Subordinate Brownian Motions and Their Applications

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Outline

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A process $B = \{B_t : t \geq 0\}$ taking values in $\mathbb{R}^d$ is called a d-dimensional Brownian motion if (1) $B_0 = 0$; (2) B has independent increments, that is, for any $t, s > 0$, $B_{t+s} - B_t$ is independent of B_t; (3) for any $t, s > 0$, $B_{t+s} - B_t$ is a normal random variable with mean zero and covariance matrix $\sqrt{2t}\,\mathbf{I}$.

We will use $\mathbb{P}$ to denote the law of B and $\mathbb{E}$ to denote expectation wrt $\mathbb{P}$. For any $x \in \mathbb{R}^d$, we will use $\mathbb{P}_x$ to denote the law of the process $x + B = \{x + B_t : t \geq 0\}$ and $\mathbb{E}_x$ to denote expectation wrt $\mathbb{P}_x$.

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If we use $p(t,x,y)$ to denote the transition density of a d-dimensional Brownian motion, then by definition,

$$p(t,x,y) = (4\pi t)^{d/2} \exp(-\frac{|y-x|^2}{4t}).$$

The generator of Brownian motion is the Laplacian Δ. In other words, for any $y \in \mathbb{R}^d$, $(t,x) \mapsto p(t,x,y)$ is a solution of the (Fokker-Planck) equation:

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x).$$

Brownian motion has many appealing statistical features: (1) It has finite moment of all orders; (2) it has continuous sample paths (or trajectories); and (3) it satisfies a self-similarity (or scaling property): for any $a > 0$, $a^{-1/2}B_{at}$ has the same distribution as B_t.

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Because of appealing statistical properties and its amenability to mathematical analysis, Brownian motion has been THE model for continuous time motion and noise.

However, Brownian motion is obviously inadequate in a lot complex systems: (1) lots of real world data exhibit heavy tail behavior; (2) many systems does not evolve continuously.

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A process $X = \{X_t : t \geq 0\}$ taking values in $\mathbb{R}^d$ is called a Lévy process in $\mathbb{R}^d$ if (1) $X_0 = 0$; (2) X has independent increments, that is, for any $t, s > 0$, $X_{t+s} - X_t$ is independent of X_t; (3) for any $t, s > 0$, $X_{t+s} - X_t$ has the same distribution as X_s.

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Lévy processes are widely applied nowadays in various fields: physics, finance, operational research, economics, etc.

The simplest Lévy process is the Poisson process. For any $\lambda > 0$, a Poisson process of intensity λ can be described as follows: the process starts at the origin, it stays there an random amount ($\exp(\lambda)$) of time and then jumps to 1, it stays at 1 an random amount ($\exp(\lambda)$, independent of the stay at 0) of time and then jumps to 2, etc.

Another way to describe a Poisson process $N = \{N_t : t \geq 0\}$ of intensity λ is that it is a Lévy process such that N_t is a Poisson random variable with parameter λt.

The characteristic function of a Poisson process $N = \{N_t : t \geq 0\}$ of intensity λ is given by

$$\mathbb{E}e^{i\theta N_t} = \exp\left(-t\lambda(1 - e^{i\theta})\right), \quad \theta \in \mathbb{R}.$$

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Another example of a Lévy process is the compound Poisson process. Suppose that $\lambda > 0$ is a constant and ν is a distribution on $\mathbb{R}^d \setminus \{0\}$. A compound Poisson process with intensity λ and step distribution F can be described as follows: It starts at the origin, stays there an random amount (exp(λ)) of time and then jumps according to the distribution ν; it stays at the new position an random amount (exp(λ), independent of the stay at 0) of time and then jumps (independent of the previous jumps) according to ν, etc.

Another way to describe a compound Poisson process with intensity λ and step distribution ν is as follows. Suppose that $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity λ. Suppose that $Y_1, Y_2, \cdots$ are iid random variables with common distribution F and independent of N. Then the process $X = \{X_t : t \geq 0\}$ defined by

$$X_t = \sum_{k=1}^{N_t} Y_k, \quad t \geq 0$$

is a compound Poisson process with intensity λ and step distribution ν.

