Approximating Limit Cycles of an Autonomous Delay Differential Equation

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OUTLINE

- Introduction
- Main Existence Theorem
- Principal Parameter Estimation Algorithms
- Application to a Van der Pol Equation with Delay
- Final Observations

INTRODUCTION

- Measurement and control of machine tool errors has led to delay differential equation models. (Hanna, Tobias [4])

- Machine tool chatter has been established as a Hopf bifurcation of limit cycles from stable machining. (Gilsinn [1], Nayfeh, et al. [5])

- Chatter is self sustained limit cycles caused by the cutting tool interacting with undulations from a previous cut.

- Wish to approximate limit cycles with an analytic form and develop a computable error bound.

SOME BASIC FACTS

- **An autonomous delay differential equation (DDE) with a fixed delay will be written**

$$\dot{x}(t) = X(x(t), x(t-h)) \qquad (1)$$

with initial condition ϕ **from the space of continuous functions on [-*h*,0] .**

- **If** $X(x,y)$ **satisfies a Lipschitz condition with respect to *x* , independent of *y* , there exists a unique solution of (1) for** ϕ **on [-*h*,0] .**
- **A linear DDE with fixed delay takes the form**

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) \qquad (2)$$

with initial condition ϕ **on [-*h*,0] ,** $x \in R^n$ **a column vector. (Forward Int.)**

- **The formal adjoint equation to (2) takes the form**

$$\dot{y}(t) = -y(t)A(t) - y(t+h)B(t+h) \qquad (3)$$

for ψ **on** $[t_0, t_0 + h]$ **, some initial interval,** $\psi \in R^n$ **, a row vector. (Backward Int.)**

- **A solution** $Z(t,s)$ **of the linear DDE (2) is called a <u>fundamental solution</u> if** $Z(s,s) = I,\ Z(t,s) = 0,\ t < s.$

<u>OBJECTIVE</u>

- **Find periodic solution of autonomous delay differential equation (DDE)**

$$\dot{x}(t) = X\left(x(t), x(t-h)\right) \qquad (4)$$

with initial condition ϕ **from the space of continuous functions on** $[-h, 0]$

- **Period** $T = 2\pi / \omega$ **is also unknown.**
- **Introduce** t/ω **for *t* to get**

$$\omega\dot{x}(t) = X\left(x(t), x(t-\omega h)\right) \qquad (5)$$

- **Look for periodic solutions of fixed period** 2π
- **<u>Desired Result</u>: Given an approximate** 2π**-periodic solution and frequency,** $(\overline{\omega}, \overline{x}(t))$**, of (2), wish to show that if they satisfy a certain <u>noncriticality condition</u> then there exists an exact frequency and** 2π**-periodic solution,** $(\omega^*, x^*(t))$**, in a <u>computable neighborhood</u> of** $(\overline{\omega}, \overline{x}(t))$

NOTATION

$$x_\omega = (x(t), x(t-\omega h))$$
$$dX(x_\omega;\phi) = X_1(x_\omega)\phi_1 + X_2(x_\omega)\phi_2, \qquad \phi = (\phi_1, \phi_2)$$

- **If** $(\overline{\omega}, \overline{x})$ **are an approximate frequency and** 2π **- periodic solution then**

$$\overline{\omega}\dot{\overline{x}}(t) = X(\overline{x}_{\overline{\omega}}) + k(t), \quad k(t+2\pi) = k(t) \quad (6)$$

- **The variational equation with respect to the approximate solution is**

$$\overline{\omega}\dot{z}(t) = dX(\overline{x}_{\overline{\omega}}; z_{\overline{\omega}}) \qquad (7)$$

Let *A(t)* = $X_1(\overline{x}_{\overline{\omega}})$ **, *B(t)* =** $X_2(\overline{x}_{\overline{\omega}})$

- ρ **is a <u>characteristic multiplier</u> of the linear system (2) if there exists a non-trivial solution of (2) such that**

$$x(t+2\pi) = \rho x(t) \qquad (8)$$

NONCRITCAL APPROXIMATE SOLUTION

- **The pair $(\overline{\omega}, \overline{x})$ is *noncritical* with respect to $\omega \dot{x} = X(x_\omega)$ if the variational equation about the approximate solution (7) has a characteristic multiplier ρ of multiplicity one with the remaining multipliers unequal to one. If $v(t),\ |v|_2 = 1$ is the 2π periodic solution of the adjoint corresponding to ρ then**

$$\int_0^{2\pi} v(t) J(\dot{\overline{x}}, \overline{\omega})(t)\,dt \neq 0 \qquad (13)$$

where

$$J(\dot{\overline{x}}, \overline{\omega})(t) = \dot{\overline{x}}_t + \overline{\omega} h X_2(\overline{x}_{\overline{\omega}})\dot{\overline{x}}_{t,\overline{\omega}} \qquad (14)$$

(Hale [3], Stokes [7])

IMPORTANT LEMMAS

LEMMA (Halanay [2]): When the linear DDE coefficients A(t), B(t) are periodic the linear and adjoint systems have the same finite number of independent solutions.

