Bifurcations and chaos in high-speed milling

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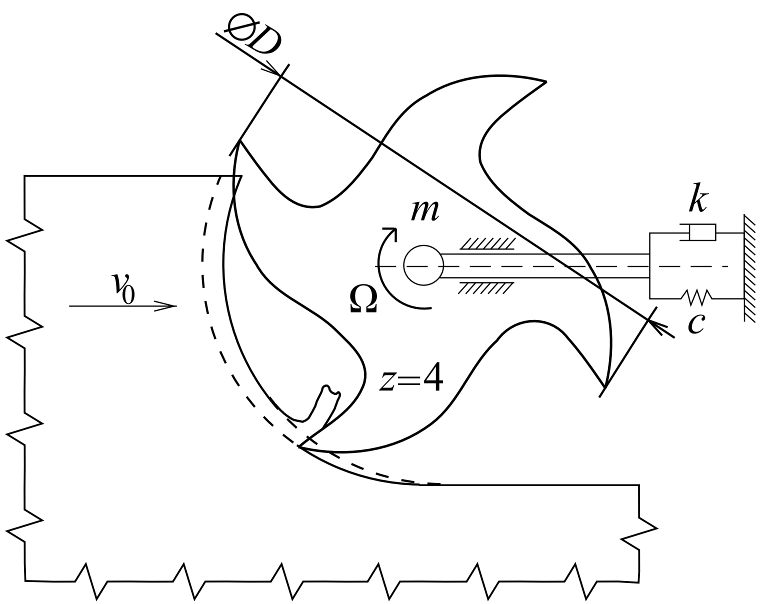
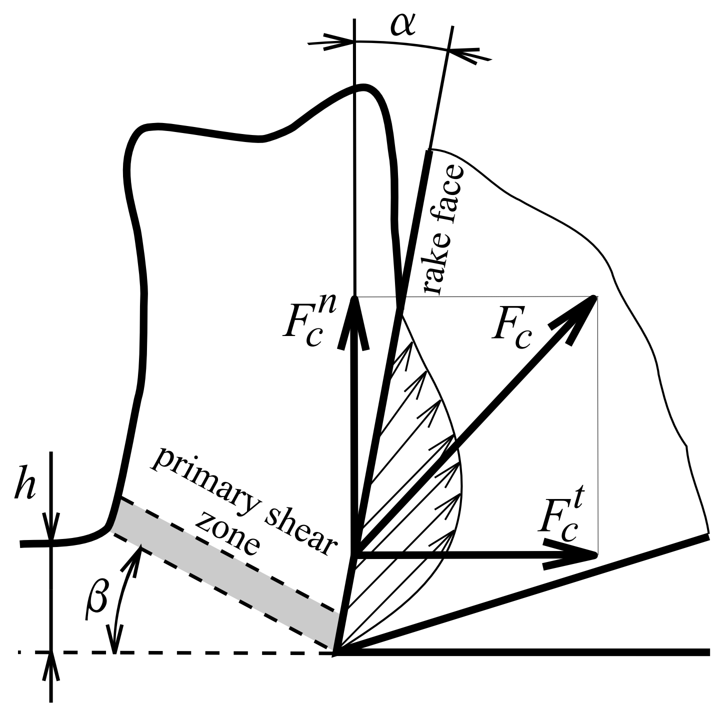
Contents

High-speed milling (standard model)





Calculation of the cutting force:

$$F_{\mathrm{c}}^{\mathrm{t}} = K^{\mathrm{t}} w h^{3/4}(t) \quad \text{and}$$

$$F_{\mathrm{c}}^{\mathrm{n}} = K^{\mathrm{n}} w h^{3/4}(t),$$

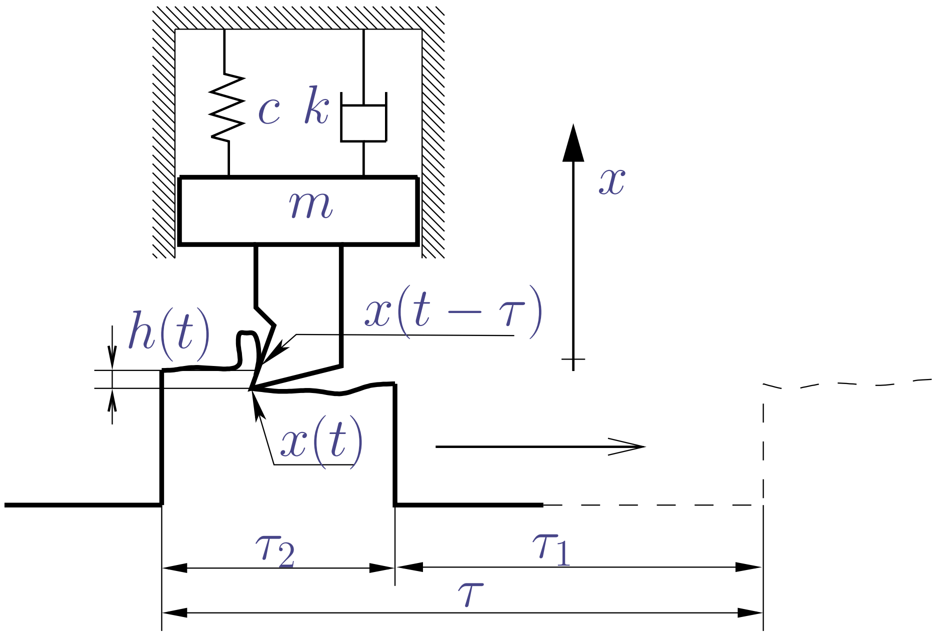
[Tlusty, 2000], [Burns and Davies, 2002].

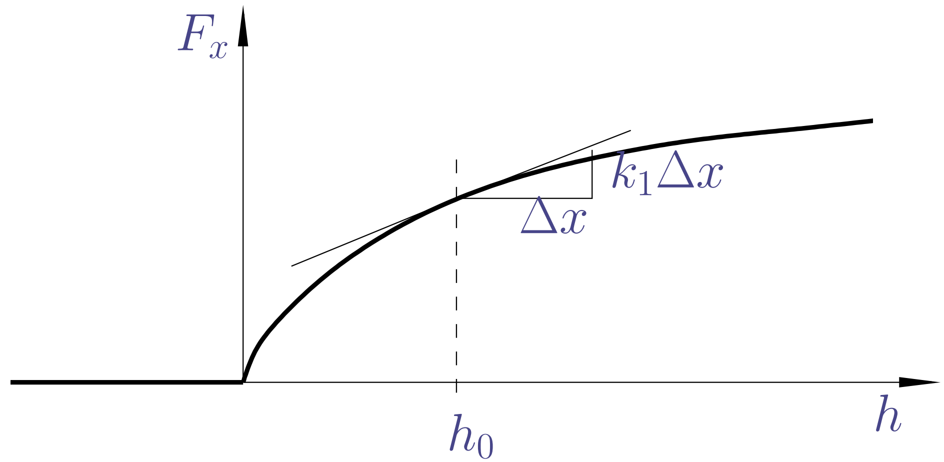
History

Mostly stability results and simulation.

- Averaging and harmonic balance techniques [Minis, Y. Altintas]
- Semi-discretization [T. Insperger and G. Stepan]
- Time finite element analysis [B. Mann and P. Bayly]
- Heuristic assumptions for period-doubling boundaries [W. Corpus and W. Endres]
- Discrete time model [Davies and T. Burns]
- Analytical stability chart [R. Szalai and G. Stepan]

Mechanical model





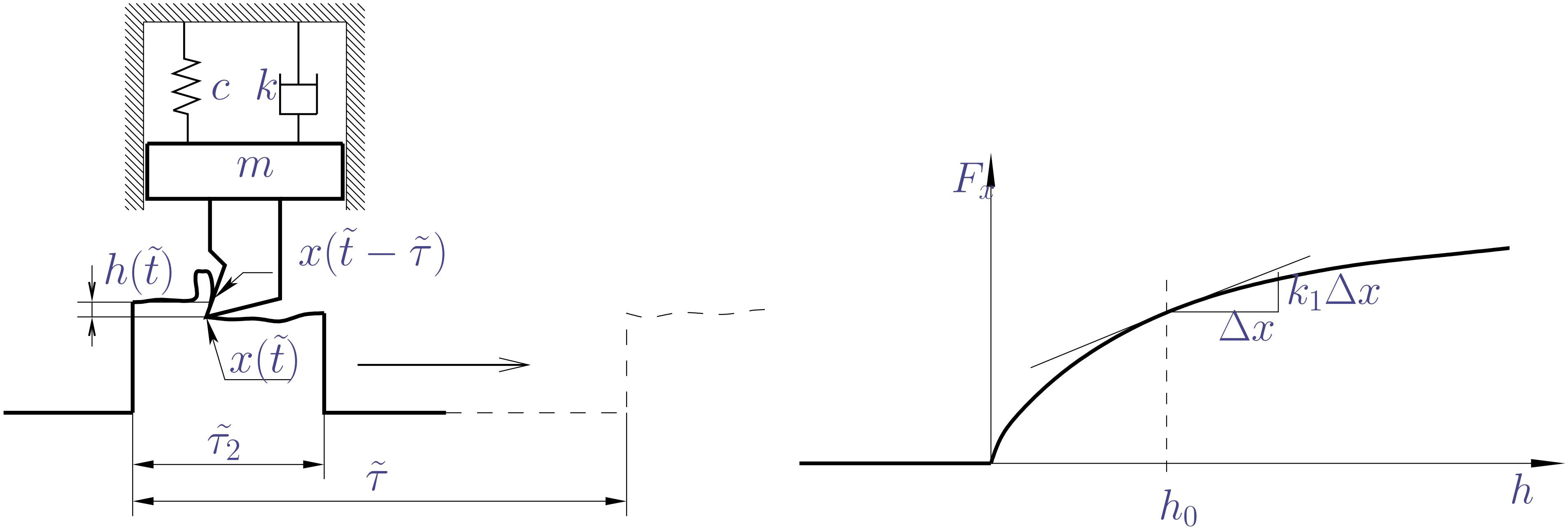
Equation of motion:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = g(t)\frac{K\,w}{m}\left(h_0 + x(t-\tau) - x(t)\right)^{3/4},$$

where

$$g(t) = \begin{cases} 0, \text{ if } & k\tau \le t < k\tau + \tau_1 \\ 1, \text{ if } & k\tau + \tau_1 \le t < (k+1)\tau \end{cases}, \quad k \in \mathbb{Z}.$$

Discrete-time model



$$m\,\ddot{x}(\tilde{t}) + k\,\dot{x}(\tilde{t}) + s\,x(\tilde{t}) = 0, \quad \tilde{t} \in [\tilde{t}_0, \tilde{t}_0 + \tilde{\tau}_1]$$

$$m\left(\dot{x}(\tilde{t}) - \dot{x}(\tilde{t} - \tilde{\tau}_2)\right) = \tilde{\tau}_2 F_c(h(\tilde{t})), \quad \tilde{t} \in [\tilde{t}_0 + \tilde{\tau}_1, \tilde{t}_0 + \tilde{\tau}],$$

where $h(\tilde{t}) = h_0 + x(\tilde{t} - \tilde{\tau}) - x(\tilde{t})$, $h_0 = v_0 \tilde{\tau}$,

and $F_c(h(t)) = K w\, h^{3/4}(t)$ is the cutting force.

Mathematical model

Natural eigenfrequency: $\omega_n = \sqrt{s/m}$

Relative damping: $\zeta = k/(2\sqrt{s\,m})$

Dimensionless time: $t = \omega_n \tilde{t}$

Dimensionless eigenfrequency: $\hat{\omega}_d = \sqrt{1-\zeta^2}$.

