Finite frames and Sigma-Delta quantization

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Outline and collaborators

1. Finite frames
2. Sigma-Delta quantization – theory and implementation
3. Sigma-Delta quantization – number theoretic estimates

Collaborators: Matt Fickus (frame force); Alex Powell and Özgür Yilmaz ($\Sigma-\Delta$ quantization); Alex Powell, Aram Tangboondouangjit, and Özgür Yilmaz ($\Sigma-\Delta$ quantization and number theory).

Finite Frames

Frames

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- If $\{e_n\}_{n=1}^N$ is a finite unit norm tight frame (FUN-TF) for $\mathbb{K}^d$, with frame constant A, then $A = N/d$.
- Let $\{e_n\}$ be an A-unit norm TF for any separable Hilbert space H. $A \geq 1$, and $A = 1 \Leftrightarrow \{e_n\}$ is an ONB for H (*Vitali*).

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- FUN-TFs can be characterized as minimizers of a “frame potential function” (with Fickus) analogous to
- *Coulomb’s Law*.

Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a,b) = p(\|a-b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a,b) = (a-b)/\|a-b\|^3, \qquad f(x) = 1/x^3$$

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$$CF(a,b) = (a-b)/\|a-b\|^3, \qquad f(x) = 1/x^3$$

- Frame force

$$FF(a,b) = <a,b> (a-b), \qquad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \Sigma_{m=1}^{N} \Sigma_{n=1}^{N} |<x_m, x_n>|^2$$

Characterization of FUN-TFs

For the Hilbert space $H = \mathbb{R}^d$ and N, consider $\{x_n\}_1^N \in S^{d-1} \times ... \times S^{d-1}$ and

$$TFP(\{x_n\}) = \Sigma_{m=1}^N \Sigma_{n=1}^N | < x_m, x_n > |^2.$$

- **Theorem** Let $N \leq d$. The minimum value of TFP, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

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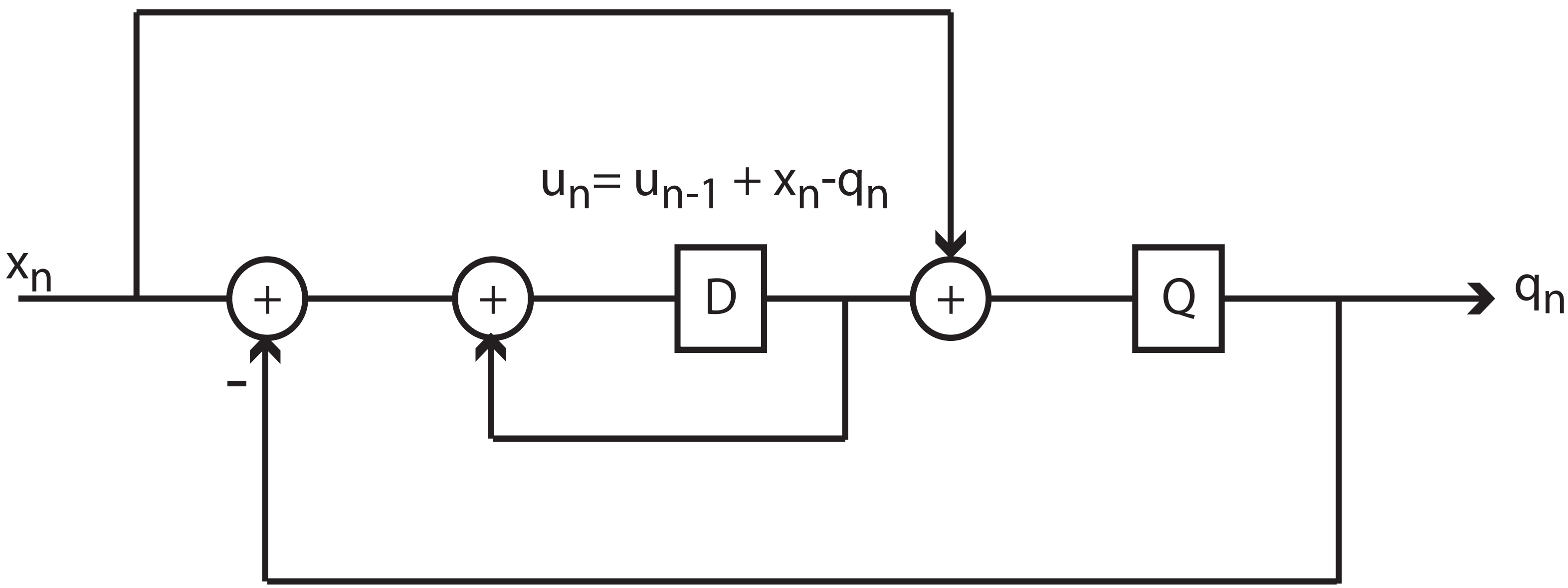
- **Theorem** Let $N \geq d$. The minimum value of TFP, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUN-TFs of N elements for $\mathbb{R}^d$.
- **Problem** Find FUN-TFs analytically, effectively, computationally.

Sigma-Delta quantization – theory and implementation

Given u_0 and $\{x_n\}_{n=1}$

$$u_n = u_{n-1} + x_n - q_n$$
$$q_n = Q(u_{n-1} + x_n)$$



First Order ΣΔ

A quantization problem

Qualitative Problem Obtain *digital* representations for class X, suitable for storage, transmission, recovery.