LEMMA (Hale [3], Halanay [2]): If ρ is a simple characteristic multiplier of the linear DDE (2), p(t) a nontrivial 2π -periodic solution of the linear DDE (2), q(t) a nontrivial 2π - periodic solution of the adjoint (3) and

$$J(p,\omega)(t) = p(t) + \omega h B(t) p(t-\omega h) \qquad (9)$$

then

$$\int_0^{2\pi} q(t) J(p,\omega)(t) dt \neq 0 \qquad (10)$$

LEMMA (Halanay [2]): If $(\bar{\varpi},\bar{x})$ is noncritical, f, 2π-periodic, such that

$$\int_0^{2\pi} v\ (t) f(t) dt = 0 \qquad (11)$$

then there exists a unique 2π -periodic solution of

$$\bar{\varpi}\dot{z}(t) = dX(\bar{x}_{\bar{\varpi}}; z_{\bar{\varpi}}) + f(t) \qquad (12)$$

which satisfies $|z| \le M\,|f|_2$ **for some** $M > 0$**, independent of** f

FUNDAMENTAL THEOREM

THEOREM (Stokes [7]): Let $(\overline{\omega}, \overline{x})$ **satisfy**

$$\overline{\omega}\dot{\overline{x}}(t) = X(\overline{x}_{\overline{\omega}}) + k(t) \qquad (15)$$

and let $|k| \le r$ **. Suppose there exist** K_1 **and** K **such that for** ψ_1, ψ_2

$$\left|dX(\overline{x}_{\overline{\omega}};\phi_{\overline{\omega}})\right| \le K_1|\phi| \qquad (16)$$

$$\left|dX(\overline{x}_{\overline{\omega}}+\psi_1;\phi_{\overline{\omega}}) - dX(\overline{x}_{\overline{\omega}}+\psi_2;\phi_{\overline{\omega}})\right| \le K|\psi_1-\psi_2||\phi|$$

Assume $(\overline{\omega}, \overline{x})$ **is noncritical (in the delay sense) and let** v **be the appropriate solution of the adjoint to the variational equation,** $|v| = 1$**.**
Let

$$\alpha = \left[\frac{1}{2\pi}\left|\int_0^{2\pi} v\;(t) J(\dot{\overline{x}}_{\overline{\omega}}, \overline{\omega})(t)dt\right|\right]^{-1} \qquad (17)$$

If *M* is the constant from the previous lemma, let

$$\lambda_1 = M(1+\alpha\left|J(\dot{\overline{x}}, \overline{\omega})\right|) \qquad (18)$$

$$\lambda_2 = (1+MK_1)\left(\frac{\lambda_1}{M}\right)$$

Finally, if there is a function C of computable parameters such that

$$rC(K, K_1, h, \alpha, \lambda_1, \lambda_2, \dot{\overline{x}}, \ddot{\overline{x}}, \overline{\omega}) < 1 \qquad (19)$$

then there exists an exact 2π**-periodic solution** x^* **and an exact frequency** ω^* **so that**

$$\left|x^* - \overline{x}\right| \le 4\lambda_1 r$$

$$\left|\omega^* - \overline{\omega}\right| \le 2\alpha r \qquad (20)$$

PROOF (OUTLINE)

Goal: Find $\beta,\ z(t)\ 2\pi$ **-periodic so that**

$$\omega = \overline{\omega} + \beta,\ \ x(t) = \overline{x}(t) + \frac{\overline{\omega}}{\overline{\omega}+\beta} z(t) \qquad (21)$$

is and exact solution of

$$\omega \dot{x}(t) = X\,(x(t), x(t-\omega h)) \qquad (22)$$

Substituting (21) into (22)

$$\overline{\omega}\dot{z}(t) = dX\,(\overline{x}_{\overline{\omega}}; z_{\overline{\omega}}) + R(z,\beta) - \beta J\,(\dot{\overline{x}}, \overline{\omega})(t) - k(t) \qquad (23)$$

where

$$\overline{\omega}\dot{\overline{x}} = X\,(\overline{x}_{\overline{\omega}}) + k(t) \qquad (24)$$

and $R(z,\beta)$ **is a function of computable parameters and** (z,β).

