $N(t) = N(\tau(t), \Omega(t))$ be such that $\langle f \cdot \mathbb{W}_k, \Psi_{k,n} \rangle = 0$ Then, $f(t) \approx f_{\mathcal{P}}(t)$, where

$$f_{\mathcal{P}}(t) = \sum_{k=-\infty}^{\infty} \left[\sum_{n=-N(t)}^{N(t)} \langle f \cdot \mathbb{W}_k, \Psi_{k,n} \rangle \Psi_{k,n}(t) \right].$$

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- Adaptive "Gabor-Type" System for Time-Frequency Analysis.

Time-Frequency Analysis, Cont'd

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- Recall the Haar Wavelet System

Time-Frequency Analysis, Cont'd

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- Recall the Haar Wavelet System
- Adaptive "Wavelet-Type" System for Time-Frequency Analysis.

Signal Adaptive Frame Theory

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Recall : Let $\mathbb{H}$ be a Hilbert Space. A **Reisz basis** $\mathcal{B}$ for $\mathbb{H}$ is a bounded unconditional basis. As is well known, $\mathcal{B}$ is a Reisz basis if and only if it is equivalent to $\mathcal{E}$, an orthonormal basis for $\mathbb{H}$.

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Definition

A sequence of elements $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathbb{H}$ is a **frame** in there exist constants A and B such that

$$A\|f\| \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \leq B\|f\| \, .$$

Signal Adaptive Frame Theory

- If we work with the ON windowing system $\{\mathbb{W}_k(t)\}$, let $\{\Psi_{k,j}\}$ be an orthonormal basis that preserves orthogonality between adjacent windows. Let $f \in \mathbb{PW}_\Omega$ and $N = N(T, \Omega)$ be such that $\langle f \cdot \mathbb{W}_k, \Psi_{k,n} \rangle = 0$ for all $n > N$.

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- Then

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- This also gives

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} \left[\sum_{n \in \mathbb{Z}} |\langle f \cdot \mathbb{W}_k, \Psi_{k,n} \rangle|^2 \right].$$

Signal Adaptive Frame Theory, Cont'd

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Theorem (Almost Orthogonal Window Frames – Conjecture)

$$\mathcal{A}_{1-\delta}\|f\|^2 \leq \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}|\langle f\cdot\mathbb{A}_k, \Psi_{n,k}\rangle|^2\right] \leq \mathcal{A}_{1+\delta}\|f\|^2 .$$

Moreover, this $\longrightarrow$ *Normalized Tight Frame as* $\delta \longrightarrow 0$.

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