Quantitative Problem Find dictionary $\{e_n\} \subseteq X$:

1. Sampling [continuous range $\mathbb{K}$ is not digital]

$$\forall x \in X, \ \ x = \sum x_n e_n, \ \ x_n \in \mathbb{K} \ (\mathbb{R} \text{ or } \mathbb{C}).$$

2. Quantization. Construct finite alphabet $\mathcal{A}$ and

$$Q: \ X \to \{\sum q_n e_n : \ q_n \in \mathcal{A} \subseteq \mathbb{K}\}$$

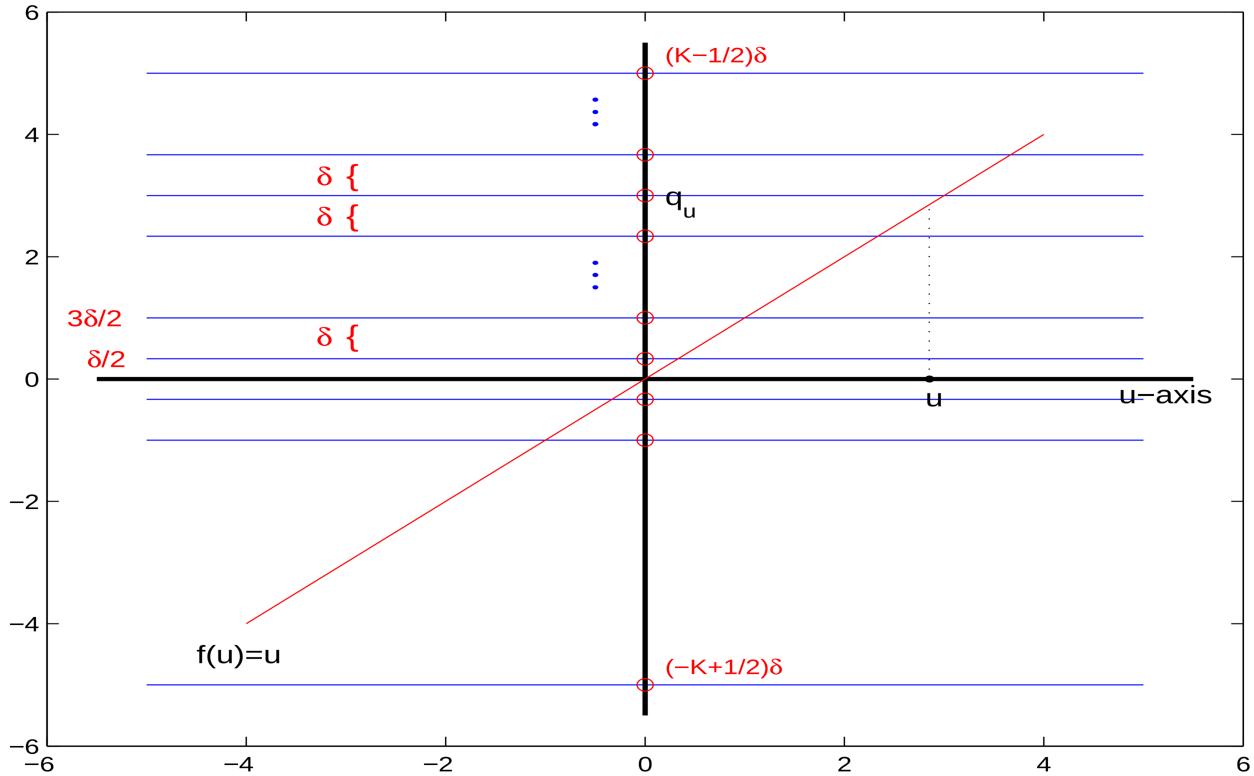
such that $|x_n - q_n|$ and/or $\|x - Qx\|$ small.

Methods Fine quantization, e.g., PCM. Take $q_n \in \mathcal{A}$ close to given x_n. Reasonable in 16-bit (65,536 levels) digital audio.

Coarse quantization, e.g., $\Sigma\Delta$. Use fewer bits to exploit redundancy.

Quantization

$$\mathcal{A}_K^\delta = \{(-K+1/2)\delta, (-K+3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K-1/2)\delta\}$$



$$Q(u) = \arg\min\{|u-q| : q \in \mathcal{A}_K^\delta\} = q_u$$

Setting

Let $x \in \mathbb{R}^d$, $\|x\| \leq 1$. Suppose $F = \{e_n\}_{n=1}^N$ is a FUN-TF for $\mathbb{R}^d$. Thus, we have

$$x = \frac{d}{N} \sum_{n=1}^{N} x_n e_n$$

with $x_n = \langle x, e_n \rangle$. Note: $A = N/d$, and $|x_n| \leq 1$.

