Bifurcations and chaos in high-speed milling

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- Discrete-time model
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- Delay-differential equation model
High-speed milling (standard model)

Calculation of the cutting force:

\[ F^t_c = K^t w h^{3/4}(t) \quad \text{and} \quad F^n_c = K^n w h^{3/4}(t), \]

[Tlusty, 2000], [Burns and Davies, 2002].
History

Mostly stability results and simulation.

- Averaging and harmonic balance techniques [Minis, Y. Altintas]
- Semi-discretization [T. Insperger and G. Stepan]
- Time finite element analysis [B. Mann and P. Bayly]
- Heuristic assumptions for period-doubling boundaries [W. Corpus and W. Endres]
- Discrete time model [Davies and T. Burns]
- Analytical stability chart [R. Szalai and G. Stepan]
Mechanical model

Equation of motion:
\[ \ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = g(t) \frac{K_w}{m} \left( h_0 + x(t - \tau) - x(t) \right)^{3/4}, \]
where
\[ g(t) = \begin{cases} 
0, & \text{if } k\tau \leq t < k\tau + \tau_1 \\
1, & \text{if } k\tau + \tau_1 \leq t < (k + 1)\tau, \quad k \in \mathbb{Z}.
\end{cases} \]
Discrete-time model

\[ m\ddot{x}(\tilde{t}) + k\dot{x}(\tilde{t}) + sx(\tilde{t}) = 0, \quad \tilde{t} \in [\tilde{t}_0, \tilde{t}_0 + \tilde{\tau}_1] \]

\[ m\left(\dot{x}(\tilde{t}) - \dot{x}(\tilde{t} - \tilde{\tau}_2)\right) = \tilde{\tau}_2 F_c(h(\tilde{t})), \quad \tilde{t} \in [\tilde{t}_0 + \tilde{\tau}_1, \tilde{t}_0 + \tilde{\tau}], \]

where \( h(\tilde{t}) = h_0 + x(\tilde{t} - \tilde{\tau}) - x(\tilde{t}), \ h_0 = v_0\tilde{\tau}, \)

and \( F_c(h(t)) = Kw h^{3/4}(t) \) is the cutting force.
Mathematical model

Natural eigenfrequency: \( \omega_n = \sqrt{s/m} \)
Relative damping: \( \zeta = k/(2\sqrt{sm}) \)
Dimensionless time: \( t = \omega_n \tilde{t} \)
Dimensionless eigenfrequency: \( \hat{\omega}_d = \sqrt{1 - \zeta^2}. \)
State transition between \( t_j = t_0 + j\tau \) and \( t_{j+1} \) is described by

\[
\begin{pmatrix}
  x_{j+1} \\
  v_{j+1}
\end{pmatrix} = A
\begin{pmatrix}
  x_j \\
  v_j
\end{pmatrix} + \begin{pmatrix}
  0 \\
  \frac{K\omega_\tau^2}{m\omega_n^2} (h_0 + (1 - A_{11})x_j - A_{12}v_j)^{3/4}
\end{pmatrix}.
\]

where \( x_j = x(t_j), \) \( v_j = \dot{x}(t_j) \) and

\[
A = \exp\left(\begin{pmatrix}
  0 & 1 \\
  -1 & \zeta
\end{pmatrix}\right) \tau_1
\]
Stability

The linearized equation around the fixed point

\[
\begin{pmatrix}
 x_{j+1} \\
 v_{j+1}
\end{pmatrix} = \begin{pmatrix}
 A_{11} & A_{12} \\
 A_{21} + \hat{w} (1 - A_{11}) & A_{22} - \hat{w} A_{12}
\end{pmatrix} \begin{pmatrix}
 x_j \\
 v_j
\end{pmatrix}.
\]