State transition between $t_j = t_0 + j\tau$ and t_{j+1} is described by

$$\begin{pmatrix} x_{j+1} \\ v_{j+1} \end{pmatrix} = A \begin{pmatrix} x_j \\ v_j \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{Kw\tau_2}{m\omega_n^2}\left(h_0 + (1-A_{11})x_j - A_{12}v_j\right)^{3/4} \end{pmatrix}.$$

where $x_j = x(t_j)$, $v_j = \dot{x}(t_j)$ and

$$A = \exp \begin{pmatrix} 0 & 1 \\ -1 & \zeta \end{pmatrix} \tau_1$$

Stability

The linearized equation around the fixed point

$$\begin{pmatrix} x_{j+1} \\ v_{j+1} \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} + \hat{w}\,(1 - A_{11}) & A_{22} - \hat{w}\,A_{12} \end{pmatrix}}_{B} \begin{pmatrix} x_j \\ v_j \end{pmatrix}.$$

Stability boundaries:

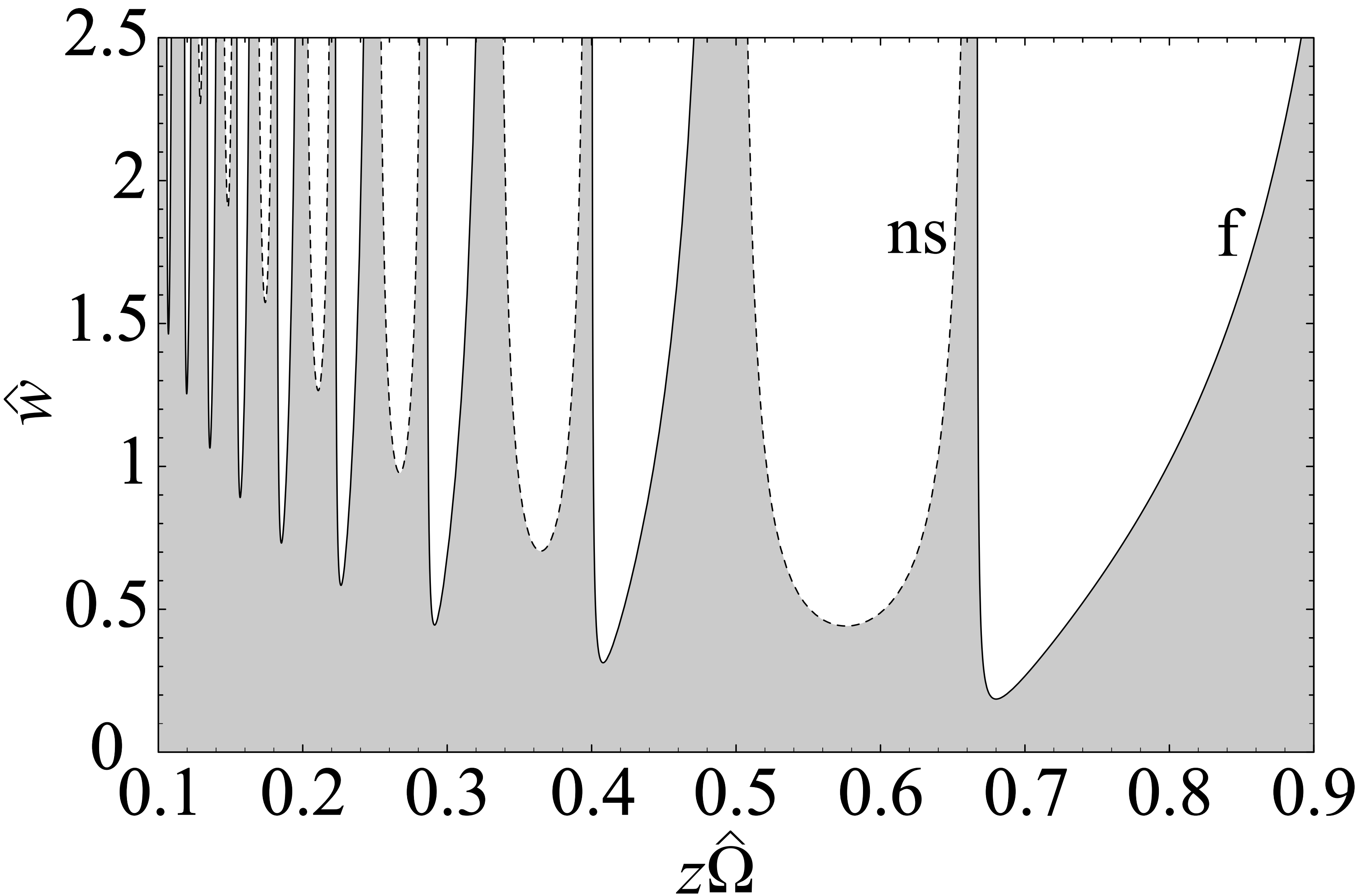
$$\hat{w}^{\mathrm{f}}_{\mathrm{cr}} = \frac{\det A + \mathrm{tr} A + 1}{2A_{12}} = \hat{\omega}_d \frac{\cos(\hat{\omega}_d \tau) + \cosh(\zeta\tau)}{\sin(\hat{\omega}_d \tau)}$$

$$\hat{w}^{\mathrm{ns}}_{\mathrm{cr}} = \frac{\det A - 1}{A_{12}} = -2\hat{\omega}_d \frac{\sinh(\zeta\tau)}{\sin(\hat{\omega}_d \tau)},$$

where

$$\hat{w} = \frac{3}{4h_0^{1/4}} \frac{K\tau_2}{m\omega_n^2} w$$

Stability chart



Flip Bifurcation

Consider the following perturbation of the linear system around the fixed point in the basis of the eigenvectors

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} -1 + a^f \Delta\hat{w} & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + \begin{pmatrix} \sum_{i+j=2,3} c_{ij}\xi_n^i \eta_n^j \\ \sum_{i+j=2,3} d_{ij}\xi_n^i \eta_n^j \end{pmatrix},$$

Using center manifold and normal form reduction we find that there is a period two orbit on the center manifold

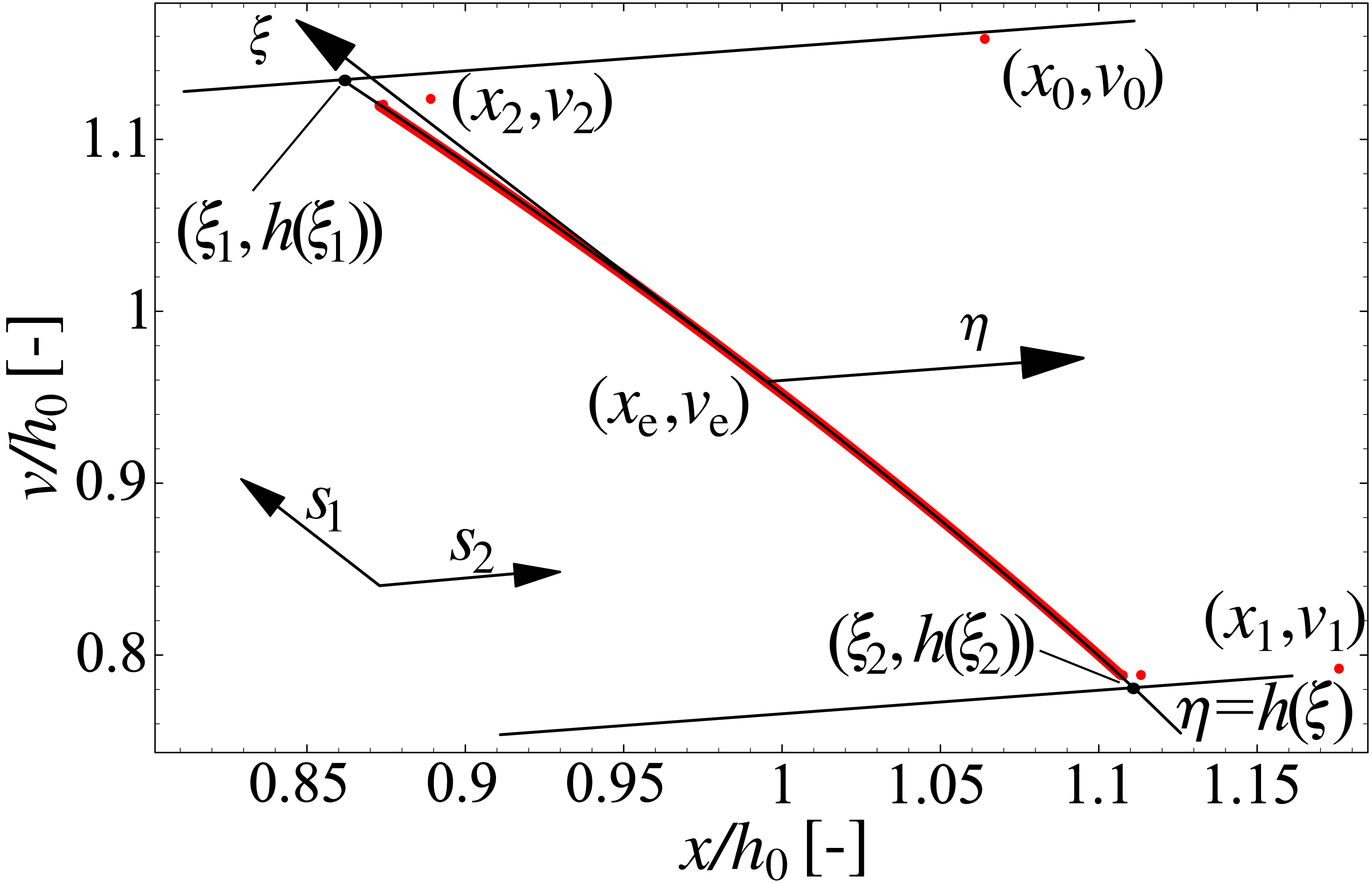
$$\xi_{1,2} = \sqrt{-\frac{\Delta\hat{w}a^f}{\delta}},$$

where

$$\delta = -\frac{5}{12h_0^2}\frac{\cosh(\zeta\tau) + \cos(\hat{\omega}_d\tau)}{\cosh(\zeta\tau) + 2\sinh(\zeta\tau) + \cos(\hat{\omega}_d\tau)} < 0.$$

Hence, the bifurcation is subcritical!

Simulation



ξ
(x_0, v_0)
(x_2, v_2)
$(\xi_1, h(\xi_1))$
η
(x_e, v_e)
s_1
s_2
(x_1, v_1)
$(\xi_2, h(\xi_2))$
$\eta = h(\xi)$
1.1
1
0.9
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
v/h_0 [-]
x/h_0 [-]

Neimark-Sacker bifurcation

Similarly, the Taylor expansion of the system in the eigenbasis

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = (1+|a^h|\Delta\hat{w}) \begin{pmatrix} \mathrm{e}^{i\varphi} & 0 \\ 0 & \mathrm{e}^{-i\varphi} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + \begin{pmatrix} \sum\limits_{i+j=2,3} c_{ij}\xi_n^i \eta_n^j \\ \sum\limits_{i+j=2,3} d_{ij}\xi_n^i \eta_n^j \end{pmatrix}.$$

The radius of the invariant circle (in the eigenbasis)

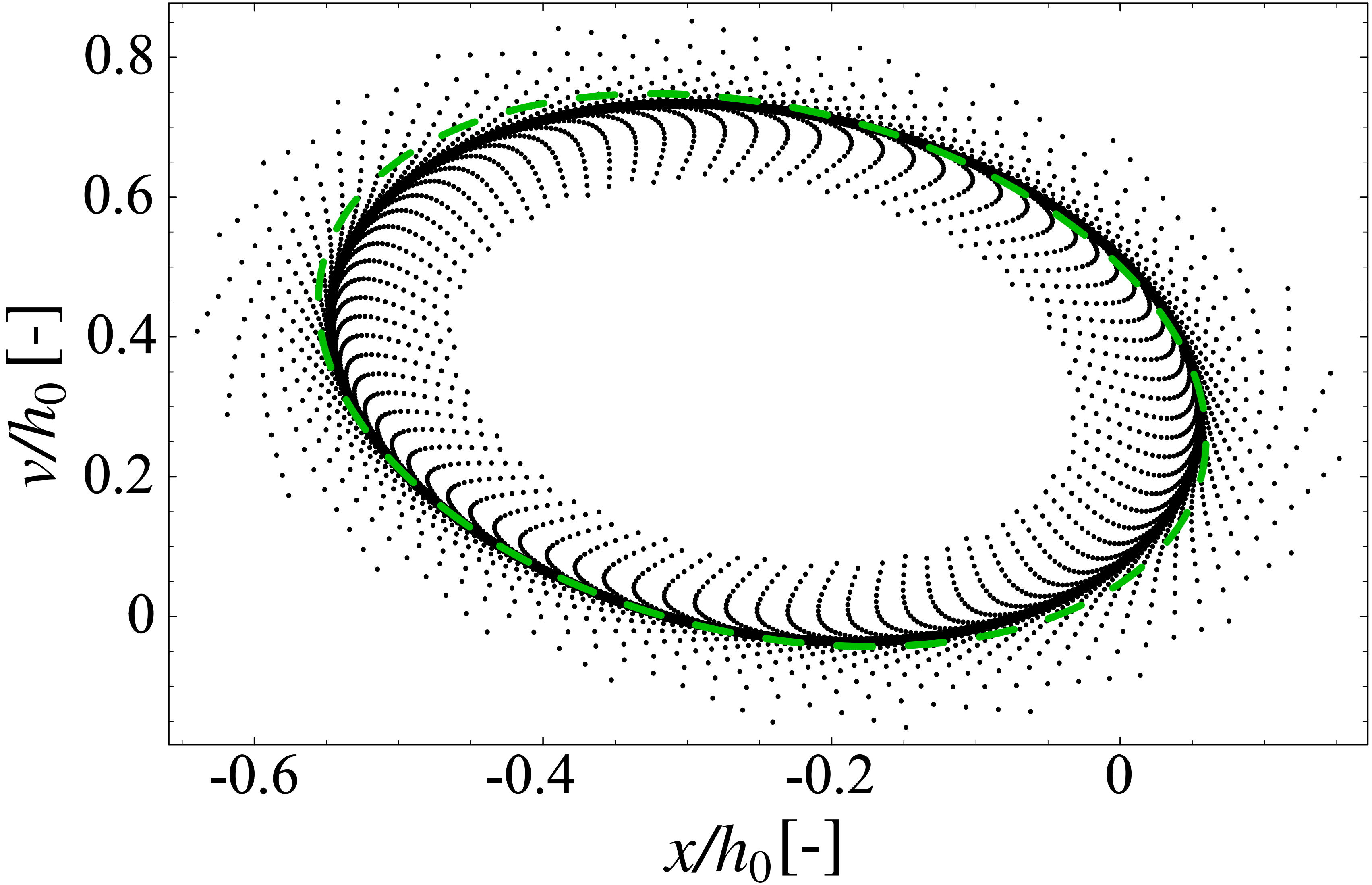
$$R = \sqrt{-\frac{2|a^h|\,\Delta\hat{w} + |a^h|^2\,\Delta\hat{w}^2}{2(1+|a^h|\,\Delta\hat{w})\delta}} \approx \sqrt{-\frac{|a^h|\,\Delta\hat{w}}{\delta}},$$

where

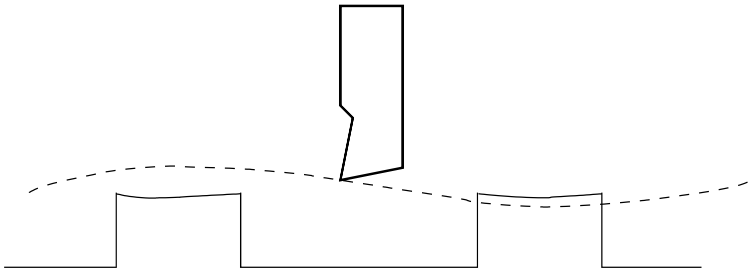
$$\delta = \frac{\mathrm{e}^{-5\zeta\tau_1}(4\mathrm{e}^{4\zeta\tau_1} - 3\mathrm{e}^{2\zeta\tau_1} - 1)(\cosh(\zeta\tau_1) - \cos(\omega_d\tau_1))}{32h_0^2},$$

This is subcritical, too!