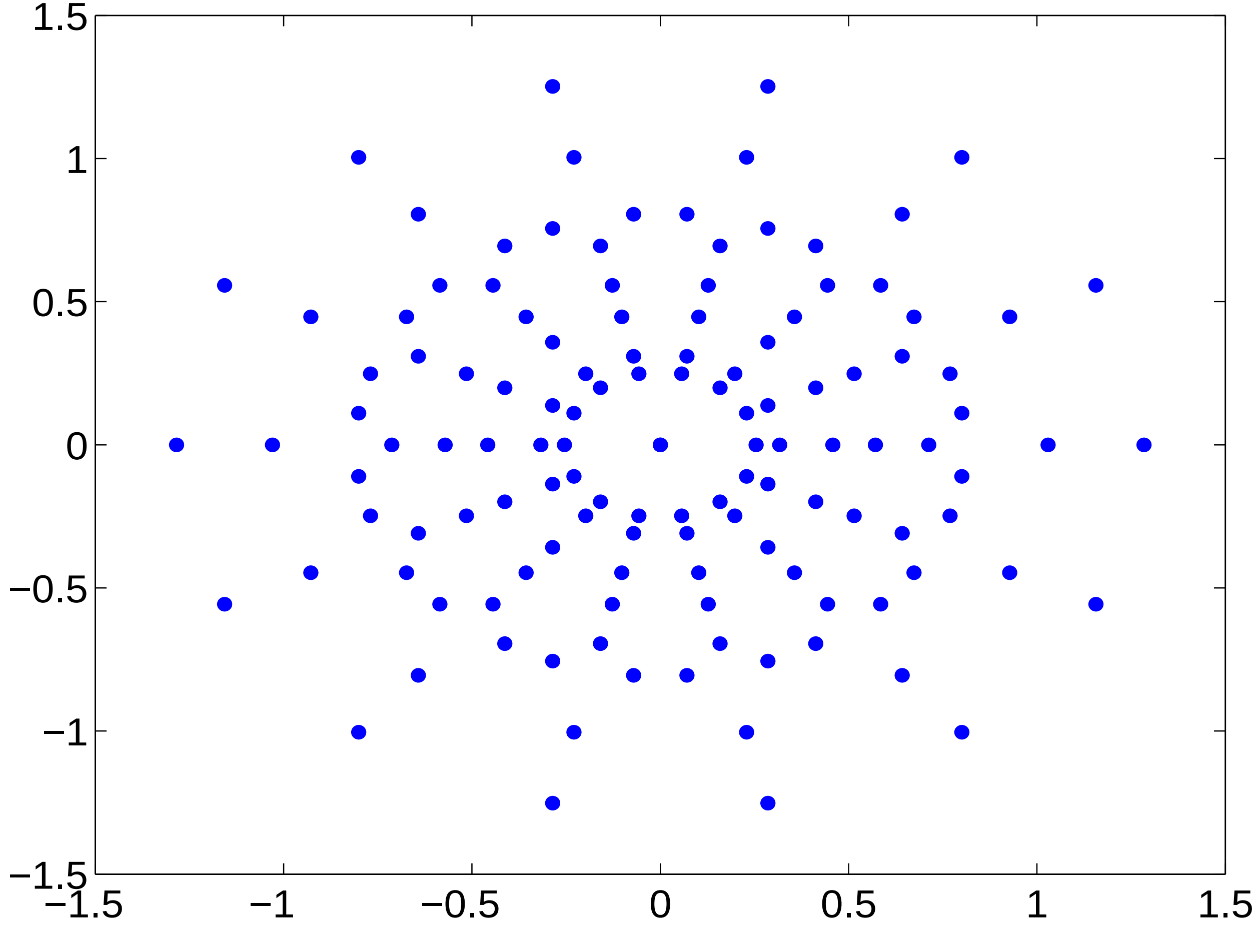
Goal Find a "good" quantizer, given

$$\mathcal{A}_K^\delta = \{(-K + \frac{1}{2})\delta, (-K + \frac{3}{2})\delta, \ldots, (K - \frac{1}{2})\delta\}.$$

Example Consider the alphabet $\mathcal{A}_1^2 = \{-1, 1\}$, and $E_7 = \{e_n\}_{n=1}^7$, with

$e_n = (\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))$.

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7



$$\Gamma_{\mathcal{A}_1^2}(E_7) = \{\tfrac{2}{7}\sum_{n=1}^{7} q_n e_n : \; q_n \in \mathcal{A}_1^2\}$$

PCM

Replace $x_n \leftrightarrow q_n = \arg\{\min|x_n - q| : q \in \mathcal{A}_K^\delta\}$. Then $\tilde{x} = \frac{d}{N}\sum_{n=1}^{N} q_n e_n$ satisfies

$$\|x - \tilde{x}\| \leq \frac{d}{N}\|\sum_{n=1}^{N}(x_n - q_n)e_n\| \leq \frac{d}{N}\frac{\delta}{2}\sum_{n=1}^{N}\|e_n\| = \frac{d}{2}\delta.$$

Not good!

Bennett's "white noise assumption"

Assume that $(\eta_n) = (x_n - q_n)$ is a sequence of independent, identically distributed random variables with mean 0 and variance $\frac{\delta^2}{12}$. Then the mean square error (MSE) satisfies

$$\text{MSE} = E\|x - \tilde{x}\|^2 \leq \frac{d}{12A}\,\delta^2 = \frac{(d\delta)^2}{12N}$$

Remarks

1. Bennett's "white noise assumption" is not rigorous, and not true in certain cases.
2. The MSE behaves like C/A. In the case of $\Sigma\Delta$ quantization of bandlimited functions, the MSE is $O(A^{-3})$ (Gray, Güntürk and Thao, Bin Han and Chen). PCM does not utilize redundancy efficiently.
3. The MSE only tells us about the average performance of a quantizer.

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.