Stability boundaries:

\[
\hat{w}_f^{cr} = \frac{\det A + \text{tr}A + 1}{2A_{12}} = \hat{w}_d \frac{\cos(\hat{w}_d \tau) + \cosh(\zeta \tau)}{\sin(\hat{w}_d \tau)},
\]

\[
\hat{w}_{ns}^{cr} = \frac{\det A - 1}{A_{12}} = -2\hat{w}_d \frac{\sinh(\zeta \tau)}{\sin(\hat{w}_d \tau)},
\]

where

\[
\hat{w} = \frac{3}{4h_0^{1/4}} \frac{K \tau_2}{m \omega_n^2} w.
\]
Flip Bifurcation

Consider the following perturbation of the linear system around the fixed point in the basis of the eigenvectors

\[
\begin{pmatrix}
\xi_{n+1} \\
\eta_{n+1}
\end{pmatrix} = \begin{pmatrix}
-1 + a^f \Delta \hat{w} & 0 \\
0 & \lambda_2
\end{pmatrix}
\begin{pmatrix}
\xi_{n} \\
\eta_{n}
\end{pmatrix} + \begin{pmatrix}
\sum_{i+j=2,3} c_{ij} \xi_i^i \eta_j^j \\
\sum_{i+j=2,3} d_{ij} \xi_i^i \eta_j^j
\end{pmatrix},
\]

Using center manifold and normal form reduction we find that there is a period two orbit on the center manifold

\[
\begin{align*}
\hat{\xi}_{1,2} &= \sqrt{-\frac{\Delta \hat{w} a^f}{\delta}}, \\
\delta &= -\frac{5}{12 h_0^2} \frac{\cosh(\hat{\zeta} \tau) + \cos(\hat{\omega}_d \tau)}{\cosh(\hat{\zeta} \tau) + 2 \sinh(\hat{\zeta} \tau) + \cos(\hat{\omega}_d \tau)} < 0.
\end{align*}
\]

Hence, the bifurcation is subcritical!
Neimark-Sacker bifurcation

Similarly, the Taylor expansion of the system in the eigenbasis

\[
\begin{pmatrix}
\xi_{n+1} \\
\eta_{n+1}
\end{pmatrix} = (1 + |a^h| \Delta \hat{w}) \begin{pmatrix}
e^{i\varphi} & 0 \\
0 & e^{-i\varphi}
\end{pmatrix} \begin{pmatrix}
\xi_n \\
\eta_n
\end{pmatrix} + \begin{pmatrix}
\sum_{i+j=2,3} c_{ij} \xi_i \eta_j \\
\sum_{i+j=2,3} d_{ij} \xi_i \eta_j
\end{pmatrix}.
\]

The radius of the invariant circle (in the eigenbasis)

\[
R = \sqrt{-\frac{2|a^h| \Delta \hat{w} + |a^h|^2 \Delta \hat{w}^2}{2(1 + |a^h| \Delta \hat{w})\delta}} \approx \sqrt{-\frac{|a^h| \Delta \hat{w}}{\delta}},
\]

where

\[
\delta = e^{-5\zeta T_1} \frac{(4e^{4\zeta T_1} - 3e^{2\zeta T_1} - 1)(\cosh(\zeta T_1) - \cos(\omega_0 T_1))}{32h_0^2},
\]

This is subcritical, too!
Simulation

![Graph](image-url)
The motion exists if:

\[ \hat{w} > \frac{3\omega_d}{2^{7/4}} \frac{\cos(\hat{\omega}_d \tau) + \cosh(\zeta \tau)}{\sin(\hat{\omega}_d \tau)} \]

It is stable when

\[ \hat{w} < 2^{1/4} \omega_d \frac{\cos(2\omega_d \tau_1) + \cosh(2\zeta \tau_1)}{\sin(2\omega_d \tau_1)} \]

or

\[ \hat{w} < -2^{7/4} \omega_d \frac{\sinh(2\zeta \tau_1)}{\sin(2\omega_d \tau_1)}. \]
Bifurcation diagram 1