Simulation



Period-2 motion with 'fly-overs'



The motion exists if:

$$\hat{w} > \frac{3\hat{\omega}_d}{2^{7/4}} \frac{\cos(\hat{\omega}_d \tau) + \cosh(\zeta\tau)}{\sin(\hat{\omega}_d \tau)}$$

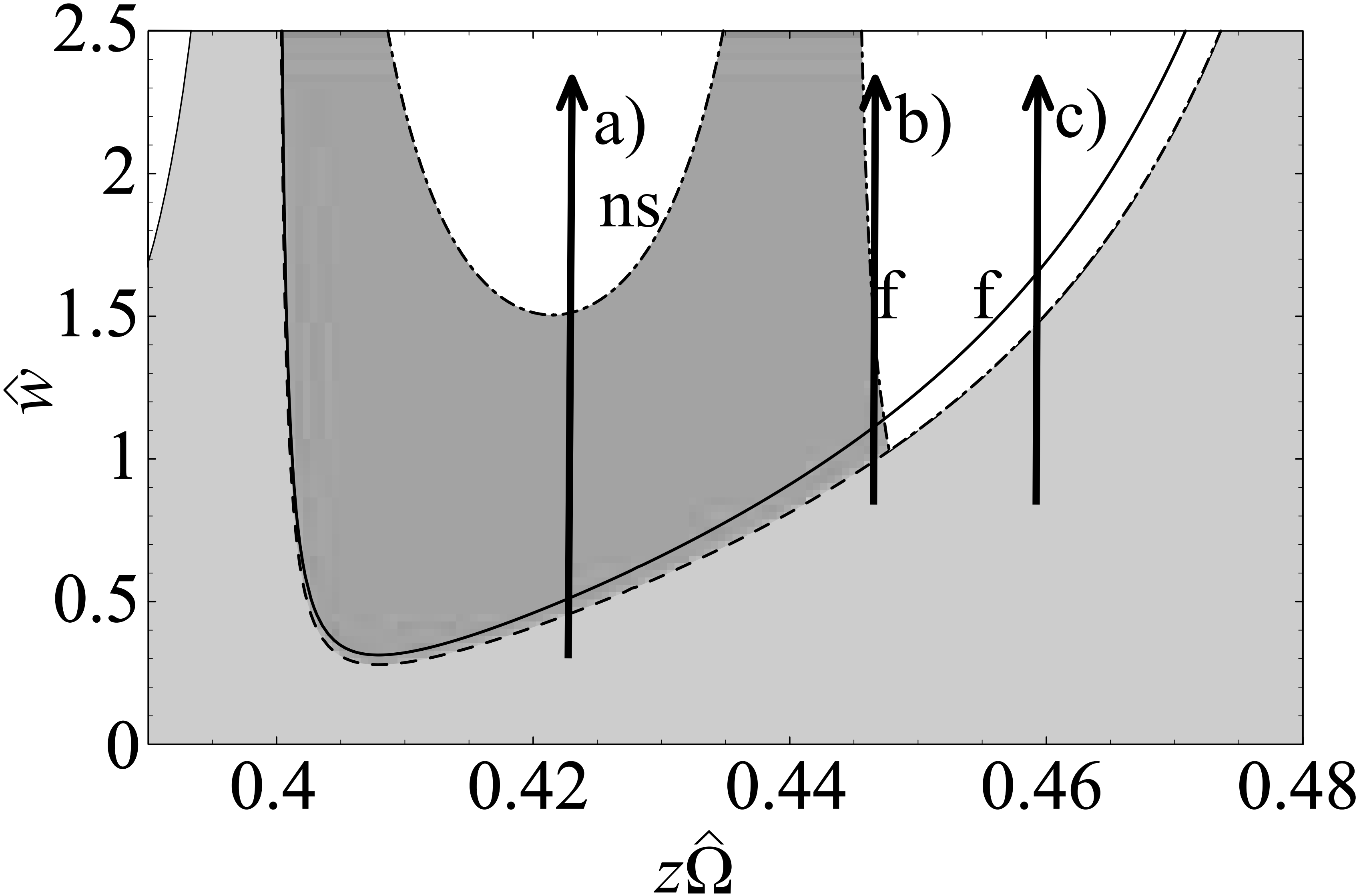
It is stable when

$$\hat{w} < 2^{1/4}\omega_d \frac{\cos(2\omega_d\tau_1) + \cosh(2\zeta\tau_1)}{\sin(2\omega_d\tau_1)}$$

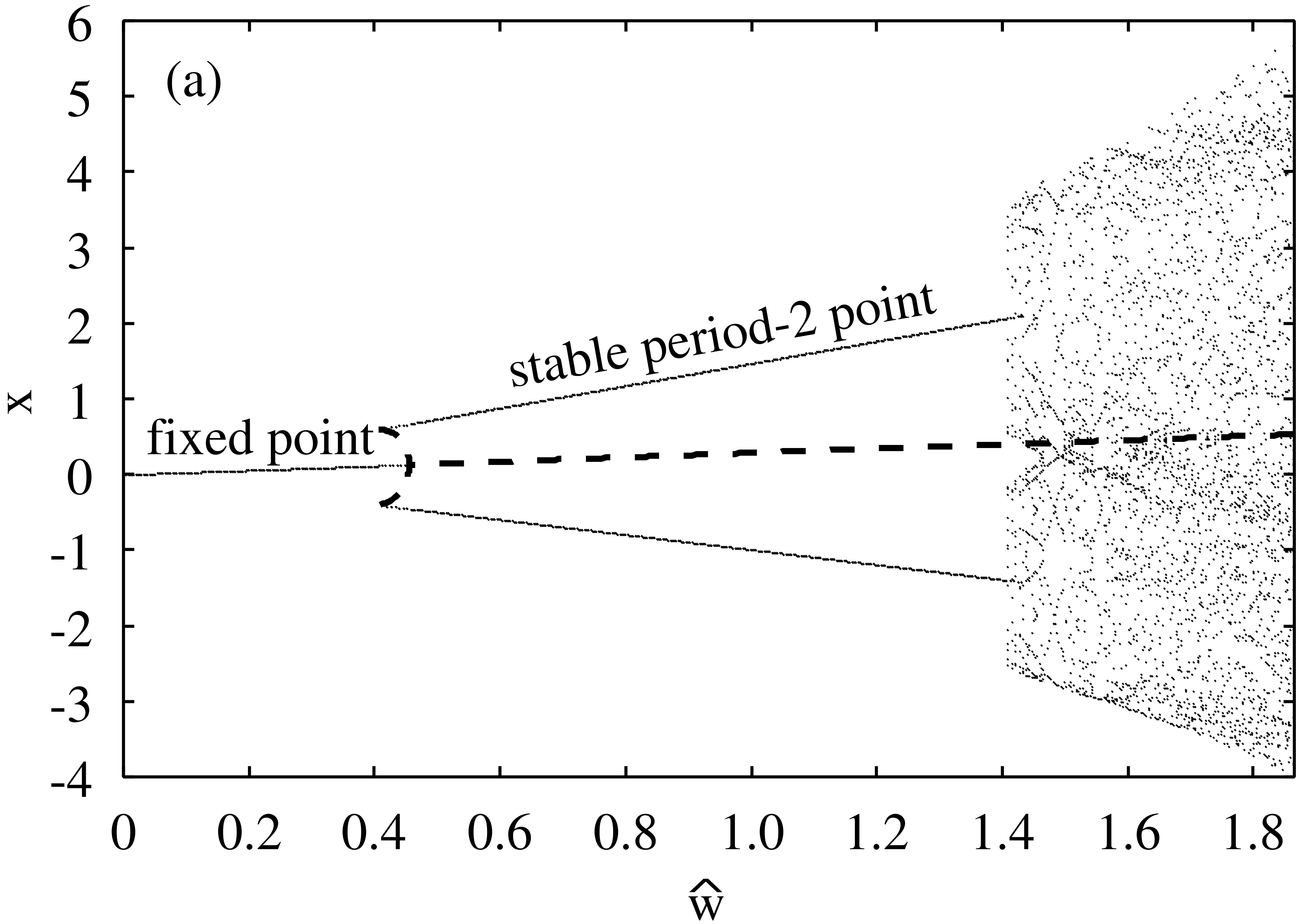
or

$$\hat{w} < -2^{7/4}\omega_d \frac{\sinh(2\zeta\tau_1)}{\sin(2\omega_d\tau_1)}.$$