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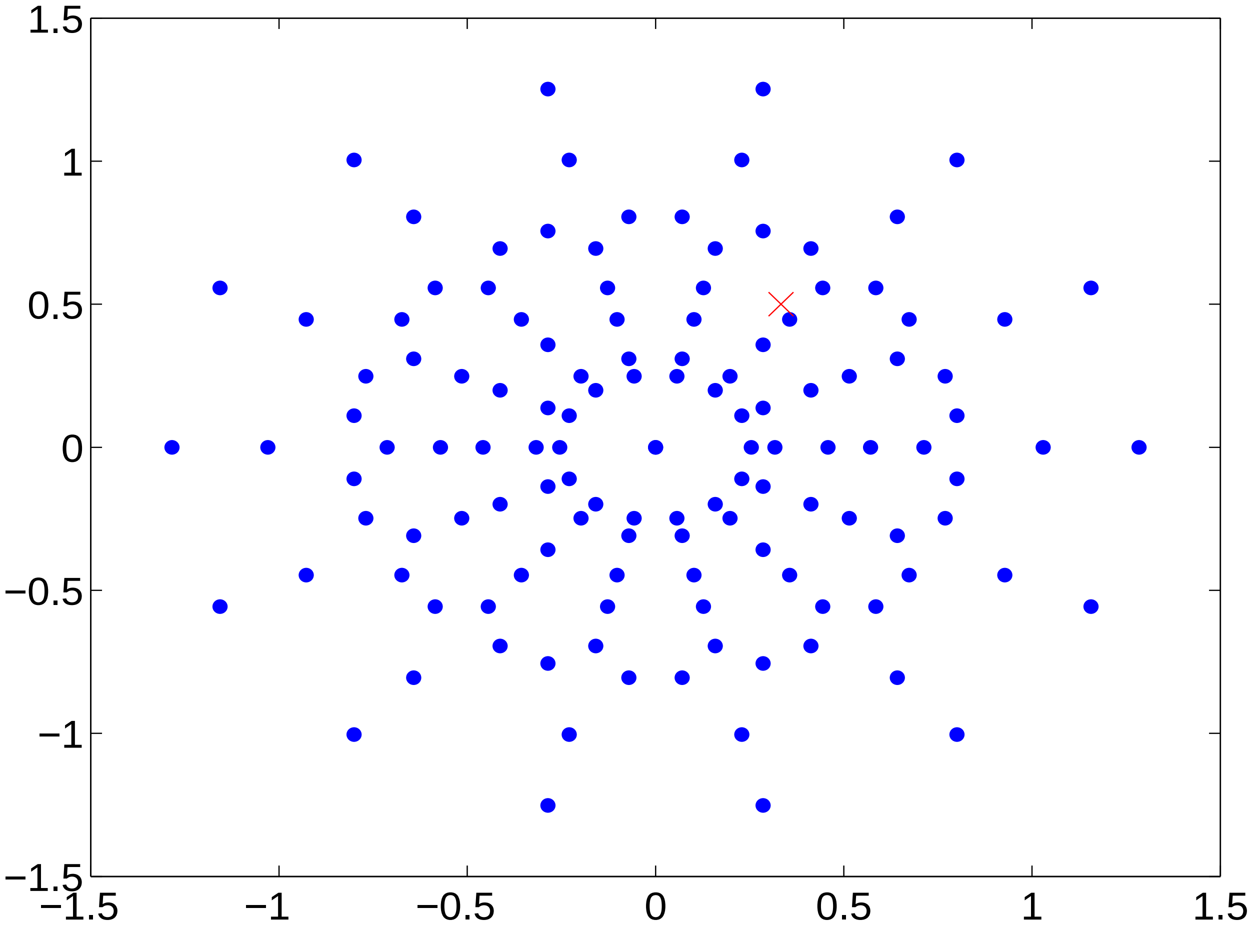

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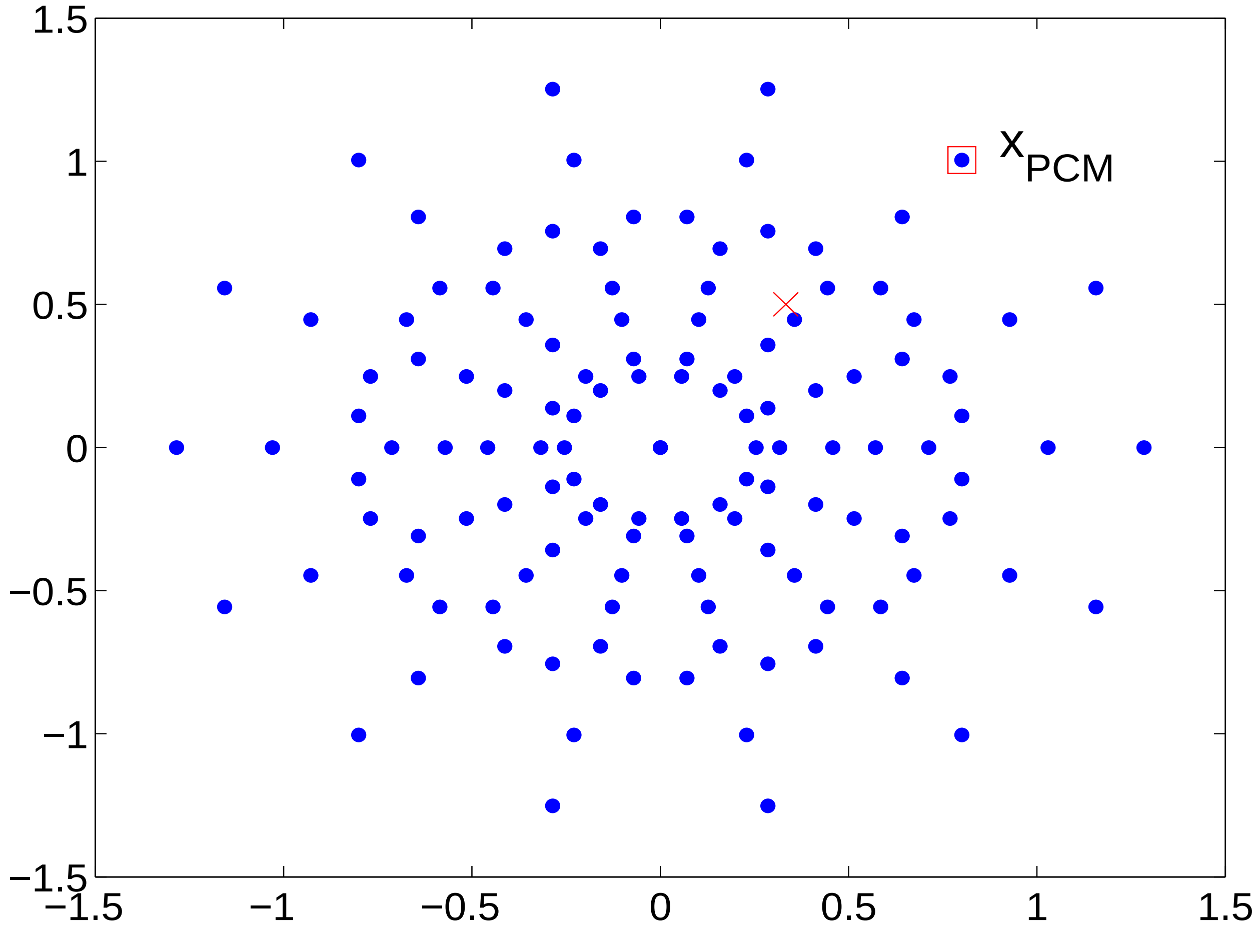