The characteristic function of a compound Poisson process with intensity λ and step distribution ν is given by

$$\mathbb{E}e^{i\theta X_t} = \exp\left(-t\lambda \int_{-\infty}^{\infty} (1 - e^{i\theta s})\nu(ds)\right), \quad \theta \in \mathbb{R}.$$

A Poisson process with intensity λ is a compound Poisson process with intensity λ and step distribution $\nu(ds) = \delta_1(ds)$.

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A Lévy process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ can be described by its characteristic function

$$\mathbb{E}[\exp\{i\xi \cdot (X_t - X_0)\}] = \exp(-t\Psi(\xi)), \quad \xi \in \mathbb{R}^d, t > 0,$$

where Ψ, called the characteristic exponent or Lévy exponent of the process, is given by the Lévy-Khintchine formula

$$\Psi(\xi) = ia \cdot \xi + \frac{1}{2}\xi \cdot Q\xi + \int_{\mathbb{R}^d} \left(1 - e^{i\xi \cdot x} + i\xi \cdot x 1_{\{|x|<1\}}\right) \Pi(dx).$$

Here a is a vector in $\mathbb{R}^d$, Q is a non-negative definite $d \times d$ matrix, Π is a measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int (1 \wedge |x|^2)\Pi(dx) < \infty$. a is called the linear coefficient, Q the diffusion matrix and Π the Lévy measure of the process. (a, Q, Π) is called the generating triplet of X.

The infinitesimal generator of the above Lévy process is given by

$$\begin{aligned}\mathcal{A}f(x) &= -a\cdot\nabla f(x)+\frac{1}{2}\sum_{i,j}Q_{ij}f_{ij}(x)\\ &\quad+\int_{\mathbb{R}^d}(f(x+y)-f(x)-1_{\{|y|<1\}}y\cdot\nabla f(x))\Pi(dx).\end{aligned}$$

Or equivalently, for any bounded continuous function f, the function $u(t,x)=\mathbb{E}_x f(X_t)$ is the solution of the equation

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For any $a \in \mathbb{R}^d$, any non-negative definite $d \times d$ matrix Q and any measure Π on $\mathbb{R}^d \setminus \{0\}$ such that $\int (1 \wedge |x|^2)\Pi(dx) < \infty$, there is a Lévy process X with generating triplet (a, Q, Π). Here is a way of constructing such a Lévy process:

Let $X^{(1)}$ be the following BM with drift:

$$X_t^{(1)} = \sqrt{Q} B_t + at.$$

Let $X^{(2)}$ be a compound Poisson process with intensity $\Pi(B(0,1)^c)$ and step distribution $\Pi(B(0,1)^c)^{-1}\Pi(\cdot)|_{B(0,1)^c}$.

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For any $n \geq 1$, let $Y^{(n)}$ be a compound Poisson process with intensity $\Pi(B(1/n, 1))$ and step distribution $\Pi(B(\frac{1}{n}, 1))^{-1}\Pi(\cdot)|_{B(1/n,1)}$. Let $Z^{(n)}$ be defined by

$$Z_t^{(n)} = Y_t^{(n)} - t\Pi(B(1/n, 1)).$$

Then it can be shown that, as $n \to \infty$, $Z^{(n)}$ has a limit and we call this limit $X^{(3)}$.

If $X^{(1)}, X^{(2)}, X^{(3)}$ are independent, then

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When $a = 0$, $Q = 0$, $\Pi(dx) = c|x|^{d+\alpha}$ for some $\alpha \in (0,2)$, we have $\Psi(\xi) = c_1|\xi|^{\alpha}$. The corresponding process is called a symmetric α-stable process on $\mathbb{R}^d$. Its generator is the fractional Laplacian $-(-\Delta)^{\alpha/2}$.

The transition density $p(t,x,y)$ of a symmetric α-stable process X satisfies

$$p(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right).$$

X has infinite variance and, when $\alpha \le 1$, it also has infinite mean.