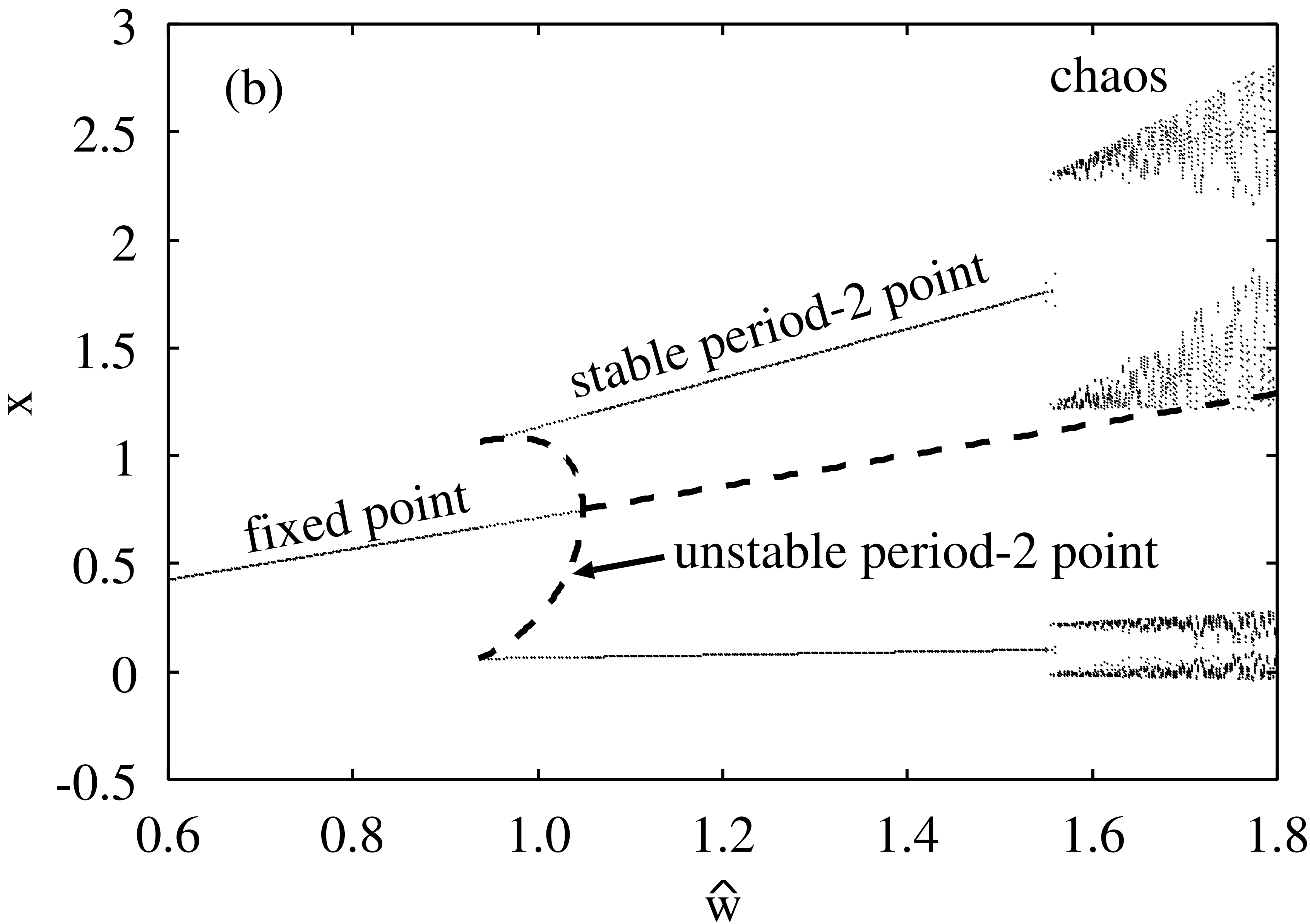
Stability chart



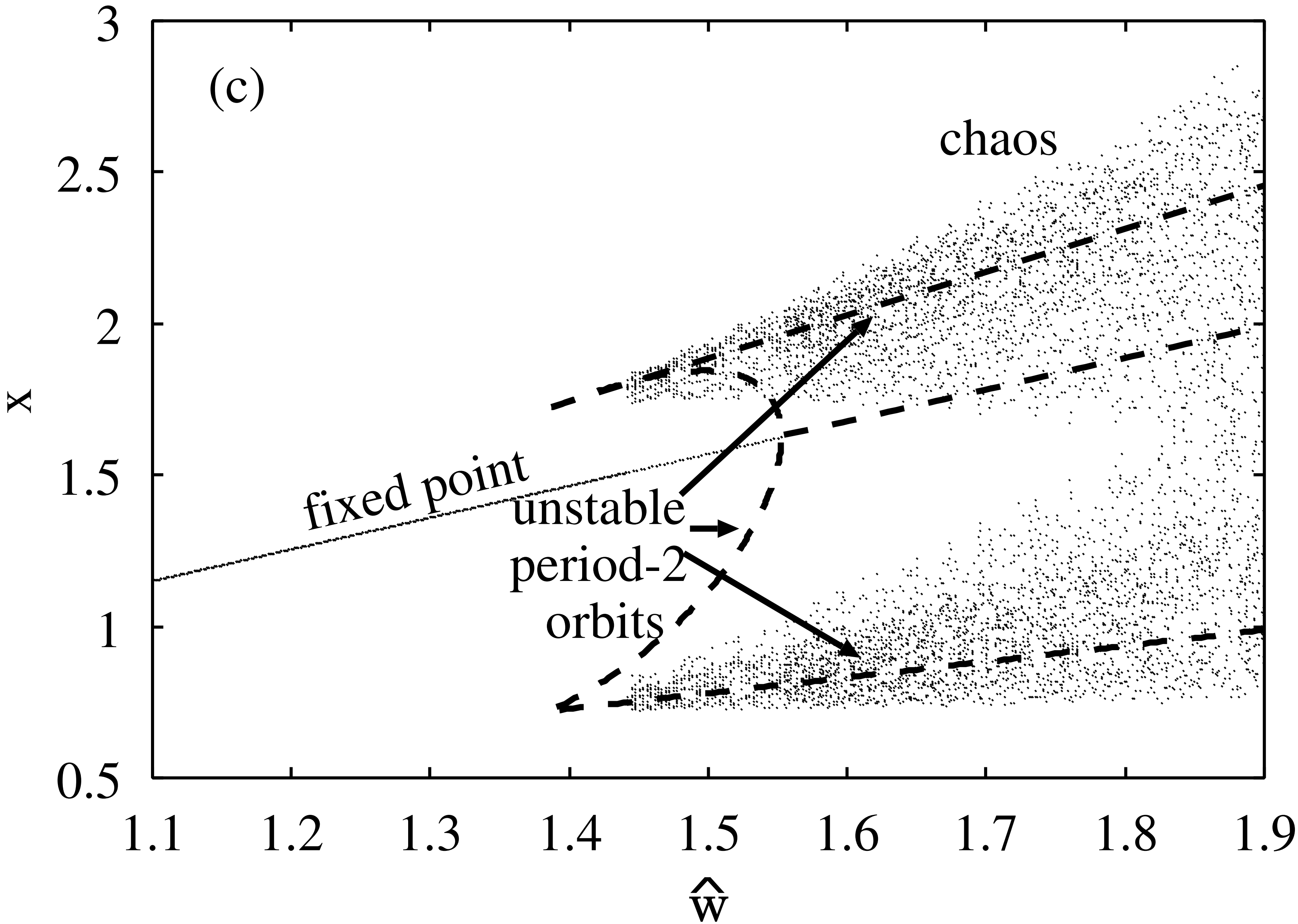
Bifurcation diagram 1



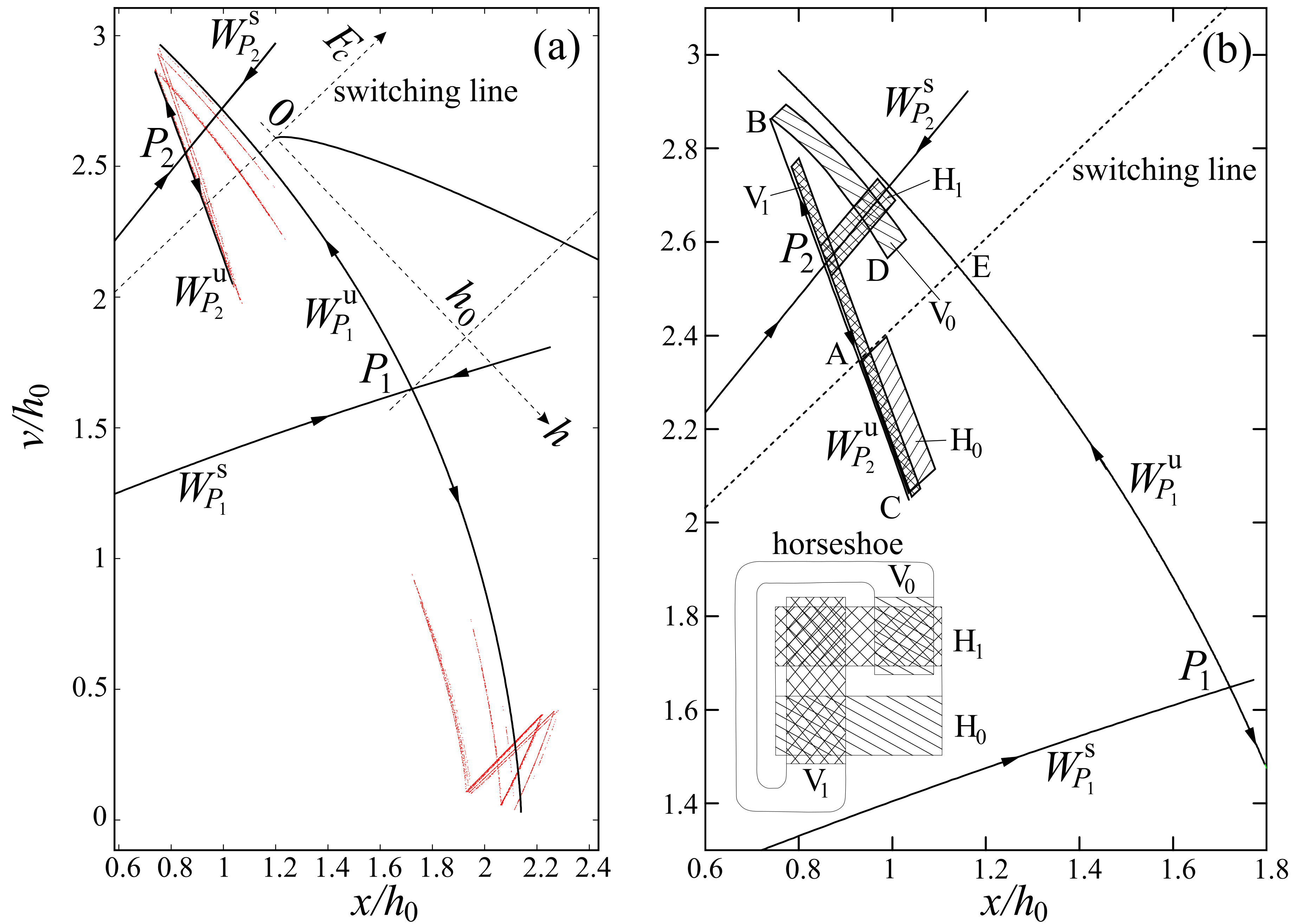
Bifurcation diagram 2



Bifurcation diagram 3



Smale horseshoe



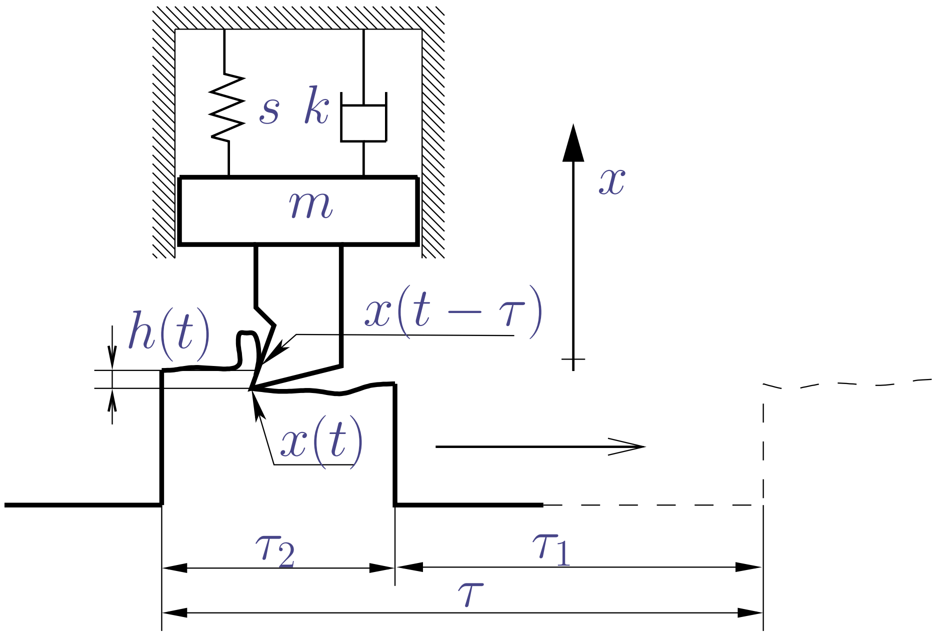
Chaos

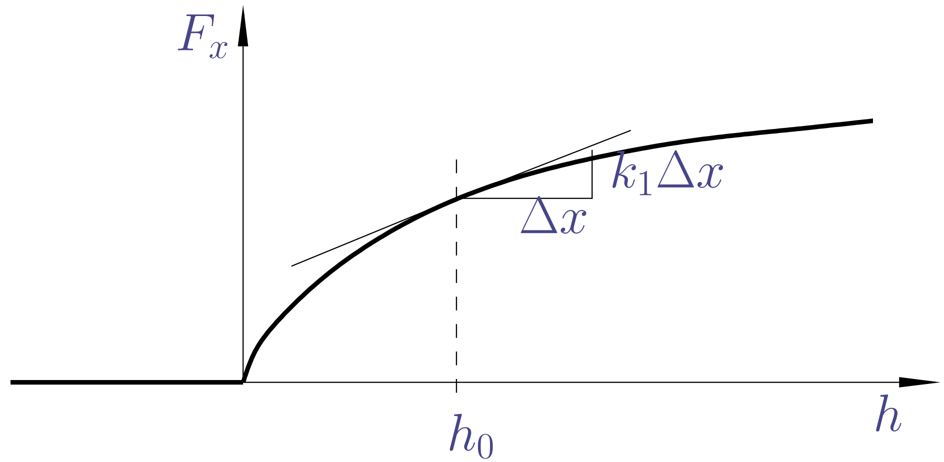
The transition matrix of the symbolic dynamics:

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad T^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \Longrightarrow T \text{ irreducible}$$