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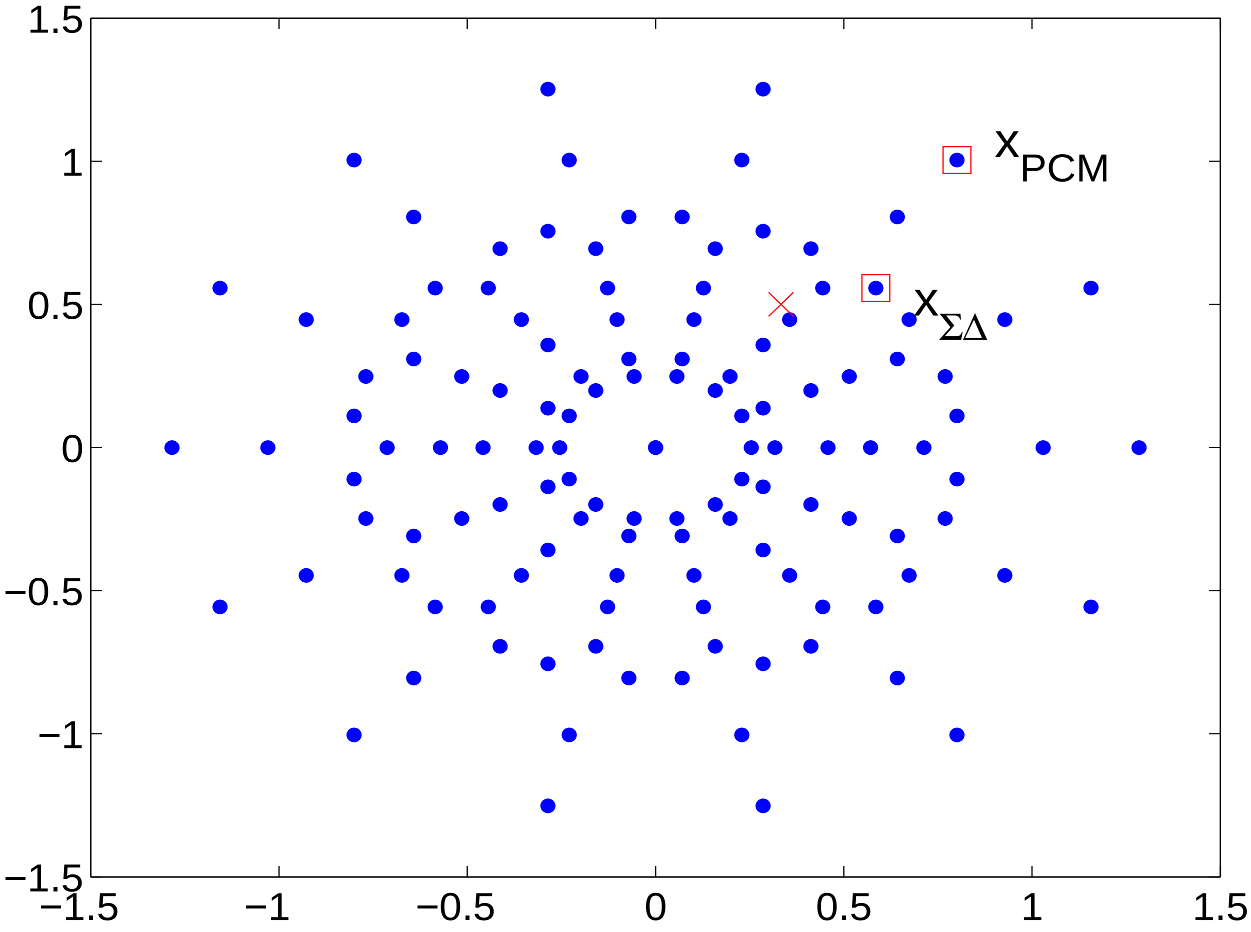

$\Sigma\Delta$ quantizers for finite frames

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, $x \in \mathbb{R}^d$.

Define $x_n = \langle x, e_n \rangle$.

Fix the ordering p, a permutation of $\{1, 2, \ldots, N\}$.

Quantizer alphabet $\mathcal{A}_K^\delta$

Quantizer function $Q(u) = \arg\{\min |u - q| : q \in \mathcal{A}_K^\delta\}$

Define the *first-order* $\Sigma\Delta$ *quantizer* with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ by means of the following recursion.

$$\begin{aligned} u_n - u_{n-1} &= x_{p(n)} - q_n \\ q_n &= Q(u_{n-1} + x_{p(n)}) \end{aligned}$$

where $u_0 = 0$ and $n = 1, 2, \ldots, N$.

Stability

The following stability result is used to prove error estimates.