Strategy: Wish to find a fixed point of a map $g = R(z(g), \beta(g))$ **such that the perturbation term on the right of (23) is orthogonal to the solution of the adjoint in the noncriticality definition. The Lemma can then be used To solve for a** $z(t)$

PROOF (CONTINUED)

Define the sets $P = \left\{ g \mid g \in C^0, g(t+2\pi) = g(t) \right\}$

$$P_1 \subset P \cap C^1$$

$$N = \left\{ g \in P \mid \left| g \right| \leq \delta \right\}$$

Construct map S **as a composition** $S = L(T)$

<u>Define</u>: $L : N \rightarrow R \times P_1$

Given $g \in N$ **solve for unique** $\beta > 0$ **so that**

$$g - \beta J(\dot{\bar{x}}, \bar{\omega})(t) - k(t) \perp \nu(t)$$

$\nu(t)$ **solution of adjoint for** $(\bar{\omega}, \bar{x})$ **noncritical.**

By Lemma there exists a unique $z \in P_1$ **such that**

$$\bar{\omega}\dot{z}(t) = dX(\bar{x}_{\bar{\omega}}; z_{\bar{\omega}}) + g(t) - \beta J(\dot{\bar{x}}, \bar{\omega})(t) - k(t)$$

$$\left| z \right| \leq M \left| g - \beta J(\dot{\bar{x}}, \bar{\omega}) - k \right|_2$$

Now define $L(g) = (\beta(g), z(g))$

<u>Define</u>: $T : R \times P_1 \rightarrow N$

Given $\beta > 0,\ (\beta, z) \in R \times P_1,$

$$T(\beta, z) = R(z, \beta)$$

<u>Define</u>: $S : N \rightarrow N, \qquad S(g) = R(z(g), \beta(g))$

PROOF (CONTINUED)

For $\delta = r$ **in the definition of *N* there exists a bounded function**

$$C(K, K_1, h, \alpha, \lambda_1, \lambda_2, \dot{\overline{x}}, \overline{\omega})$$

Such that $rC < 1$ **, which implies** $S : N \to N$ **is a contraction.**
Therefore there exists a fixed point $g^* = R(z(g^*), \beta(g^*))$
The exact solution is then given by

$$\omega^* = \overline{\omega} + \beta(g^*)$$

$$x^*(t) = \overline{x}(t) + \frac{\overline{\omega}}{\overline{\omega} + \beta(g^*)} z(g^*)(t)$$

Finally, we can show that

$$\left| x^* - \overline{x} \right| \le 4\lambda_1 r$$

$$\left| \omega^* - \overline{\omega} \right| \le 2\alpha r$$

<u>NOTE</u>: This provides only *O(r)* estimates. These may not be optimal bounds but they are computable.

APPLICATION STEPS

- **Compute the approximation pair** $(\overline{\omega}, \overline{x})$

- **Verify that the pair is noncritical**

- **Compute M and** α

A quote from Stokes [7]

“The computational difficulties here are considerably greater than in the case of ordinary differential equations,…, but they are not insurmountable”

He never produced an example. This talk describes the first application.

APPROXIMATE SOLUTION

- **Develop solution as a trigonometric polynomial**

$$\overline{x}(t) = a_1 + a_2 \cos(t) + \sum_{k=2}^{n} \left(a_{2k-1} \cos(kt) + a_{2k} \sin(kt) \right)$$

- **Set coefficient of one term to zero, say *sin(t),* in order to estimate $\overline{\omega}$**

- **Can develop Galerkin projection equations using, e.g. MAPLE, although the expansions are nontrivial (typical over 135 terms). Not recommended in general.**

- **Summer student, Chris Copeland, and I have developed a a fast projection procedure in MATLAB based on some FFT ideas. Subject for another talk.**

VARIATION OF CONSTANTS FORMULAS

- **From now on we revert to classic notation where we set**

$$A(t) = X_1(\overline{x}_{\varpi}), \qquad B(t) = X_2(\overline{x}_{\varpi})$$

- $A(t)$ **and** $B(t)$ **are** 2π**-periodic.**
- **The variation of constants formula for the linear system**

$$\dot{x}(t) = A(t)x(t) + B(t)X(t - \varpi)$$

is

$$z(t) = Z(t,0)\phi(0) + \int_{-\varpi}^{0} Z(t, \alpha + \varpi) B(\alpha + \varpi)\phi(\alpha) d\alpha$$

($Z(t,0)$ **solution of linear system with** $Z(0,0) = I, Z(t,0) = 0, t < 0$ **) ,**$\phi \in C^o[-\varpi, 0]$

- **h has been normalized to 1.**
- **The variation of constants formula for the adjoint**

$$\dot{y}(t) = -y(t)A(t) - y(t + \varpi)B(t + \varpi)$$

is

$$y(t) = \psi(2\pi)Z(2\pi, t) + \int_{2\pi}^{2\pi+\varpi} \psi(s)B(s)Z(s - \varpi, t)ds$$

$$\psi \in [2\pi, 2\pi + \varpi]$$

- **These formulas are developed in Halanay [2]. Significance of the adjoint formula is that it only requires a forward integration.**