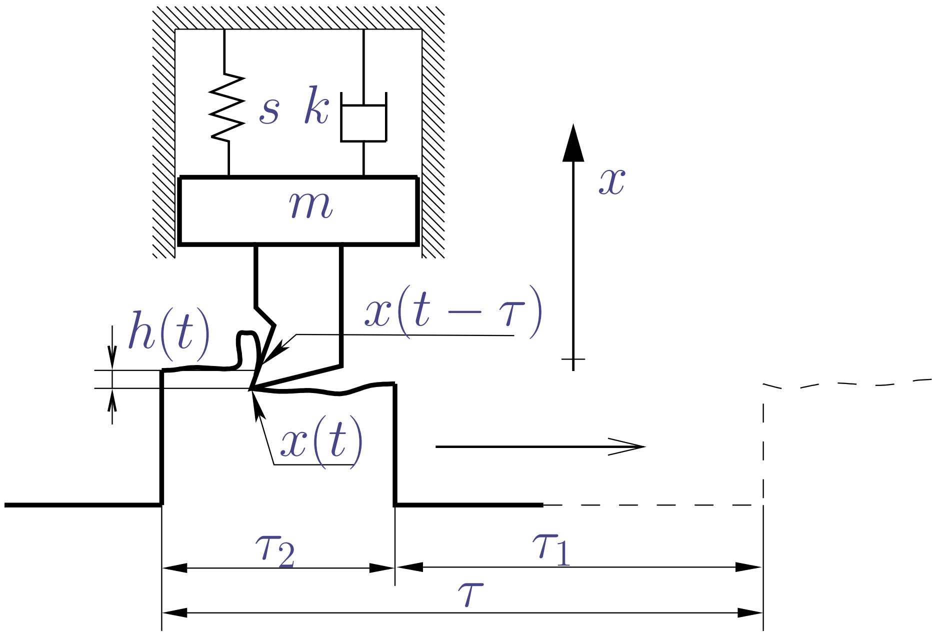
Delay differential equation model

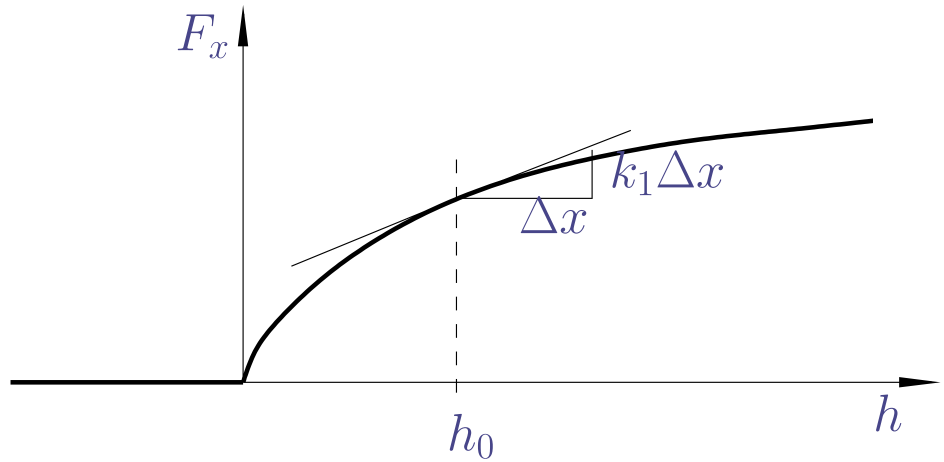
Mechanical model





Mechanical model





Equation of motion:

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where

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Variational system

Linearized equation with dimensionless time $(\hat{t} = \omega_n t)$:

$$\ddot{x}(\hat{t}) + 2\zeta\dot{x}(\hat{t}) + x(\hat{t}) = g(\hat{t})\hat{w}\left(x(\hat{t} - \hat{\tau}) - x(\hat{t})\right),$$

where $\hat{w} = 3Kw/(4h_0^{1/4}m\omega_n^2)$ is the dimenzionless chip width.

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where $\hat{w} = 3Kw/(4h_0^{1/4}m\omega_n^2)$ is the dimenzionless chip width.

Rewritten into 1st order form $(\mathbf{x}(\hat{t}) = (x(\hat{t}), \dot{x}(\hat{t}))^T)$:

$$\dot{\mathbf{x}}(\hat{\tau}) = \mathbf{A}(\hat{t})\mathbf{x}(\hat{t}) + \mathbf{B}(\hat{t})\mathbf{x}(\hat{t}-\hat{\tau}),$$

where

$$\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ -1 - g(t)\hat{w} & -2\zeta \end{pmatrix}, \quad \mathbf{B}(t) = \begin{pmatrix} 0 & 0 \\ g(t)\hat{w} & 0 \end{pmatrix}.$$

Stability analysis

- $\mathrm{e}^{\lambda\hat{\tau}}$ characteristic multiplier

  $\Updownarrow$

  $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}(t)$ satisfies the equation such that $\mathbf{v}(t) = \mathbf{v}(t + \hat{\tau})$

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Stability analysis

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- The periodic solution is assimptotically stable if for each characteristic multiplier $|e^{\lambda\hat{\tau}}| < 1$.

Exploiting the first condition we are left with the BVP

$$\dot{\mathbf{v}}(t) = \left(\mathbf{A}(t) + e^{-\lambda\hat{\tau}}\mathbf{B}(t) - \lambda\mathbf{I}\right)\mathbf{v}(t)$$

$$\mathbf{v}(0) = \mathbf{v}(\hat{\tau}).$$

Stability analysis

The BVP is solvable iff

$$0 = f(\mu) := \det\left(\mathbf{\Phi}(\hat{\tau}) - \mathbf{I}\right),$$

where

$$\mathbf{\Phi}(\hat{\tau}) = \mu \mathrm{e}^{(\mathbf{A}_2+\mu\mathbf{B}_2)\hat{\tau}_2}\mathrm{e}^{(\mathbf{A}_1+\mu\mathbf{B}_1)\hat{\tau}_1},$$

$$\mu = \mathrm{e}^{-\lambda\hat{\tau}},$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

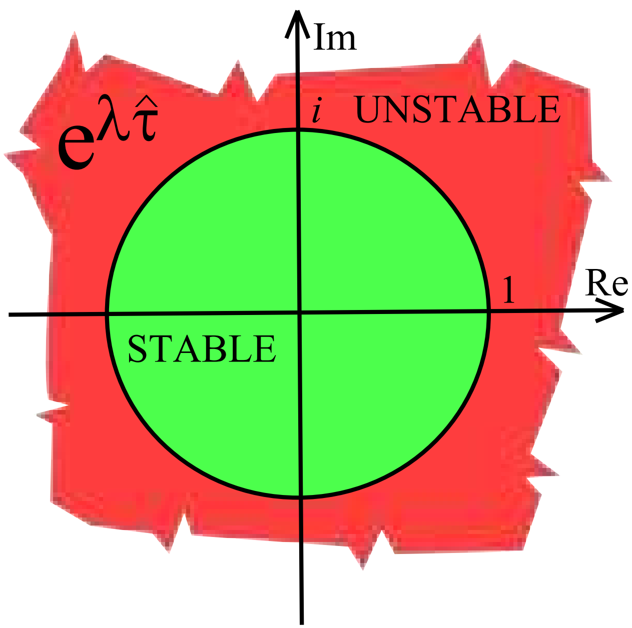
$$\mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ -1-\hat{w} & -2\zeta \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & 0 \\ \hat{w} & 0 \end{pmatrix}.$$

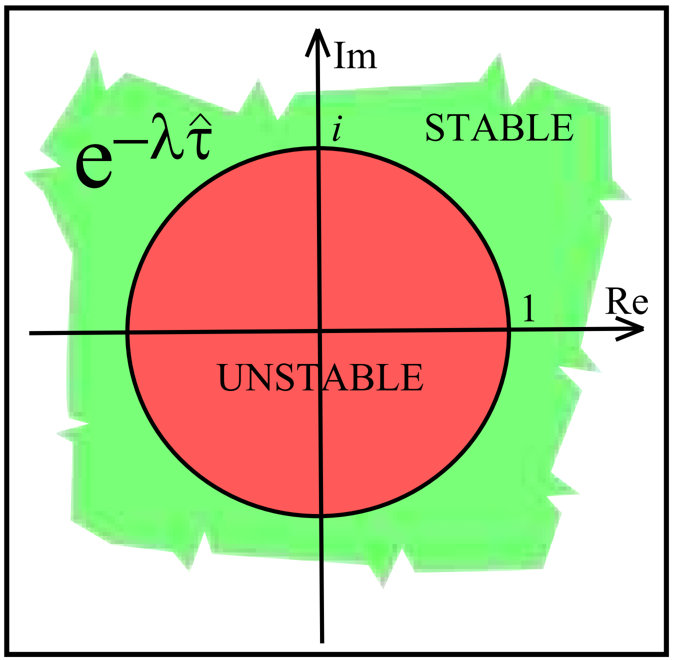
Argument principle

Roots for which $\mu = (\mathrm{e}^{\lambda\hat{\tau}})^{-1}\ f\ |\mu| < 1$ cause instability. They can be easily counted

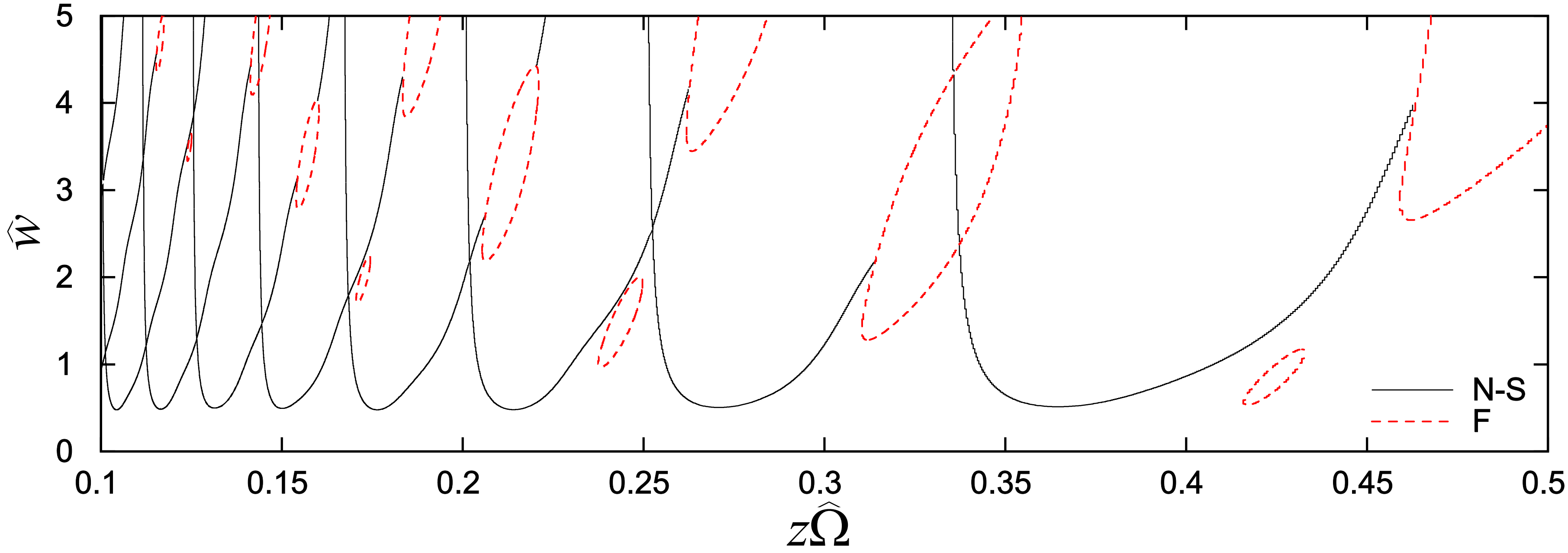
$$N = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \frac{1}{2\pi}\int_\gamma \mathrm{d}\arg f$$

$$= \frac{1}{2\pi}\sum_{j=1}^{n} \arg \frac{f(\exp(j\frac{2\pi}{n}i))}{f(\exp((j-1)\frac{2\pi}{n}i))}$$