- Fixed point
- Stable period-2 point

\[ w^p \]
Smale horseshoe

(a) Switching line

(b) Horseshoe

\[ \frac{x}{h_0}, \frac{v}{h_0} \]

\[ W^s_{P_2}, W^u_{P_2}, W^s_{P_1}, W^u_{P_1} \]

\[ P_1, P_2, V_0, V_1, H_0, H_1 \]

\[ O \]

\[ \text{switching line} \]
The transition matrix of the symbolic dynamics:

\[ T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \implies T \text{ irreducible} \]
Delay differential equation model
Equation of motion:

\[ x(t) \frac{d^2}{dt^2} + 2x(t) \frac{d}{dt} + 4x(t) = g(t) \]

where

\[ g(t) = \begin{cases} 0 & \text{if } k_1 t < k_1 + 1 \\ k_1 & \text{if } k_1 + 1 \leq t < (k_1 + 1) \end{cases} \]
Equation of motion:

\[ \ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = g(t) \frac{K w}{m} (h_0 + x(t - \tau) - x(t))^{3/4}, \]

where

\[ g(t) = \begin{cases} 
0, & \text{if} \quad k\tau \leq t < k\tau + \tau_1 \\
1, & \text{if} \quad k\tau + \tau_1 \leq t < (k + 1)\tau, 
\end{cases} \quad k \in \mathbb{Z}. \]
Variational system

Linearized equation with dimensionless time (\( \hat{t} = \omega_n t \)):
\[
\ddot{x}(\hat{t}) + 2\zeta \dot{x}(\hat{t}) + x(\hat{t}) = g(\hat{t}) \hat{w} (x(\hat{t} - \hat{\tau}) - x(\hat{t})),
\]
where \( \hat{w} = 3K w/(4h_0^{1/4} m \omega_n^2) \) is the dimensionless chip width.
Variational system

Linearized equation with dimensionless time ($\hat{t} = \omega n t$):

$$\ddot{x}(\hat{t}) + 2\zeta \dot{x}(\hat{t}) + x(\hat{t}) = g(\hat{t})\hat{w} \left( x(\hat{t} - \hat{\tau}) - x(\hat{t}) \right),$$

where $\hat{w} = 3Kw/(4h_0^{1/4}m\omega_n^2)$ is the dimensionless chip width.

Rewritten into 1$^{\text{st}}$ order form ($x(\hat{t}) = (x(\hat{t}), \dot{x}(\hat{t}))^T$):

$$\dot{x}(\hat{\tau}) = A(\hat{t})x(\hat{t}) + B(\hat{t})x(\hat{t} - \hat{\tau}),$$

where

$$A(\hat{t}) = \begin{pmatrix} 0 & 1 \\ -1 - g(\hat{t})\hat{w} & -2\zeta \end{pmatrix}, \quad B(\hat{t}) = \begin{pmatrix} 0 & 0 \\ g(\hat{t})\hat{w} & 0 \end{pmatrix}.$$
Stability analysis

$e^{\lambda \tau}$ characteristic multiplier

$x(t) = e^{\lambda t} v(t)$ satisfies the equation such that $v(t) = v(t + \tau)$
Stability analysis

- $e^{\lambda\hat{\tau}}$ characteristic multiplier

\[ \iff \]
\[ x(t) = e^{\lambda t}v(t) \text{ satisfies the equation such that } v(t) = v(t + \hat{\tau}) \]

- The periodic solution is asymptotically stable if for each characteristic multiplier $|e^{\lambda\hat{\tau}}| < 1$. 

\[ \varepsilon \]
\[ x(t) = e^{\lambda t}v(t) \text{ satisfies the equation such that } v(t) = v(t + \hat{\tau}) \]

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\[ \varepsilon \]

\[ \varepsilon \]
Stability analysis

- $e^{\lambda \hat{\tau}}$ characteristic multiplier

\[ x(t) = e^{\lambda t} v(t) \]

satisfies the equation such that $v(t) = v(t + \hat{\tau})$

The periodic solution is asymptotically stable if for each characteristic multiplier $|e^{\lambda \hat{\tau}}| < 1$.