A symmetric α-stable process satisfies the following self-similarity (scaling property): for any $c > 0$, $c^{-1/\alpha}X_{ct}$ has the same distribution as X_t.

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Lévy processes form a very rich class of processes. However, general Lévy processes are not very tractable. Subordinate Brownian motions are obtained from Brownian motion by replacing its time parameter t by an independent subordinator, i.e., an increasing Lévy process starting from 0. Subordinate BMs form a very large class of Lévy processes. Yet, they are much more tractable.

Before we define subordinate Brownian motions, we first say a few things about subordinators.

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A subordinator is a just a nonnegative Lévy process starting from 0, which is necessarily increasing. A subordinator $S = (S_t : t \geq 0)$ is usually characterized by its Laplace transform

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \qquad \forall\, t, \lambda > 0.$$

The function ϕ is called the Laplace exponent of the subordinator.

The Laplace exponent of a subordinator can be written in the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$. b is called the drift coefficient and μ the Lévy measure of the subordinator.

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$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\,\mu(dt)$$

is the Laplace exponent of some subordinator.

A function $\phi : (0, \infty) \to (0, \infty)$ is the Laplace exponent of some subordinator if and only if $\phi(0+) = 0$ and

$$(-1)^n \phi^{(n)}(t) \leq 0, \quad t > 0, n = 1, 2, \ldots.$$

A function satisfying the properties above is called a Bernstein function.

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For technical reasons, we will sometimes assume that the Lévy measure μ of ϕ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \geq 0$ for every non-negative integer $n \geq 1$. (This is equivalent to saying that ϕ is a complete Bernstein function.)

When the assumption above is satisfied, the mean occupation time measure of S

$$U(A) = \mathbb{E} \int_0^\infty 1_A(S_t)dt, \quad A \subset [0, \infty)$$

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Outline

Let $B = (B_t : t \geq 0)$ be a d-dimensional Brownian motion, and let $S = (S_t : t \geq 0)$ be an independent subordinator. The process $X = (X_t : t \geq 0)$ defined by $X_t := B_{S_t}$, $t \geq 0$ is called a subordinate Brownian motion.

Subordinate Brownian motions form a large class of symmetric Lévy processes, yet it is much more tractable than general symmetric Lévy processes. Subordinate Brownian motions are used in mathematical finance, as the subordintaor can be thought of as the “operational time” or “intrinsic time”.

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If the Laplace exponent of S is ϕ, then the Lévy exponent of the subordinate Brownian motion X is given by $\Phi(\xi) = \phi(|\xi|^2)$. The infinitesimal generator can be written as $-\phi(-\Delta)$.

When $\phi(\lambda) = \lambda^{\alpha/2}$, the resulting subordinate Brownian motion turns out to be a symmetric α-stable process. The infinitesimal generator of this process can be written as $-(-\Delta)^{\alpha/2}$.

When $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, the resulting subordinate Brownian motion turns out to be a relativistic α-stable process with mass m. The infinitesimal generator of this process can be written as $m - (-\Delta + m^{2/\alpha})^{\alpha/2}$.

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The linear coefficient a of X is always 0, and the diffusion matrix of X is $b\,\mathbf{I}$.

The Lévy measure of the process X has a density J, called the Lévy density, given by

$$J(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \mu(t)dt, \qquad x \in \mathbb{R}^d.$$

Thus $J(x) = j(|x|)$ with

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t)dt, \qquad r > 0.$$

Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$.

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X has a transition density given by

$$p(t, x, y) = \int_{[0,\infty)} p^0(s, x, y)\mathbb{P}(S_t \in ds)$$

where

$$p^0(s, x, y) = (4\pi s)^{-d/2} \exp(-\frac{|x-y|^2}{4s}).$$

Analytically, $p(t, x, y)$ is the fundamental solution of $\partial_t u = -\phi(-\Delta)$, so it is also called the heat kernel of $-\phi(-\Delta)$.

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For any open subset $D \subset \mathbb{R}^d$, we use X^D to denote the subprocess of X killed upon exiting D. The infinitesimal generator of X^D is $-\phi(-\Delta)|_D$.