TESTING NONCRITICALITY CONDITION

- **Test characteristic multiplier of multiplicity one and compute** α

- ρ **is a characteristic multiplier of the linear system if there is a solution** $z(t)$ **such that** $z(t+2\pi) = \rho z(t)$

- **Halanay [2] shows that the eigenvalues of the following operator are the multipliers of the variational equation for the linear system**

$$(U\phi)(s) = Z(s+2\pi,0)\phi(0) + \int_{-\varpi}^{0} Z(s+2\pi,\alpha+\varpi)B(\alpha+\varpi)\phi(\alpha)d\alpha, \quad s \in [-\sigma,0]$$

- **Called the monodromy operator, defined formally by**

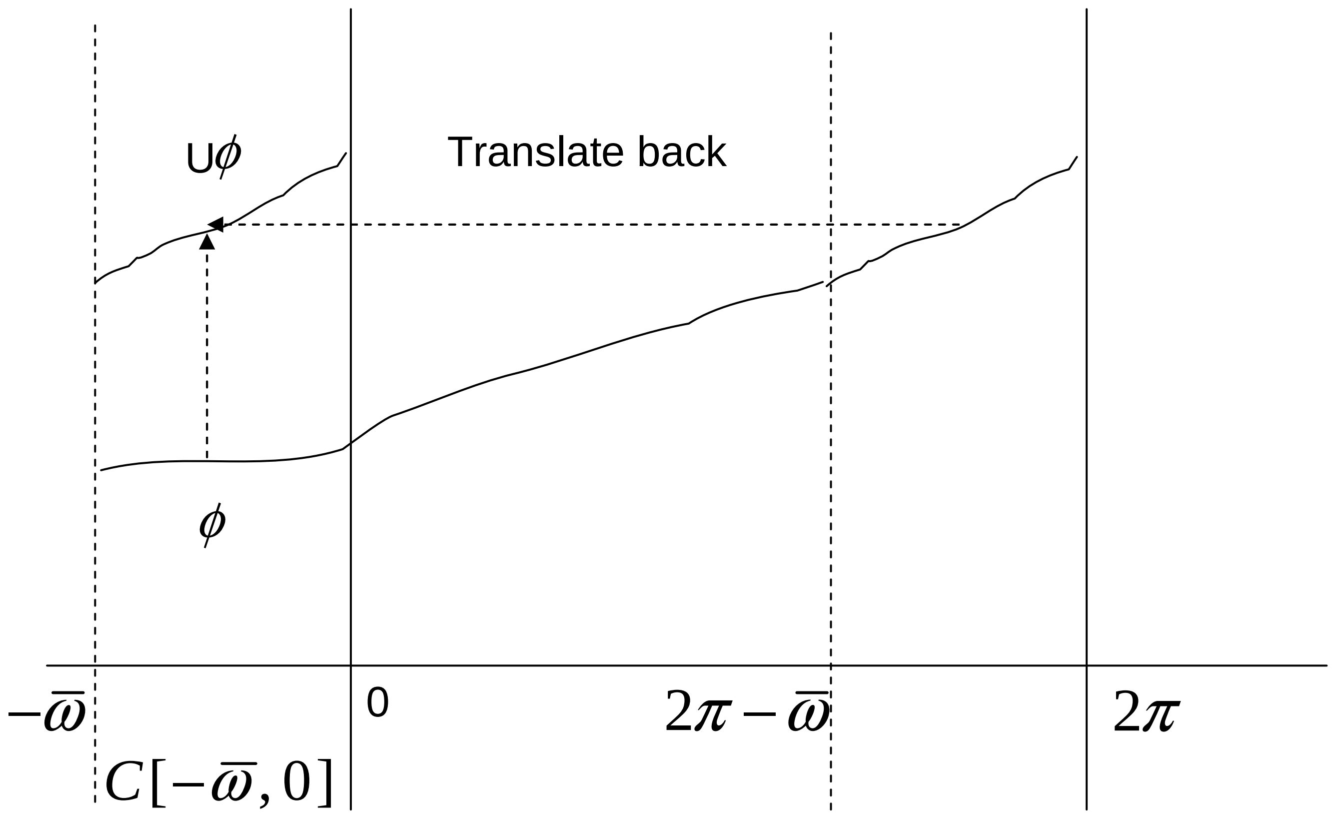
$$(U\phi)(t) = x(t+2\pi;\phi)$$

where $\phi \in C^0[-\sigma,0].$

- ***U* is compact with at most a countable number of eigenvalues with 0 the only possible limit point.**