Proposition If the frame coefficients $\{x_n\}_{n=1}^N$ satisfy

$$|x_n| \leq (K - 1/2)\delta, \quad n = 1, \cdots, N,$$

then the state sequence $\{u_n\}_{n=0}^N$ generated by the first-order $\Sigma\Delta$ quantizer with alphabet $\mathcal{A}_K^\delta$ satisfies $|u_n| \leq \delta/2, n = 1, \cdots, N.$

- The first-order $\Sigma\Delta$ scheme is equivalent to

$$u_n = \sum_{j=1}^{n} x_{p(j)} - \sum_{j=1}^{n} q_j, \quad n = 1, \cdots, N.$$

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- Stability results lead to tiling problems for higher order schemes.

Error estimate

- **Definition** Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, and let p be a permutation of $\{1, 2, \ldots, N\}$. The *variation* $\sigma(F, p)$ is

$$\sigma(F,p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|.$$

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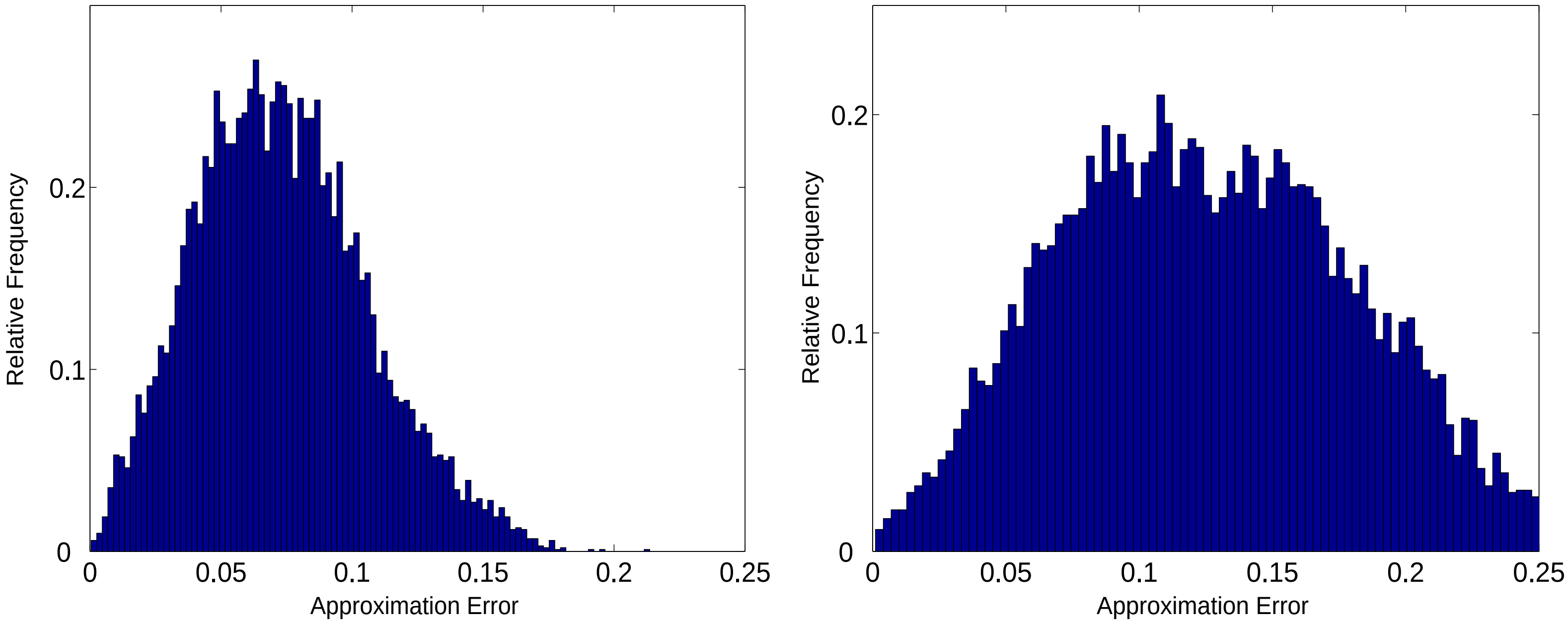
- **Theorem** Let $F = \{e_n\}_{n=1}^N$ be an A-FUN-TF for $\mathbb{R}^d$. The approximation

$$\tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

generated by the first-order $\Sigma\Delta$ quantizer with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ satisfies

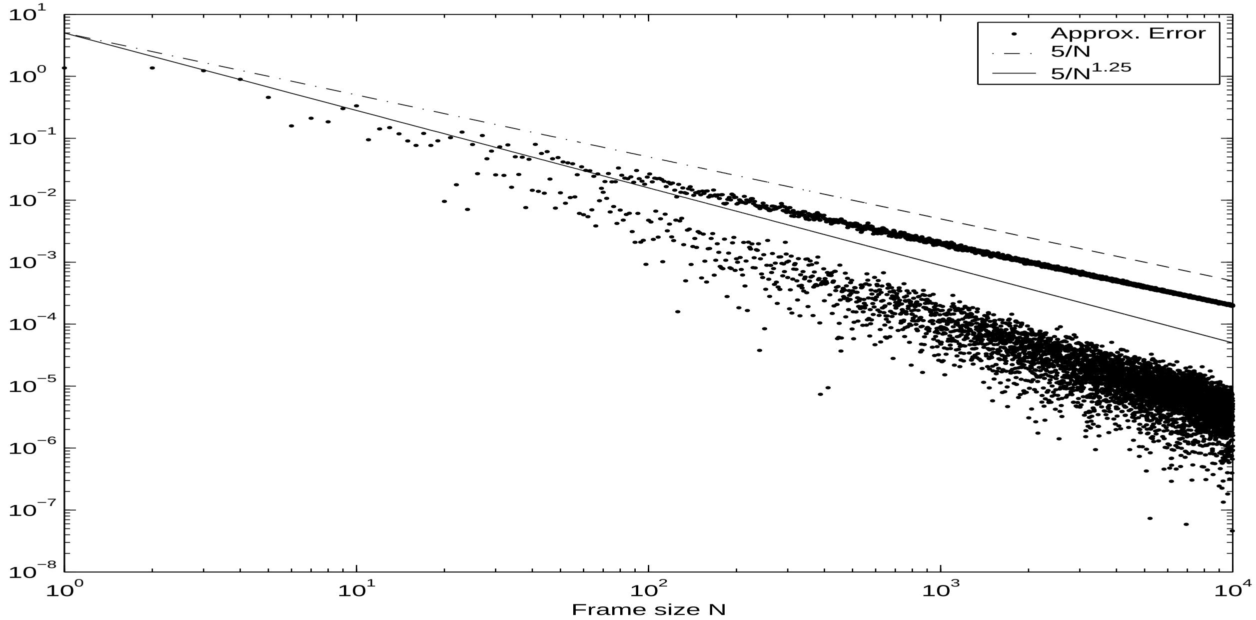
$$\|x - \tilde{x}\| \leq \frac{(\sigma(F,p)+1)d}{N} \frac{\delta}{2}.$$