X^D has a continuous transition density $p_D(t, x, y)$ with respect to the Lebesgue measure. Analytically, $p_D(t, x, y)$ is the fundamental solution of $\partial_t u = -\phi(-\Delta)|_D$. Recently we have succeeded in establishing sharp two-sided estimates on $p(t, x, y)$ and $p_D(t, x, y)$ for a few classes of subordinate Brownian motions. We are working to deal with the general case.

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The function

$$G_D(x,y)=\int_0^\infty p_D(t,x,y)dt$$

is called the Green function of X^D. Analytically, $G_D(\cdot,y)$ is the solution of $\phi(-\Delta)|_D u=\delta_y$.

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In the remainder of this talk, I will always assume that S is a complete subordinator with Laplace exponent ϕ and that X is a subordinate Brownian motion via S. There are two classes of subordinate Brownian motions: pure jump (without Brownian compnent) subordinate Brownian motions and subordinate Brownian motions with Brownian components.

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Outline

In this part, we will always assume that the Laplace exponent ϕ of S is a complete Bernstein function satisfying

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda)\,, \quad \lambda \to \infty$$

where ℓ is a slowly varying function at infinity, $0 < \alpha < 2 \wedge d$. This is just an assumption on the asymptotic behavior of ϕ at infinity.

It is easy to check that, when $d \geq 3$, the subordinate Brownian motion is transient. When X is transient, the Green function $G(x, y)$ of X

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When $d \le 2$, X may not be transient. However, under the following assumption, X will be also transient for $d \le 2$.

H: there exists $\gamma \in [0, d/2)$ such that

$$\liminf_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^\gamma} > 0.$$

By spatial homogeneity we may write $G(x, y) = G(x - y)$ where the function G is given by the following formula

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} u(t) dt, \qquad x \in \mathbb{R}^d,$$

where u is the potential density of S. Using this formula we see that G is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$.

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By using our standing assumption, we can apply the Tauberian theorem and the monotone density theorem to get asymptotic behaviors of u and μ at 0. Using these, one can get the following asymptotic behaviors of G and J at the origin.

Theorem(Song and Vondracek)

$$\begin{aligned} G(x) &\asymp \frac{1}{|x|^d \phi(|x|^{-2})}, & |x| \to 0 \\ j(r) &\asymp \frac{\phi(|x|^{-2})}{|x|^d}. & r \to 0 \end{aligned}$$

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Using the results above and some very complicated analysis, we can prove the following result.

Theorem (Kim, Song and Vondracek)

Let D a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R, Λ). Then there exists $C = C(\mathrm{diam}(D), R, \Lambda) > 1$ such that

$$C^{-1}\left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{1}{|x-y|^d\,\phi(|x-y|^{-2})}$$

$$\leq \quad G_D(x,y) \leq C\left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{1}{|x-y|^d\,\phi(|x-y|^{-2})}$$

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Suppose that $d \geq 2$ and $\alpha \in (0,2)$. Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and let G_D be the Green function of the symmetric α-stable process in D. Then

$$G_D(x,y) \asymp \left(1 \wedge \frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x-y|^\alpha}\right) \frac{1}{|x-y|^{d-\alpha}}, \quad x,y \in D.$$

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Suppose that $d \geq 2$, $\alpha \in (0,2)$ and $m > 0$. Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and let G_D be the Green function of the Lévy process with Lévy exponent $\Phi(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$ in D. Then

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Outline

In this part, we will always assume that the Laplace exponent ϕ of S is a complete Bernstein function with a positive drift and, without loss of generality, we shall assume that the drift of S is equal to 1. Thus the Laplace exponent of S can be written as

$$\phi(\lambda) = \lambda + \psi(\lambda),$$

where

$$\psi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(t) dt.$$

The only other assumption is some control on the behavior of Lévy density μ near the origin: for any $K > 0$, there exists $c = c(K) > 1$ such that

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for $d = 2$ we define

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Theorem (Kim, Song and Vondracek)

For any bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$, there exists $C = C(D) > 1$ such that for all $x,y \in D$

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