On Finding a Minimum Toughness Condition for a k-tree to be Hamiltonian.

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- A cycle is said to be **Hamiltonian** if it uses every vertex of G exactly once.

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- A Hamilton cycle is a 2-factor.

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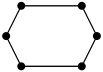

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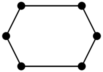

Figure: $\tau(G) = 1$ for a cycle.

A graph G is said to be t-**tough** if $\tau(G) \geq t$.

Vašek Chvátal's Conjecture

Conjecture (1973, Chvátal)

There exists a finite constant t_0 such that every t_0-tough graph is Hamiltonian.

Status of Chvátal Conjecture

Theorem (2000, D. Bauer, H. J. Broersma, H. J. Veldman)

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Theorem (1995, D. Bauer, H.J. Broersma, J. van den Heuvel, , H.J. Veldman)

If G is a t-tough graph on $n \geq 3$ with $\delta > \frac{n}{t+1} - 1$, then G is Hamiltonian.

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Let $\lambda = (v_1, \ldots\ldots, v_n)$ be a sequence of vertices of a a n vertex chordal graph G. Set $G = G_0$. If $G_i = G_{i-1} - v_i$ is a chordal graph such that $v_i \in S_1(G_{i-1})$, then λ is a **simplicial elimination ordering** of G.

Non-Hamiltonian chordal graphs

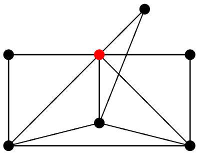

Figure: 1-tough Non-Hamiltonian Chordal Graph

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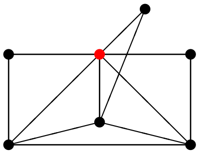

Figure: 1-tough Non-Hamiltonian Chordal Graph

Theorem (2000, D. Bauer, H. J. Broersma, H. J. Veldman)

For every $\epsilon > 0$ there exists a $(\frac{7}{4} - \epsilon)$-tough chordal non-traceable graph.

Tough Enough Chordal Graphs

Theorem (1998, G. Chen, M. S. Jacobson, A. E. Kézdy and J. Lehel)

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Conjecture

2-tough chordal graphs are Hamiltonian.

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Theorem (1985, J.M. Kiel)

1-*tough interval graphs are Hamiltonian.*

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All $\frac{3}{2}$*-tough split graphs are Hamiltonian.*

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- The smallest k-tree is a k-clique. For G with $|V(G)| > k$, G is a k-tree if and only if it contains a k-simplicial vertex v such that $G - v$ is a k-tree.

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- For $G \neq K_{k+1}$, the set $S_1(G)$ is an independent set. When $G = K_{k+1}$ every vertex is k-simplicial and we let $|S_1(G)| = 1$.
- An edge is said to be **contractible** if it is not incident with a k-simplicial vertex and the contraction of the edge yields a k-tree. We denote the set of contractible edges by $A(G)$

Branch Number

Let $\lambda = (v_1, \ldots\ldots, v_n)$ be a simplicial elimination ordering of a k-tree G where $G = G_0$ and $G_i = G - \{v_1, \ldots, v_i\}$. For $i \leq n - k$, we define v_i to be a **branch vertex** if $A(G_i) = A(G_{i-1})$. Let $\beta(\lambda)$ be the collection of branch vertices of the elimination ordering.

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For any two simplicial elimination orderings λ_1, λ_2 of G, $|\beta(\lambda_1)| = |\beta(\lambda_2)|$. Let $\beta(G) = |\beta(\lambda)|$ be the branch number of G.

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Theorem (Shook, Wei)

If G is a k-tree on n vertices, then $n = 2|S_1(G)| + |A(G)| + k - \beta(G)$.

Hamiltonian k-trees

Theorem (2007, H. J. Broersma, L. Xiong, K. Yoshimoto)

If G is a $\frac{k+1}{3}$-tough k-tree, with $k \geq 2$ and $G \neq K_2$, then G is Hamiltonian.

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There exists non-Hamiltonian k-trees with $\tau(G) = 1$. What about k-trees with $\tau(G) > 1$?

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- Is there a construction for G such that $\tau(G) \geq \frac{3}{2}$?
- Does there exist a function $f(k) \to 1$ such that every $f(k)$-tough k-tree is Hamiltonian?

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- Create an auxiliary split graph F using the partition.
- Show that if F is not $\frac{3}{2}$-tough, then G is not 2-tough.
- Use theorem to prove G is Hamiltonian.