Exploiting the first condition we are left with the BVP

\[
\dot{v}(t) = (A(t) + e^{-\lambda \hat{\tau}} B(t) - \lambda I) v(t) \\
v(0) = v(\hat{\tau}).
\]
The BVP is solvable iff

\[ 0 = f(\mu) := \det(\Phi(\hat{\tau}) - I), \]

where

\[ \Phi(\hat{\tau}) = \mu e^{(A_2 + \mu B_2)\hat{\tau}_2} e^{(A_1 + \mu B_1)\hat{\tau}_1}, \]

\[ \mu = e^{-\lambda \hat{\tau}}, \]

\[ A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]

\[ A_2 = \begin{pmatrix} 0 & 1 \\ -1 - \hat{w} & -2\zeta \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ \hat{w} & 0 \end{pmatrix}. \]
Argument principle

Roots for which $\mu = (e^{\lambda \hat{t}})^{-1} f |\mu| < 1$ cause instability. They can be easily counted

$$N = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi} \int_\gamma \, d\arg f$$

$$= \frac{1}{2\pi} \sum_{j=1}^{n} \arg \frac{f(\exp(j \frac{2\pi}{n} i))}{f(\exp((j-1) \frac{2\pi}{n} i))}$$
Stability chart
Machining with ‘fly-over’ effect

\[ \ddot{x}(t) + 2\zeta \dot{x}(t) + x(t) = g(\varphi)\hat{w} (\cos \varphi + 0.3 \sin \varphi) \times \]
\[ \times [H ((h_0 + x(t - 2\tau) - x(t - \tau))) F_c ((h_0 + x(t - \tau) - x(t)) \sin \varphi) \]
\[ + H ((h_0 + x(t - \tau) - x(t - 2\tau))) F_c ((2h_0 + x(t - 2\tau) - x(t)) \sin \varphi)] \]
Numerical method

Orthogonal collocation

mesh points

representation points

collocation points

The equation is satisfied at $c_{i,j}$

$$\dot{\varphi}(c_{i,j}) = f(c_{i,j}, \varphi(c_{i,j}), \varphi(c_{i,j} - \tau \mod T))$$

[Engelborghs and Doedel, 2001]
Consider

\[ F(X, \lambda) = 0. \]

The tangent comes from

\[ F_X(X_0, \lambda_0)X' + F_\lambda(X_0, \lambda_0)\lambda' = 0. \]

The Newton iteration

\[
\begin{pmatrix}
F_X(X_1^{(\nu)}, \lambda_1^{(\nu)}) & F_\lambda(X_1^{(\nu)}, \lambda_1^{(\nu)}) \\
X_0' & \lambda_0'
\end{pmatrix}
\begin{pmatrix}
\Delta X \\
\Delta \lambda
\end{pmatrix}
=
\begin{pmatrix}
-F(X, \lambda) \\
ds - X_0'(X_1^{(\nu)} - X_0) - \lambda_0'(\lambda_1^{(\nu)} - \lambda_0)
\end{pmatrix}.
\]
One tooth tool with diameter of 19.05mm (3/4”); Workpiece width: 6.35mm
Natural frequency: 146.8Hz; Spindle speed: 3000 - 4000 rpm
Feed $h_0 = 0.1082$mm/period

$\hat{w} = 2.9 \times 10^{-4}w$, where $w$ is the depth of cut ($z$ direction).
Stability chart

\[ \hat{W} \]

\[ z \hat{\Omega} \]
Tool trajectories
Tool trajectories
Subcriticality
Conclusions

- High amplitude periodic, quasi-periodic and chaotic vibrations were found.
- These unwanted vibrations can occur at linearly stable parameters.
- This parameter region can be large due to the fold of period-2 orbits.
- Side-product: PDDE-CONT a continuation software for periodic and autonomous DDEs
Thank you for your attention!