- **In the present case** $-\sigma$ **will be** $-\varpi$

MONODROMY OPERATOR MAP ON INITIAL SPACE



Trajectory Plot Interpreted as a Mapping on Initial Space

DISCRETIZATION OF U

- **The eigenvalues of** $U\phi = \rho\phi$ **are approximated by the eigenvalues of** $U_N \bar{\phi} = \bar{\rho}\bar{\phi}$ **where** U_N **is constructed as follows:**
- **Discretize** $[-\varpi, 0]$ **with equal intervals**

$$-\varpi = s_1 < s_2 < \cdots < s_{N+1} = 0$$

where $h = s_{i+1} - s_i = \varpi / N.$

- **Approximate the integral operator by the trapezoidal rule (other rules could be used) by setting**

$$w_1 = w_{N+1} = \frac{h}{2}, \;\; w_i = h, \;\; i = 2, \cdots, N$$

$$(U_N\phi)(t) = Z(t+2\pi, 0)\phi(0) + \sum_{j=1}^{N+1} w_j Z(t+2\pi, s_j + \varpi) B(s_j + \varpi)\phi(s_j)$$

where $Z(t,0)$ **satisfies the linear variational system about the approximate periodic solution,** $Z(0,0) = I, \; Z(t,0) = 0, \; t < 0.$

Block Matrix for U_N

- The matrix for U_N becomes

$$\begin{bmatrix} w_1 Z(s_1+2\pi, s_1+\varpi)B(s_1+\varpi) & \cdots & w_j Z(s_1+2\pi, s_j+\varpi)B(s_j+\varpi) & \cdots & Z(s_1+2\pi, S_{N+1}) + w_j Z(s_1+2\pi, s_{N+1}+\varpi)B(s_{N+1}+\varpi) \\ \vdots & \vdots & \vdots & \vdots & \\ w_1 Z(s_i+2\pi, s_1+\varpi)B(s_1+\varpi) & \cdots & w_j Z(s_i+2\pi, s_j+\varpi)B(s_j+\varpi) & \cdots & Z(s_i+2\pi, S_{N+1}) + w_j Z(s_i+2\pi, s_{N+1}+\varpi)B(s_{N+1}+\varpi) \\ \vdots & \vdots & \vdots & \vdots & \\ w_1 Z(s_{N+1}+2\pi, s_1+\varpi)B(s_1+\varpi) & \cdots & w_j Z(s_{N+1}+2\pi, s_j+\varpi)B(s_j+\varpi) & \cdots & Z(s_{N+1}+2\pi, S_{N+1}) + w_j Z(s_{N+1}+2\pi, s_{N+1}+\varpi)B(s_{N+1}+\varpi) \end{bmatrix}$$

- **It is not necessary to compute $Z(s_i+2\pi, s_j+\varpi)$ for all $i, j = 1, \cdots, N+1$. For large *N,* say 1000 or more, this would be somewhat impractical computationally.**
- **Note that $s_{N+1}+2\pi = 2\pi$ and $0 < s_i + 2\pi < 2\pi$ for $i = 1, \cdots, N.$**
- **For column *j* compute $Z(2\pi, s_j+\varpi)$ but save intermediate integration points as is done in dde23 in Matlab.**
- **Interpolate the values of $Z(s_i+2\pi, s_j+\varpi)$ up the column for $i = 1, \cdots, N.$**
- **This also applies to $Z(s_i+2\pi, 0).$**
- **This reduces the block integrations of *Z* to *N*+1 instead of (*N*+1)^2.**

Computing the α parameter - 1

- **Need to solve adjoint equation for** $t \in [0, 2\pi]$, ψ **on** $[2\pi, 2\pi + \varpi]$

$$\dot{y}(t) = -y(t)A - y(t+\varpi)B(t+\varpi)$$

- **Halanay [2] showed that the solution** $y(t)$ **satisfies**

$$y(t) = \psi(2\pi)Z(2\pi, t) + \int_{2\pi}^{2\pi+\varpi} \psi(s)B(s)Z(s-\varpi, t)ds$$

- **Significance: Requires only forward integration to compute** $Z(s,t)$
- **To find: Solution of the adjoint associated with multiplier of variational eq.**
- **For** $\tilde{\phi}$ **on** $[-\varpi, 0]$**, a row vector, define**

$$\left(\tilde{U}\tilde{\phi}\right)(s) = \tilde{\phi}(-\varpi)Z(2\pi, s+\varpi) + \int_{-\varpi}^{0} \tilde{\phi}(\eta)B(\eta+\varpi)Z(2\pi+\eta, s+\varpi)d\eta$$

- **Halanay [2] defines an associated operator for** $s \in [2\pi, 2\pi + \varpi]$

$$\left(\tilde{V}\psi\right)(s) = y(s-2\pi;\psi) = \psi(2\pi)Z(2\pi, s-2\pi) + \int_{2\pi}^{2\pi+\varpi} \psi(\eta)B(\eta)Z(\eta-\varpi, s-2\pi)d\eta$$

and showed that an eigenvalue ρ_0 **of** $\tilde{V}$ **is associated with a** $1/\rho_0$ **multiplier of the adjoint.**