Order is important



Let E_7 be the FUN-TF for $\mathbb{R}^2$ given by the 7th roots of unity. Randomly select 10,000 points in the unit ball of $\mathbb{R}^2$. Quantize each point using the $\Sigma\Delta$ scheme with alphabet $\mathcal{A}_4^{1/4}$. The figures show histograms for $||x - \widetilde{x}||$ when the frame coefficients are quantized in their natural order $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ (left) and order $x_1, x_4, x_7, x_3, x_6, x_2, x_5$ (right).

Even – odd



$E_N = \{e_n^N\}_{n=1}^N, e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$. Let $x = (\frac{1}{\pi}, \sqrt{\frac{3}{17}})$.

$$x = \frac{d}{N} \sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N \rangle.$$

Let $\widetilde{x}_N$ be the approximation given by the 1st order $\Sigma\Delta$ quantizer with alphabet $\{-1, 1\}$ and natural ordering. log-log plot of $||x - \widetilde{x}_N||$.

Improved estimates

$E_N = \{e_n^N\}_{n=1}^N$, Nth roots of unity FUN-TFs for $\mathbb{R}^2$, $x \in \mathbb{R}^2$, $||x|| \leq (K - 1/2)\delta$.

Quantize $$x = \frac{d}{N} \sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N \rangle$$

using 1st order $\Sigma\Delta$ scheme with alphabet $\mathcal{A}_K^\delta$.

Theorem If N is even and large then $||x - \widetilde{x}|| \lesssim \frac{\delta \log N}{N^{5/4}}$.

If N is odd and large then $\frac{\delta}{N} \lesssim ||x - \widetilde{x}|| \leq \frac{(2\pi+1)d}{N} \frac{\delta}{2}$.

Remark The proof uses the analytic number theory approach developed by Sinan Güntürk, and the theorem is true more generally.

Harmonic frames

Zimmermann and Goyal, Kelner, Kovačević, Thao, Vetterli.

- $H = \mathbb{C}^d$. An *harmonic frame* $\{e_n\}_{n=1}^N$ for H is defined by the rows of the Bessel map L which is the complex N-DFT $N \times d$ matrix with $N - d$ columns removed.

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- $H = \mathbb{R}^d$, d even. The harmonic frame $\{e_n\}_{n=1}^N$ is defined by the Bessel map L which is the $N \times d$ matrix whose nth row is

$$e_n^N = \sqrt{\frac{2}{d}}\left(\cos(\frac{2\pi n}{N}), \sin(\frac{2\pi n}{N}), \ldots, \cos(\frac{2\pi(d/2)n}{N}), \sin(\frac{2\pi(d/2)n}{N})\right).$$

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- Harmonic frames are FUN-TFs.
- Let E_N be the harmonic frame for $\mathbb{R}^d$ and let p_N be the identity permutation. Then

$$\forall N, \ \sigma(E_N, p_N) \leq \pi d(d+1).$$

Error estimate for harmonic frames

Theorem Let E_N be the harmonic frame for $\mathbb{R}^d$ with frame bound N/d. Consider $x \in \mathbb{R}^d$, $\|x\| \leq 1$, and suppose the approximation $\tilde{x}$ of x is generated by a first-order $\Sigma\Delta$ quantizer as before. Then

$$\|x - \tilde{x}\| \leq \frac{d^2(d+1)+d}{N}\,\frac{\delta}{2}.$$

- Hence, for harmonic frames (and all those with bounded variation),

$$\text{MSE}_{\Sigma\Delta} \leq \frac{C_d}{N^2}\,\delta^2.$$

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$$\text{MSE}_{\Sigma\Delta} \leq \frac{C_d}{N^2} \delta^2.$$

- This bound is clearly superior asymptotically to

$$\text{MSE}_{\text{PCM}} = \frac{(d\delta)^2}{12N}.$$

$\Sigma\Delta$ and "optimal" PCM

The digital encoding

$$\text{MSE}_{\text{PCM}} = \frac{(d\delta)^2}{12N}$$

in PCM format leaves open the possibility that decoding (reconstruction) could lead to

$$\text{“MSE}^{\text{opt}}_{\text{PCM}}\text{”} \ll O(\frac{1}{N}).$$

Goyal, Vetterli, Thao (1998) proved

$$\text{“MSE}^{\text{opt}}_{\text{PCM}}\text{”} \sim \frac{\tilde{C}_d}{N^2}\delta^2.$$

Theorem The first order $\Sigma\Delta$ scheme achieves the asymptotically optimal MSE_{PCM} for harmonic frames.