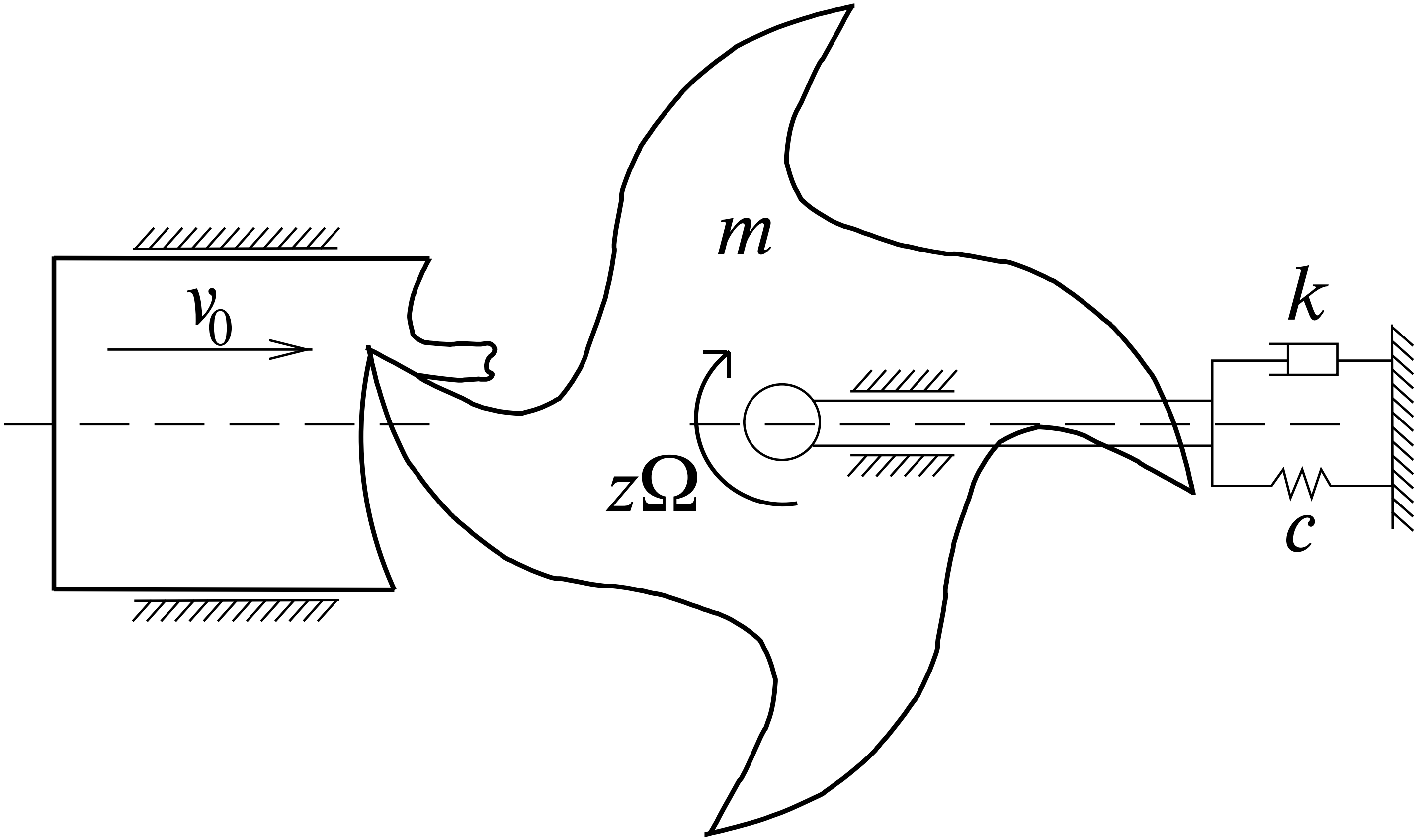




Stability chart



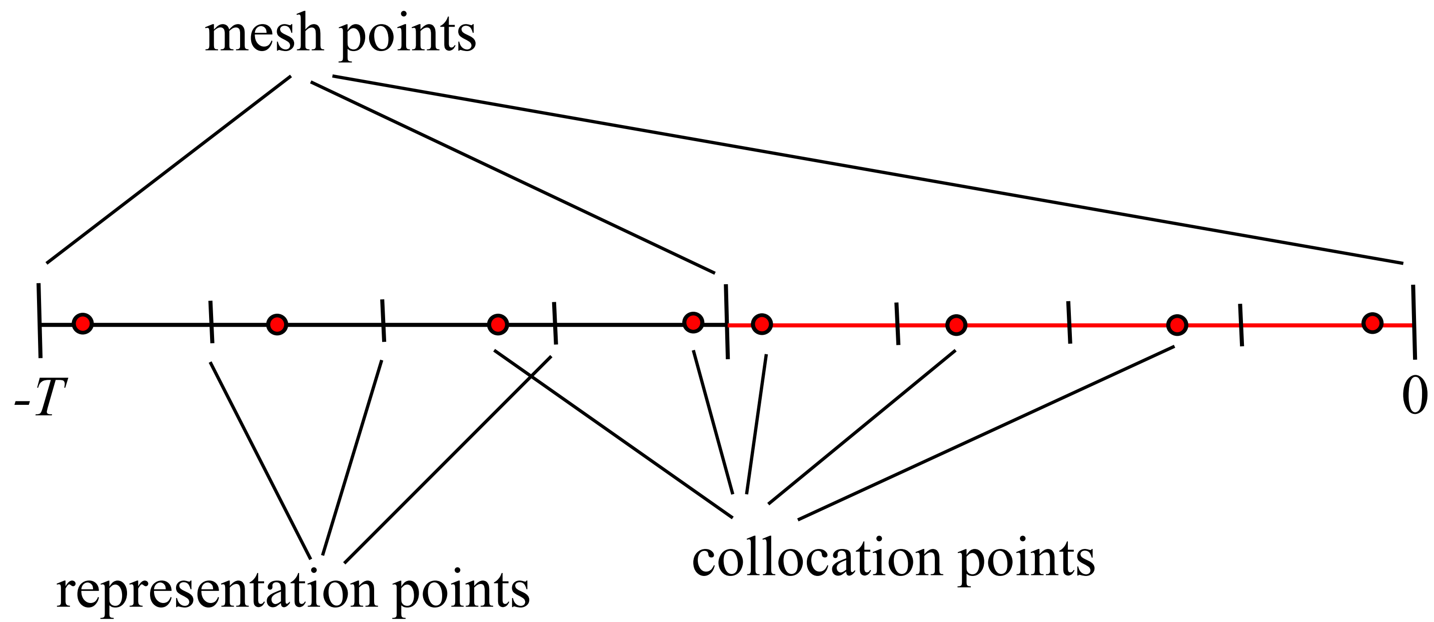
Machining with ‘fly-over’ effect



$$\ddot{x}(t) + 2\zeta\dot{x}(t) + x(t) = g(\varphi)\hat{w}\left(\cos\varphi + 0.3\sin\varphi\right) \times$$
$$\times\left[H\left((h_0 + x(t-2\tau) - x(t-\tau))\right)F_c\left((h_0 + x(t-\tau) - x(t))\sin\varphi\right)\right.$$
$$\left. + H\left((h_0 + x(t-\tau) - x(t-2\tau))\right)F_c\left((2h_0 + x(t-2\tau) - x(t))\sin\varphi\right)\right]$$

Numerical method

Orthogonal collocation



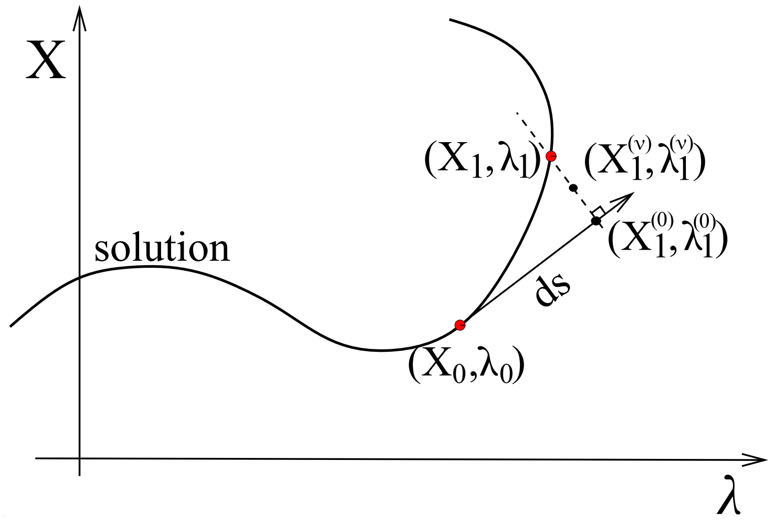
$$\tilde{\varphi}(\theta) = \sum_{j=0}^{\mathfrak{m}} \varphi(\theta_{i+\frac{j}{\mathfrak{m}}}) P_{i,j}(\theta) \quad P_{i,j}(\theta) = \prod_{r=0, r\neq j}^{\mathfrak{m}} \frac{\theta - \theta_{i+\frac{r}{\mathfrak{m}}}}{\theta_{i+\frac{j}{\mathfrak{m}}} - \theta_{i+\frac{r}{\mathfrak{m}}}}$$

The equation is satisfied at $c_{i,j}$

$$\dot{\tilde{\varphi}}(c_{i,j}) = f(c_{i,j}, \tilde{\varphi}(c_{i,j}), \tilde{\varphi}(c_{i,j} - \tau \mod T))$$

[Engelborghs and Roose], 2001]