Computing the α parameter - 2

- **Halanay [2] showed that the eigenvalues of** $U, \tilde{U}, \tilde{V}$ **are all the same and that the eigenvectors of** $\tilde{U}, \tilde{V}$ **are related by**

$$\tilde{\phi}(s) = \psi(s + 2\pi + \varpi), \qquad s \in [-\varpi, 0]$$

- **To solve for the solution of the adjoint in row form on** $[0, 2\pi]$ **we need only compute the significant eigenvalue and eigenvector of** $\tilde{U}$

- **Discretizing** $-\varpi = s_1 < \cdots < s_{N+1} = 0, \quad \Delta = \varpi / N$ **the j-th column is**

$$(\tilde{U}\tilde{\phi})(s_j) = [\tilde{\phi}(s_1), \cdots, \tilde{\phi}(s_i), \cdots, \tilde{\phi}(s_{N+1})] \begin{bmatrix} Z(2\pi, s_j + \varpi) + w_j \Delta B(s_1 + \varpi) Z(s_1 + 2\pi, s_j + \varpi) \\ \vdots \\ w_j \Delta B(s_i + \varpi) Z(s_i + 2\pi, s_j + \varpi) \\ \vdots \\ w_j \Delta B(s_{N+1} + \varpi) Z(s_{N+1} + 2\pi, s_j + \varpi) \end{bmatrix}$$

- **Then for**

$$0 = t_1 < \cdots < t_{P+1} = 2\pi, \qquad \Delta^* = 2\pi / P$$

$$y(t_j) = [\tilde{\phi}(s_1), \cdots, \tilde{\phi}(s_i), \cdots, \tilde{\phi}(s_{N+1})] \begin{bmatrix} Z(2\pi, t_j) + w_j \Delta^* B(s_1 + 2\pi + \varpi) Z(s_1 + 2\pi, t_j) \\ \vdots \\ w_j \Delta^* B(s_i + 2\pi + \varpi) Z(s_i + 2\pi, t_j) \\ \vdots \\ w_j \Delta^* B(s_{N+1} + 2\pi + \varpi) Z(s_{N+1} + 2\pi, t_j) \end{bmatrix}$$

This is the discretized form of the variation of constants formula for the adjoint equation

Estimating M such that $|z| \le M\,|f|_2$

- **Solution of nonhomogeneous system**, $t \in [0, 2\pi]$

$$\dot{z}(t) = Az(t) + B(t)z(t-\varpi) + f(t)$$

is

$$z(t) = Z(t,0)\phi(0) + \int_{-\varpi}^{0} Z(t, s+\varpi)B(s+\varpi)\phi(s)ds + \int_{0}^{t} Z(t,s)\,f(s)ds \qquad (25)$$

- 2π**-periodic initial condition for** $s \in [-\varpi, 0]$

$$\phi(s) = Z(s+2\pi,0)\phi(0) + \int_{-\varpi}^{0} Z(s+2\pi, \alpha+\varpi)\phi(\alpha)d\alpha + \int_{0}^{s+2\pi} Z(s+2\pi,\alpha)\,f(\alpha)d\alpha \qquad (26)$$

- **Solve for** $\phi(0)$ **as**

$$\phi(0) = \int_{-\varpi}^{0} (I - Z(2\pi,0))^{+} Z(2\pi, \alpha+\varpi)B(\alpha+\varpi)\phi(\alpha)d\alpha + \int_{0}^{2\pi} (I - Z(2\pi,0))^{+} Z(2\pi,\alpha)\,f(\alpha)d\alpha \qquad (27)$$

- **Insert (27) into (26), use sup norm and Schwarz inequality to get**

$$|\phi| \le m_1|\phi| + m_2\,|f|_2, \qquad |\phi| \le m_2(1-m_1)^{-1}\,|f|_2$$

- **Insert (27) into (25) to get**

$$z(t) = \int_{-\varpi}^{0} [Z(t,0)(I - Z(2\pi,0))^{+} Z(2\pi,\alpha+\varpi) + Z(t,\alpha+\varpi)]B(\alpha+\varpi)\phi(\alpha)d\alpha + \int_{0}^{2\pi} [Z(t,0)(I - Z(2\pi,0))^{+} Z(2\pi,\alpha) + Z(t,\alpha)]f(\alpha)d\alpha$$

- **Again use sup norm and Schwarz inequality to get**

$$|z| \le m_3|\phi| + m_4\,|f|_2, \qquad |z| \le [m_2 m_3(1-m_1)^{-1} + m_4]\,|f|_2 = M\,|f|_2$$