Proof of Improved Estimates theorem

- If N is even and large then $||x - \widetilde{x}|| \lesssim \frac{\delta \log N}{N^{5/4}}$.
  If N is odd and large then $\frac{\delta}{N} \lesssim ||x - \widetilde{x}|| \leq \frac{(2\pi+1)d}{N} \frac{\delta}{2}$.

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- $\forall N, \{e_n^N\}_{n=1}^N$ is a FUN-TF.

$$x - \widetilde{x}_N = \frac{d}{N}\left(\sum_{n=1}^{N-2} v_n^N (f_n^N - f_{n+1}^N) + v_{N-1}^N f_{N-1}^N + u_N^N e_N^N\right)$$

$$f_n^N = e_n^N - e_{n+1}^N, \quad v_n^N = \sum_{j=1}^{n} u_j^N, \quad \widetilde{u}_n^N = \frac{u_n^N}{\delta}$$

Proof of Improved Estimates theorem

- If N is even and large then $||x - \widetilde{x}|| \lesssim \frac{\delta \log N}{N^{5/4}}$.
  If N is odd and large then $\frac{\delta}{N} \lesssim ||x - \widetilde{x}|| \leq \frac{(2\pi+1)d}{N}\frac{\delta}{2}$.
- $\forall N, \{e_n^N\}_{n=1}^N$ is a FUN-TF.

$$x - \widetilde{x}_N = \frac{d}{N}\left(\sum_{n=1}^{N-2} v_n^N (f_n^N - f_{n+1}^N) + v_{N-1}^N f_{N-1}^N + u_N^N e_N^N\right)$$

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- To bound v_n^N.

Koksma Inequality

- **Discrepancy**
  The discrepancy D_N of a finite sequence $x_1, \ldots, x_N$ of real numbers is
  $$D_N = D_N(x_1, \ldots, x_N) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[\alpha,\beta)}(\{x_n\}) - (\beta - \alpha) \right|,$$
  where $\{x\} = x - \lfloor x \rfloor$.

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- **Koksma Inequality**
  $g : [-1/2, 1/2) \to \mathbb{R}$ of bounded variation and
  $\{\omega_j\}_{j=1}^n \subset [-1/2, 1/2) \Longrightarrow$

$$\left| \frac{1}{n} \sum_{j=1}^{n} g(\omega_j) - \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)dt \right| \le \mathrm{Var}(g) \mathrm{Disc}\Big(\{\omega_j\}_{j=1}^n \Big).$$

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- With $g(t) = t$ and $\omega_j = \widetilde{u}_j^N$, $\boxed{|v_n^N| \le n\delta \mathrm{Disc}\Big(\{\widetilde{u}_j^N\}_{j=1}^n\Big).}$

Erdös-Turán Inequality

- $\exists C > 0, \forall K, \mathrm{Disc}\Big(\{\widetilde{u}_n^N\}_{n=1}^j\Big) \leq C\left(\frac{1}{K} + \frac{1}{j}\sum_{k=1}^{K}\frac{1}{k}\Big|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\Big|\right).$

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- To approximate the exponential sum.

Approximation of Exponential Sum

(1) Güntürk'sProposition

$\forall N, \exists X_N \in \mathcal{B}_{\Omega/N}$

such that $\forall n = 0, \ldots, N$,

$$X_N(n) = u_n^N + c_n \frac{\delta}{2},\ c_n \in \mathbb{Z}$$

and $\forall t, \left| X_N'(t) - h\left(\frac{t}{N}\right) \right| \lesssim \frac{1}{N}$

(2) Bernstein's Inequality

If $x \in \mathcal{B}_\Omega$, then $\|x^{(r)}\|_\infty \leq \Omega^r \|x\|_\infty$

- $\widehat{\mathcal{B}}_\Omega = \{T \in A'(\widehat{\mathbb{R}})\ :\ \text{supp}T \subseteq [-\Omega, \Omega]\}$
- $\mathcal{M}_\Omega = \{h \in \mathcal{B}_\Omega\ :\ h' \in L^\infty(\mathbb{R})$ and all zeros of h' on $[0,1]$ are simple$\}$
- We assume $\exists h \in \mathcal{M}_\Omega$ such that $\forall N$ and $\forall\, 1 \leq n \leq N,\ h(n/N) = x_n^N$.

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(1)+(2)

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Van der Corput Lemma

- Let a, b be integers with $a < b$, and let $f \in C^2([a,b])$ with $f''(x) \geq \rho > 0$ for all $x \in [a,b]$ or $f''(x) \leq -\rho < 0$ for all $x \in [a,b]$ then

$$\Big|\sum_{n=a}^{b} e^{2\pi i f(n)}\Big| \leq \Big(\big|f'(b) - f'(a)\big| + 2\Big)\Big(\frac{4}{\sqrt{\rho}} + 3\Big).$$

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- $\forall 0 < \alpha < 1, \exists N_\alpha$ such that $\forall N \geq N_\alpha$,

$$\Big|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\Big| \lesssim N^\alpha + \frac{\sqrt{k} N^{1-\frac{\alpha}{2}}}{\sqrt{\delta}} + \frac{k}{\delta}.$$

Choosing appropriate α and K

Putting $\alpha = 3/4,\ K = N^{1/4}$ yields

$$\exists \widetilde{N} \text{ such that } \forall N \geq \widetilde{N}, \mathrm{Disc}\Big(\{\widetilde{u}_n^N\}_{n=1}^{j}\Big) \lesssim \frac{1}{N^{\frac{1}{4}}} + \frac{N^{\frac{3}{4}}\log(N)}{j}$$

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Conclusion

$$\boxed{\forall n = 1, \ldots, N, \ |v_n^N| \lesssim \delta N^{\frac{3}{4}} \log N}$$

That's all folks!

Norbert Wiener Center