Continuation



Consider

$$F(X,\lambda) = 0.$$

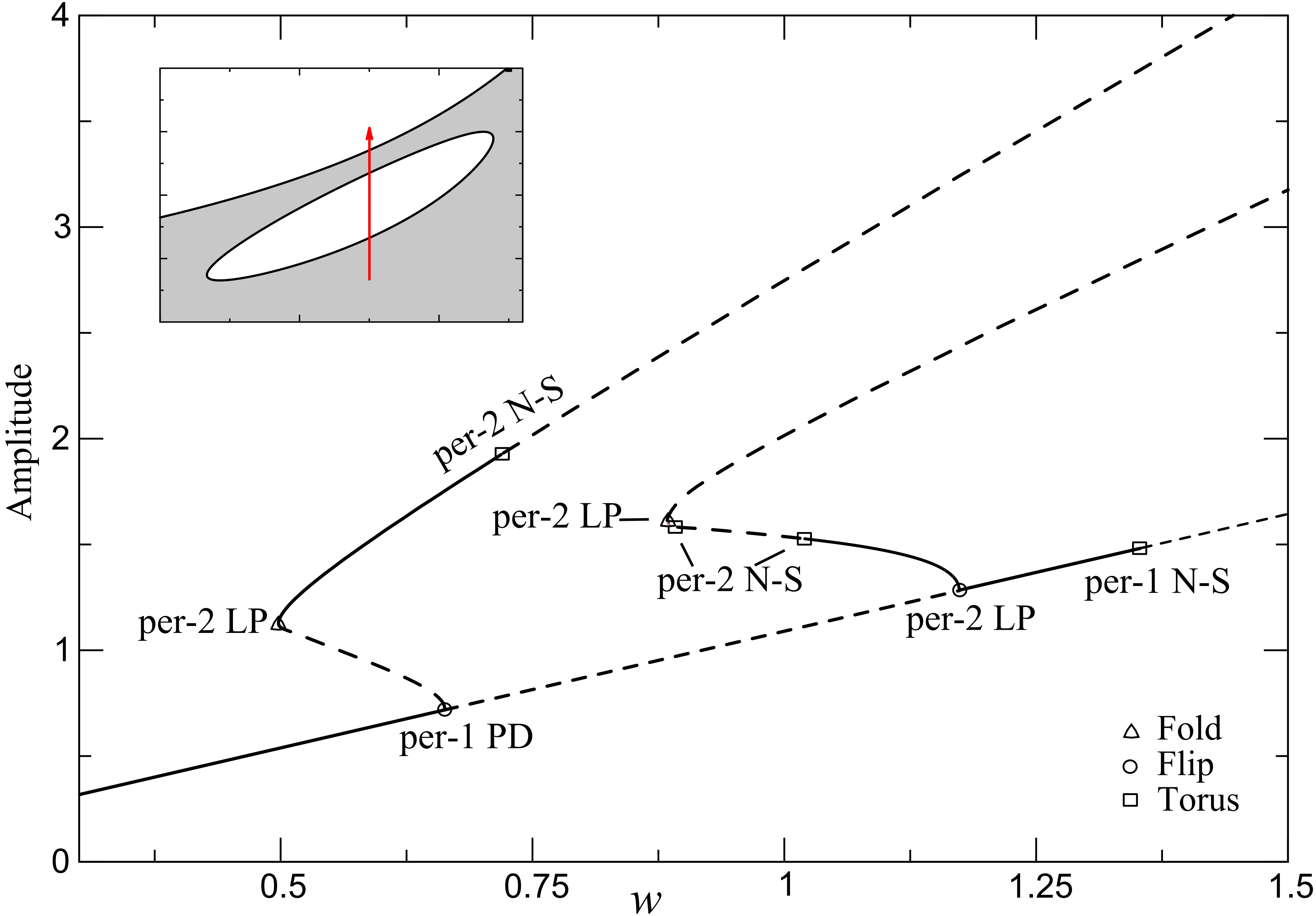
The tangent comes from

$$F_X(X_0,\lambda_0)X' + F_\lambda(X_0,\lambda_0)\lambda' = 0$$

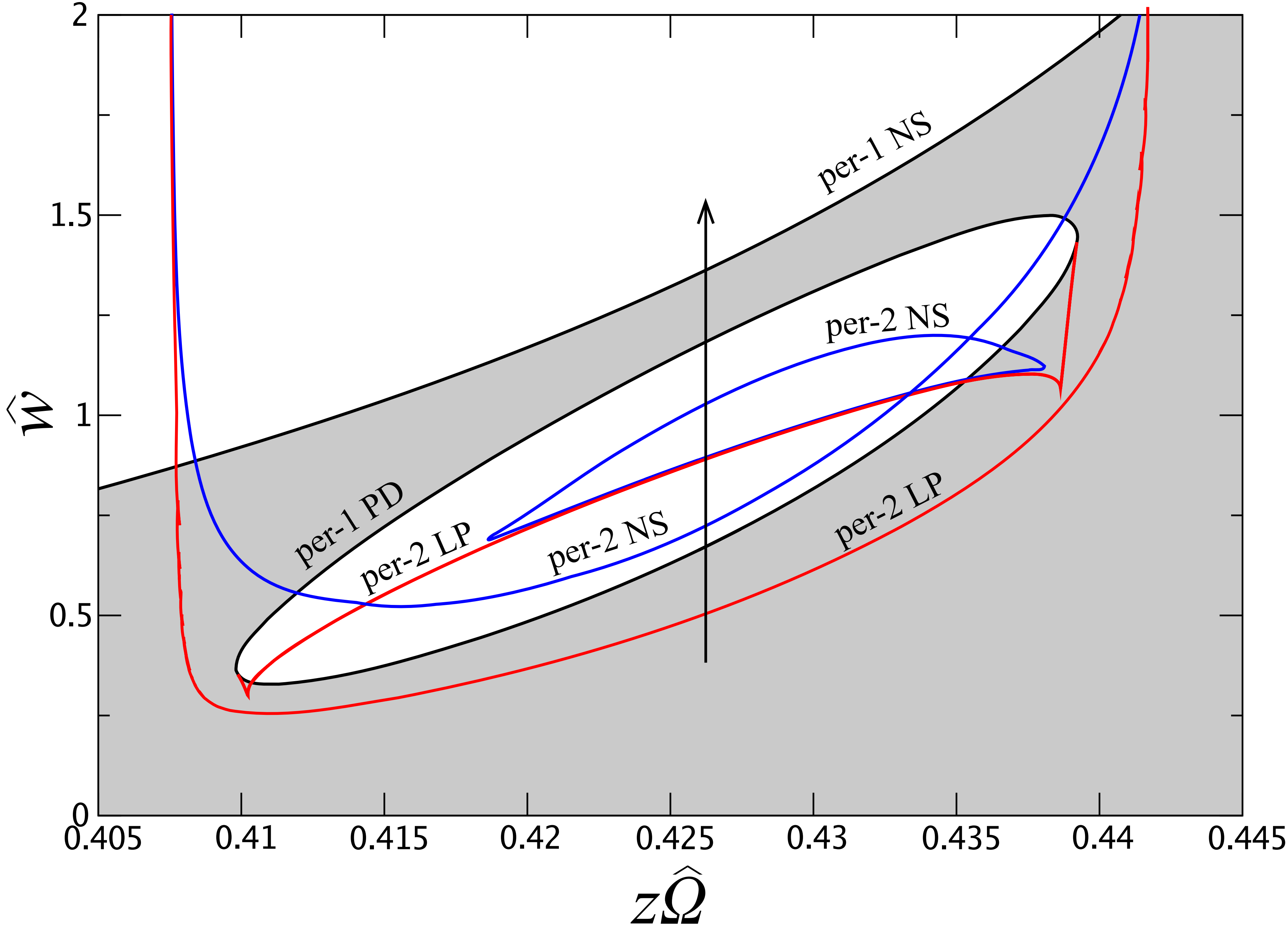
The Newton iteration

$$\begin{pmatrix} F_X(X_1^{(\nu)},\lambda_1^{(\nu)}) & F_\lambda(X_1^{(\nu)},\lambda_1^{(\nu)}) \\ X'^*_0 & \lambda'_0 \end{pmatrix} \begin{pmatrix} \Delta X \\ \Delta\lambda \end{pmatrix} = \begin{pmatrix} -F(X,\lambda) \\ ds - X'^*(X_1^{(\nu)} - X_0) - \lambda'_0(\lambda_1^{(\nu)} - \lambda_0) \end{pmatrix}$$

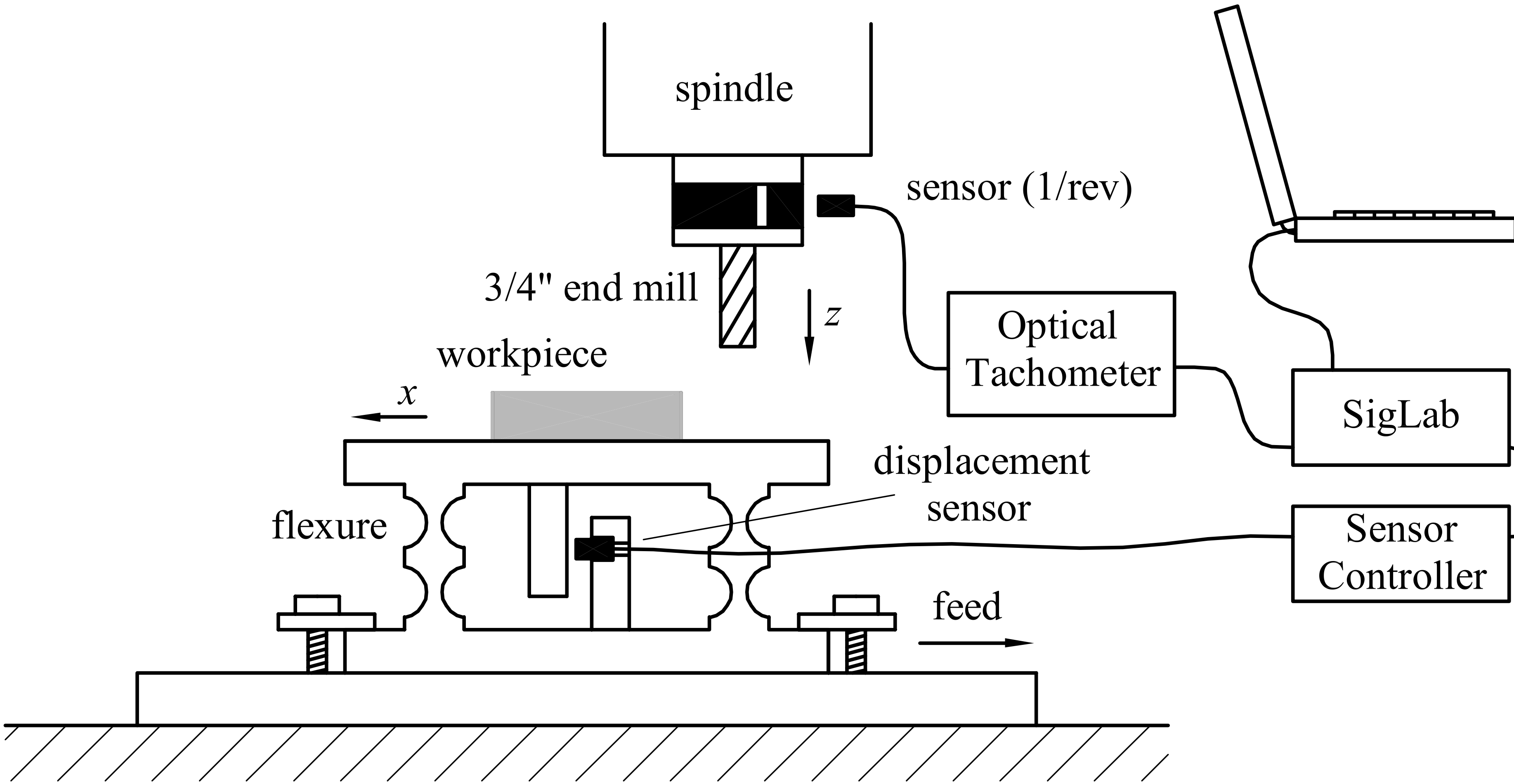
Continuation



Continuation



Experimental setup



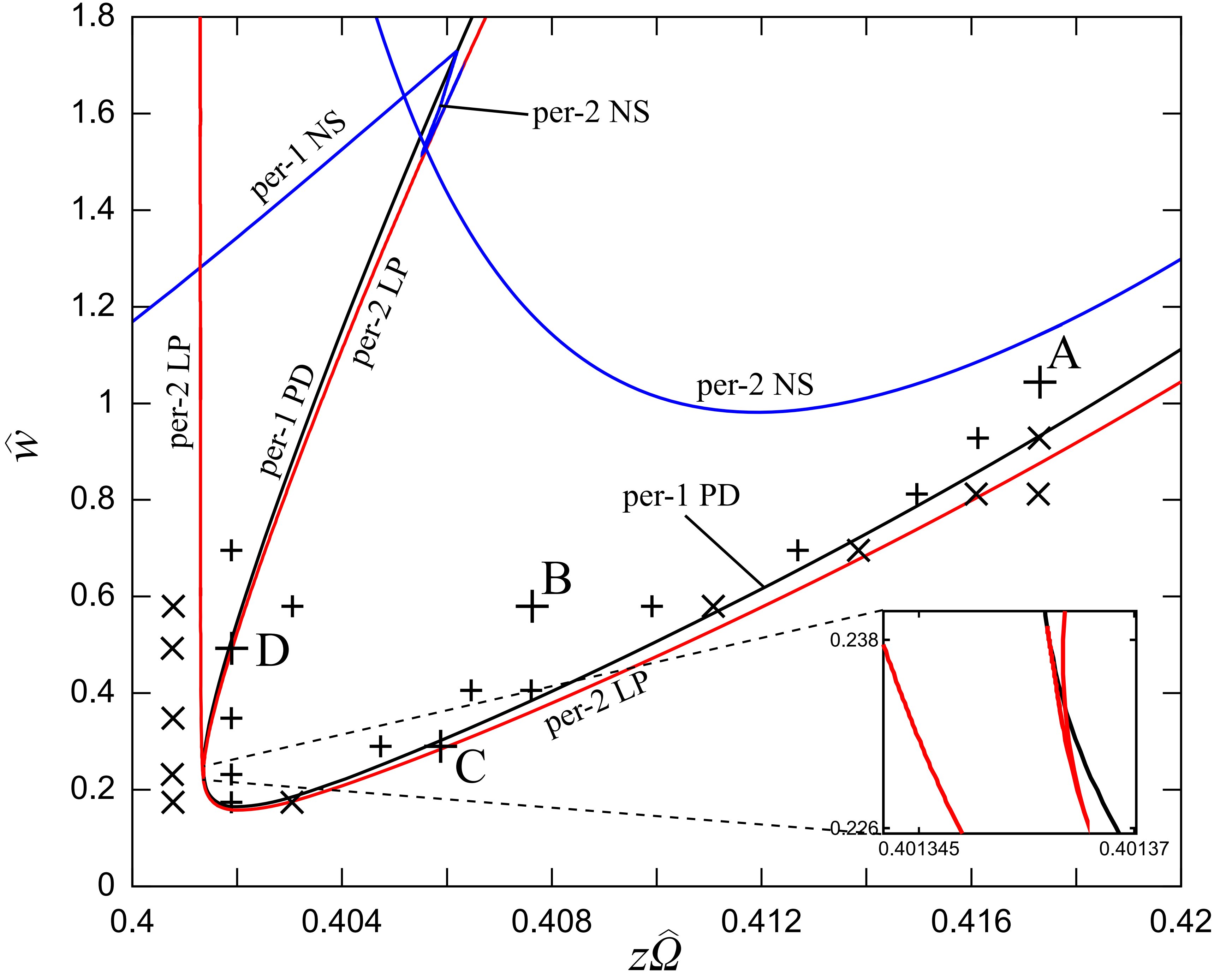
One tooth tool with diameter of 19.05mm (3/4”); Workpiece width: 6.35mm

Natural frequency: 146.8Hz; Spindle speed: 3000 - 4000 rpm

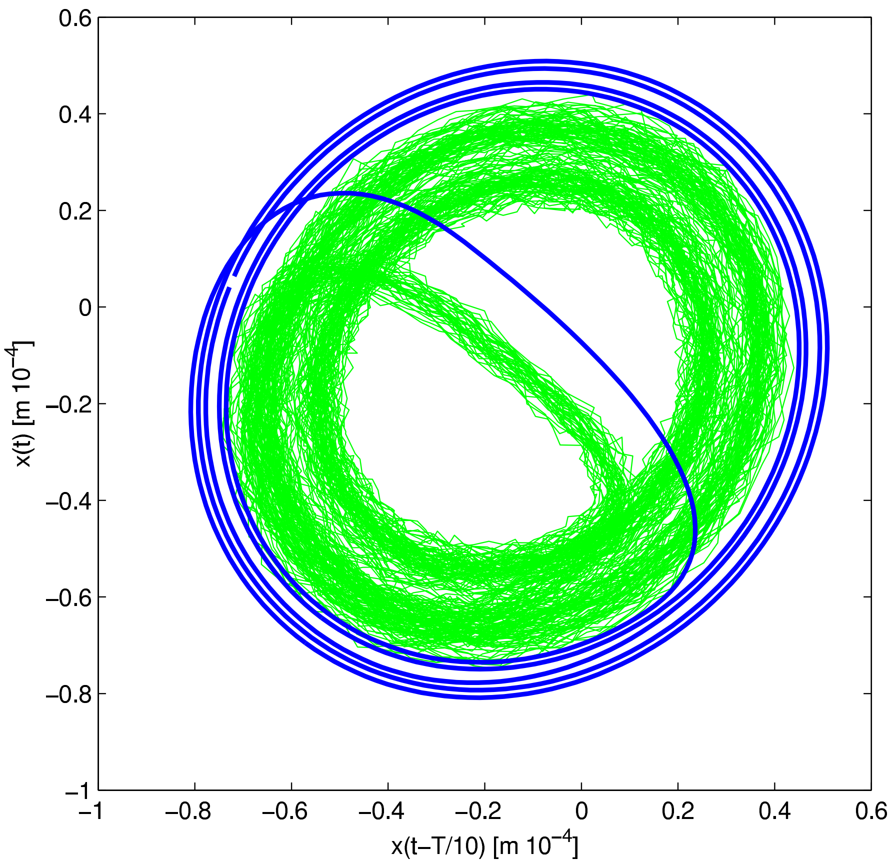
Feed $h_0 = 0.1082$mm/period

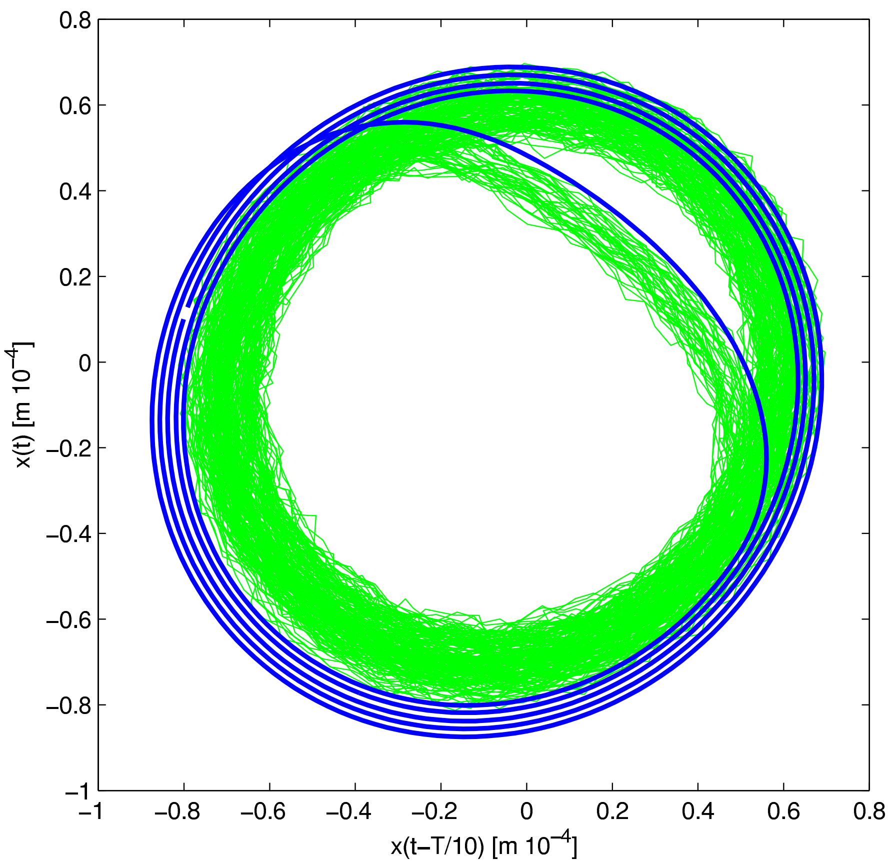
$\hat{w} = 2.9 \times 10^{-4} w$, where w is the depth of cut (z direction).