Van der Pol Equation with Unit Delay

$$\ddot{x} + \lambda(x(t-1)^2 - 1)\dot{x}(t-1) + x = 0$$

- **Introduce unknown frequency by substituting** t/ω **for** t **to get**

$$\omega^2\ddot{x} + \omega\lambda(x(t-\omega)^2 - 1)\dot{x}(t-\omega) + x = 0$$

- **In vector form with** $x_1(t) = x(t), x_2(t) = \dot{x}(t)$**, initial condition on** $[-\bar{\omega}, 0]$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1/\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda/\omega \end{pmatrix}\begin{pmatrix} x_1(t-\omega) \\ x_2(t-\omega) \end{pmatrix} + \begin{pmatrix} 0 \\ -(\lambda/\omega)\, x_1(t-\omega)^2 x_2(t-\omega) \end{pmatrix}$$

- **The variational equation can be written as**

$$\dot{z}(t) = Az(t) + B(t)Z(t-\bar{\omega})$$

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1/\bar{\omega}^2 & 0 \end{pmatrix},\ B(t) = \begin{pmatrix} 0 & 0 \\ -2(\lambda/\bar{\omega})\bar{x}_1(t-\bar{\omega})\bar{x}_2(t-\bar{\omega}) & (\lambda/\bar{\omega})\left(1-\bar{x}_1(t-\bar{\omega})^2\right) \end{pmatrix},\ B(t+2\pi) = B(t)$$

- **For this example we take** $\lambda = 0.1$

Developing an Approximate Solution

- **Selected a 7 harmonic expansion and** $\lambda = 0.1$ **in the delay equation**

$$\bar{x}(t) = a_1 + a_2 \cos(t) + \sum_{k=2}^{7} \left(a_{2k-1} \cos(kt) + a_{2k} \sin(kt) \right)$$

- **The sin*(t)* term was dropped in order to estimate** $\bar{\omega}$**.**
- **15 Galerkin projection equations developed using MAPLE.**
- **Solving the projection equations produced** $a_1 = a_3 = a_4 = a_7 = a_8 = a_{11} = a_{12} = 0$ **and**

$$a_2 = 2.0185,\ a_5 = 2.5771e-3,\ a_6 = 2.5655e-2,\ a_9 = 1.0667e-4$$
$$a_{10} = -5.2531e-4,\ a_{13} = -7.1791e-6,\ a_{14} = -2.2042e-6$$
$$\bar{\omega} = 1.0012$$

- **Using these and MAPLE to produce an expansion of the Van der Pol equation and then taking the sup norm gave residual *r* = 6.0867e-6**

Estimating the Bound K_1 and Lipschitz constant K

- **Use the fact that for matrix product** Bx **with** $|x| = \max_{1\le i\le n} |x_i|$ **then** $|B| = \max_{1\le i\le n} \sum_{j=1}^{n} |b_{ij}|$ **to show that**

$$|dX(\overline{x};\phi)| \le \left| \begin{pmatrix} 0 & 1 \\ -(1/\overline{\omega}^2) - (2\lambda/\overline{\omega})\overline{x}_1(t-\overline{\omega})\overline{x}_2(t-\overline{\omega}) & (\lambda/\overline{\omega})\left(1-\overline{x}_1(t-\overline{\omega})^2\right) \end{pmatrix} \right| |\phi|$$

$$\le \max\left\{1,\ \ 1/\overline{\omega}^2 + (2\lambda/\overline{\omega})\left(\sum_{i=1}^{14}|a_i|\right)\left(|a_2| + \sum_{i=2}^{7} i\left(|a_{2i-1}|+|a_{2i}|\right)\right) + (\lambda/\overline{\omega})\left(1+\left(\sum_{i=1}^{14}|a_i|\right)^2\right)\right\}|\phi|$$

If $\lambda = 0.1,$ **then** $K_1 = 2.3776.$

- **Also, working within the domain** $D = \left\{x \in C[0,2\pi] : |\overline{x} - x| \le 1\right\}$

$$|dX(\overline{x}_{\overline{\omega}} + \psi_1;\phi_{\tilde{\omega}}) - dX(\overline{x}_{\overline{\omega}} + \psi_2;\phi_{\tilde{\omega}})|$$

$$\le \left\{(2\lambda/\overline{\omega})\left|[\overline{x}_1(t-\overline{\omega})+\psi_{11}(t)][\overline{x}_2(t-\overline{\omega})+\psi_{21}(t)] - [\overline{x}_1(t-\overline{\omega})+\psi_{12}(t)][\overline{x}_2(t-\overline{\omega})+\psi_{22}(t)]\right|\right.$$

$$\left. +(\lambda/\overline{\omega})\left|[\overline{x}_1(t-\overline{\omega})+\psi_{11}(t)]^2 - [\overline{x}_1(t-\overline{\omega})+\psi_{12}(t)]^2\right|\right\}|\phi|$$

$$\le (6\lambda/\overline{\omega})\left(1+|\overline{x}|\right)|\psi_1-\psi_2||\phi|$$