Stability chart



Tool trajectories





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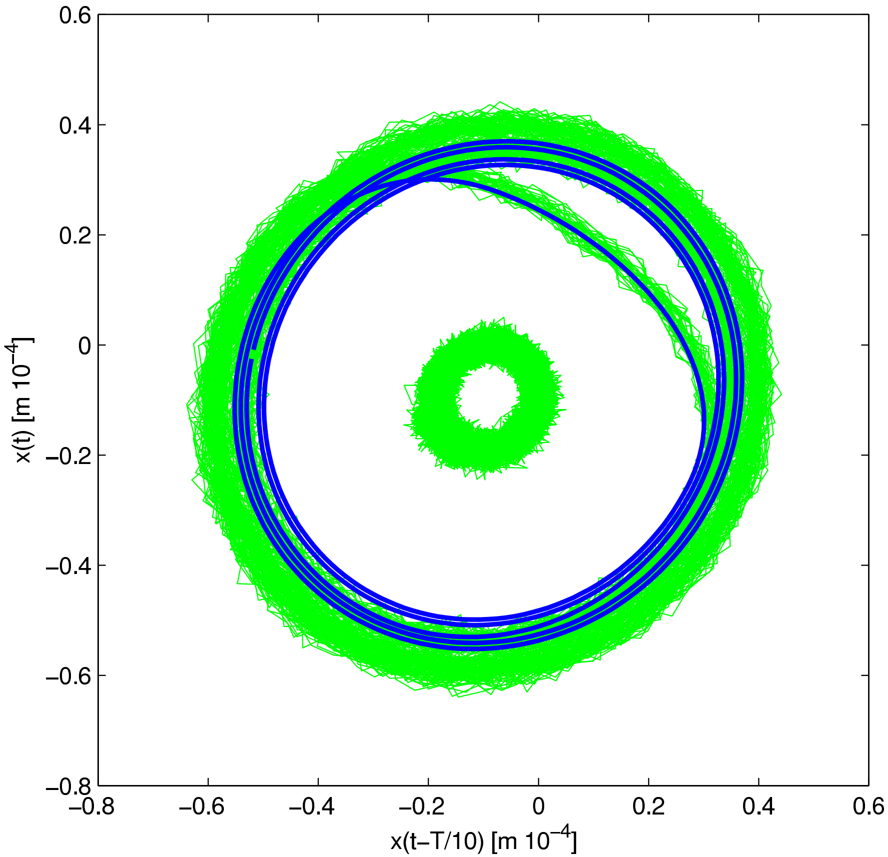
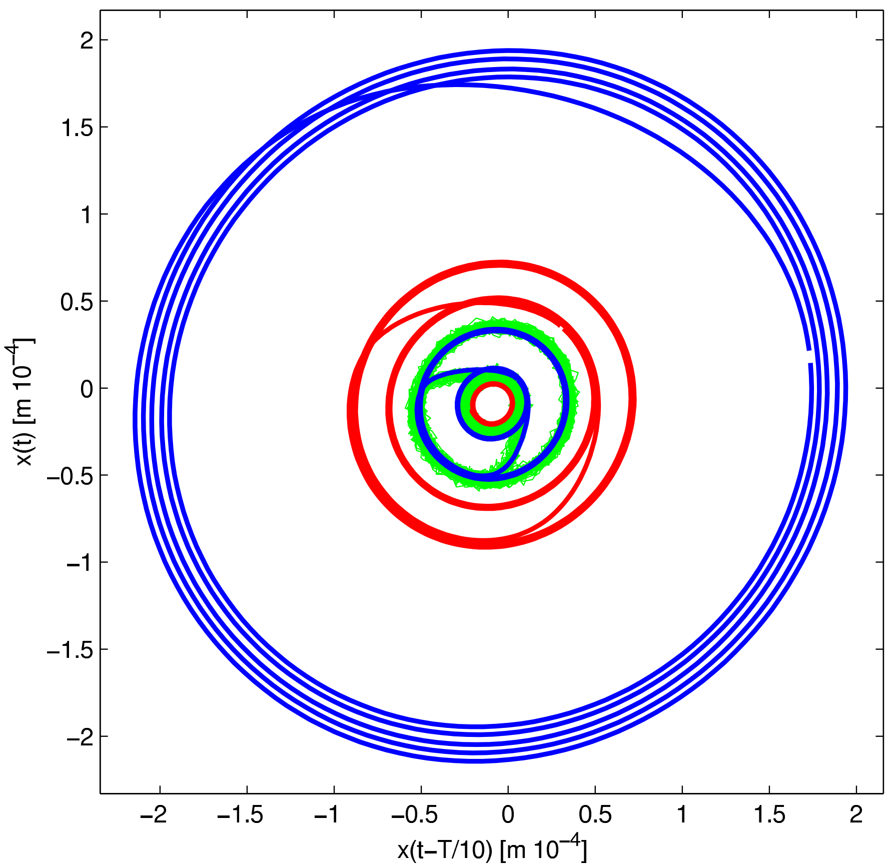
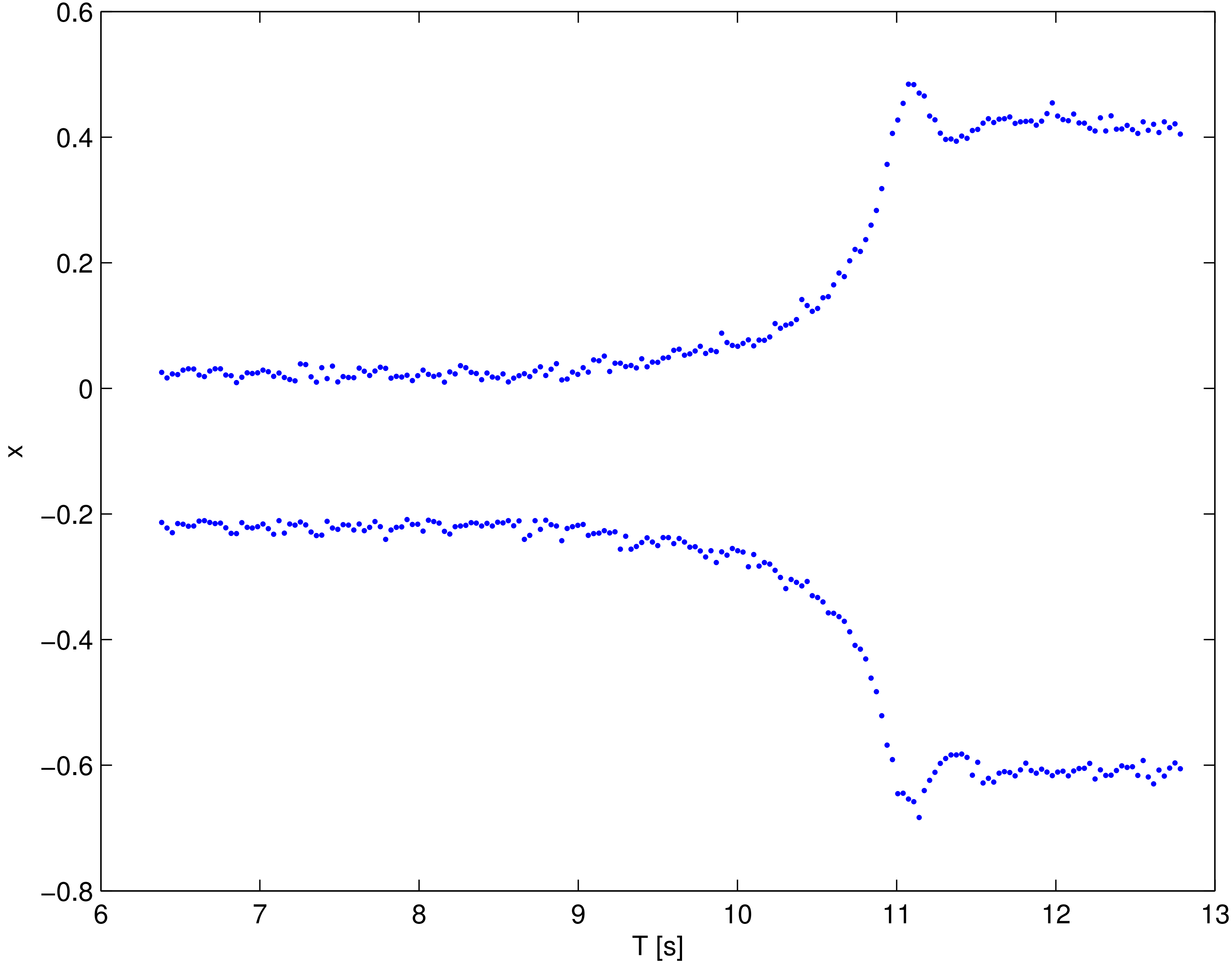


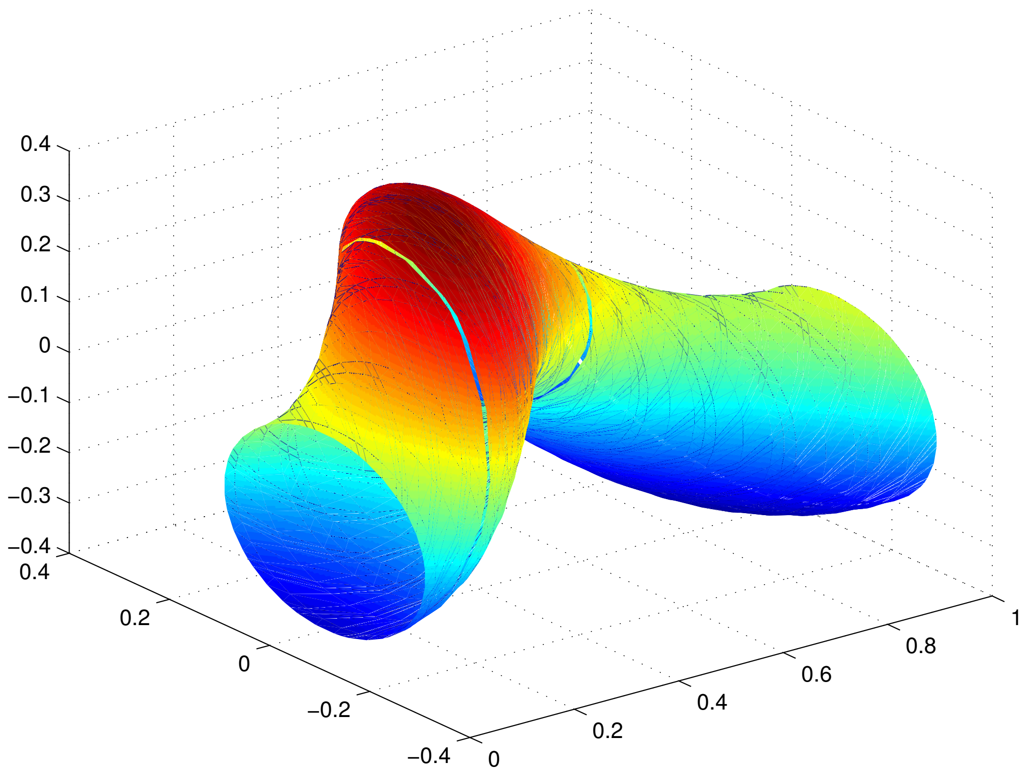


Subcriticality



Conclusions

- High amplitude periodic, quasi-periodic and chaotic vibrations were found.
- These unwanted vibrations can occur at linearly stable parameters.
- This parameter region can be large due to the fold of period-2 orbits
- Side-product: PDDE-CONT a continuation software for periodic and autonomous DDEs



Questions?

Thank you for your attention!