- On $[0,2\pi],\ \ |\overline{x}| \le 2.0225$
  If $\lambda = 0.1,$ **then** $K = 1.8113.$

Van der Pol Equation with Unit Delay

- **Using the characteristic multiplier algorithm and integrating *Z(t,0)*, etc. using dde23 in MATLAB get** $\rho = 0.99806$

- ***M* is estimated as 1.7411**

- α **is estimated as 8.0665**

- **Other estimates**

$$\left|\bar{x}\right| = 2.0225$$

$$\left|\bar{\dot{x}}\right| = 2.0258$$

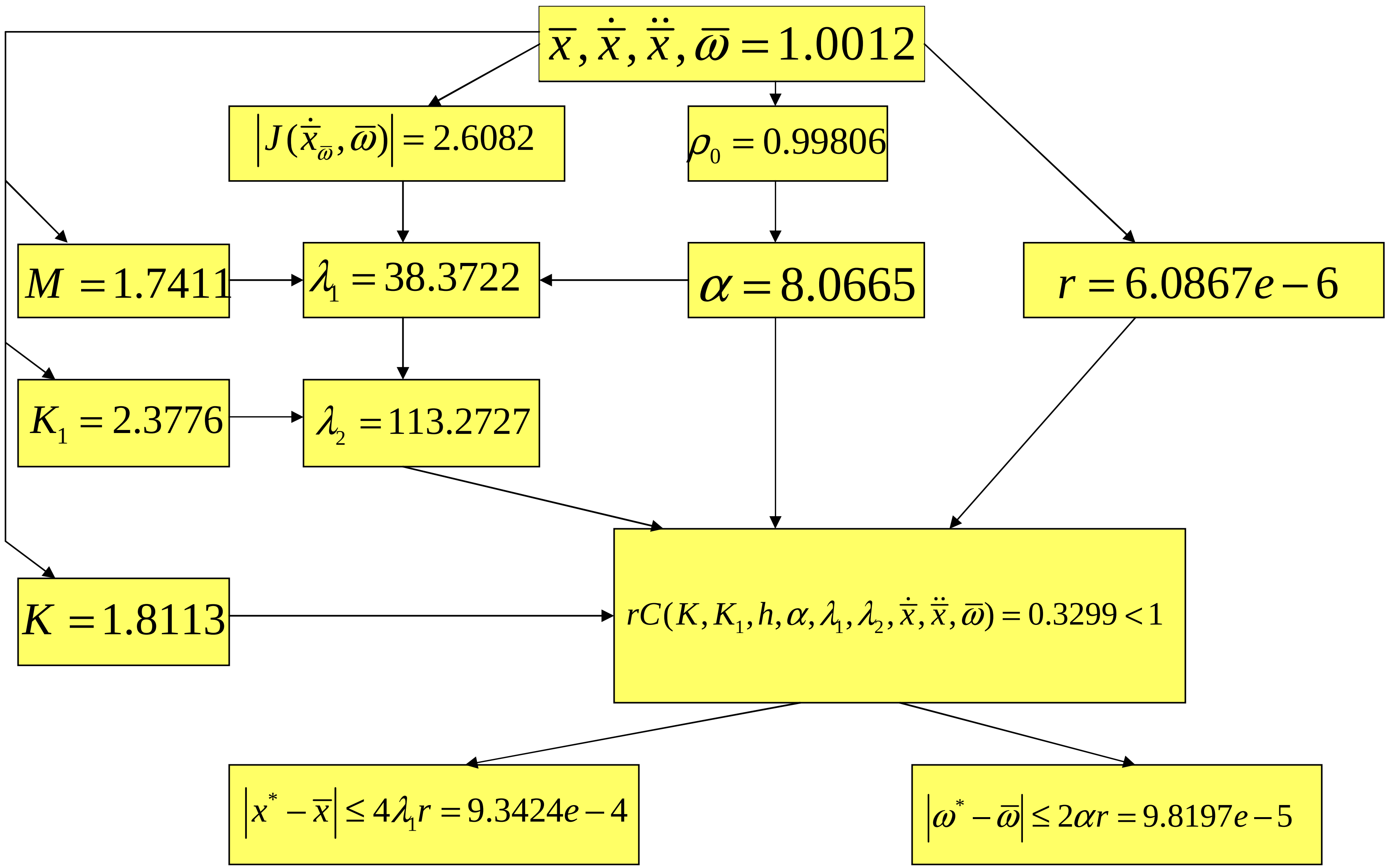
$$\left|\bar{\ddot{x}}\right| = 2.1192$$

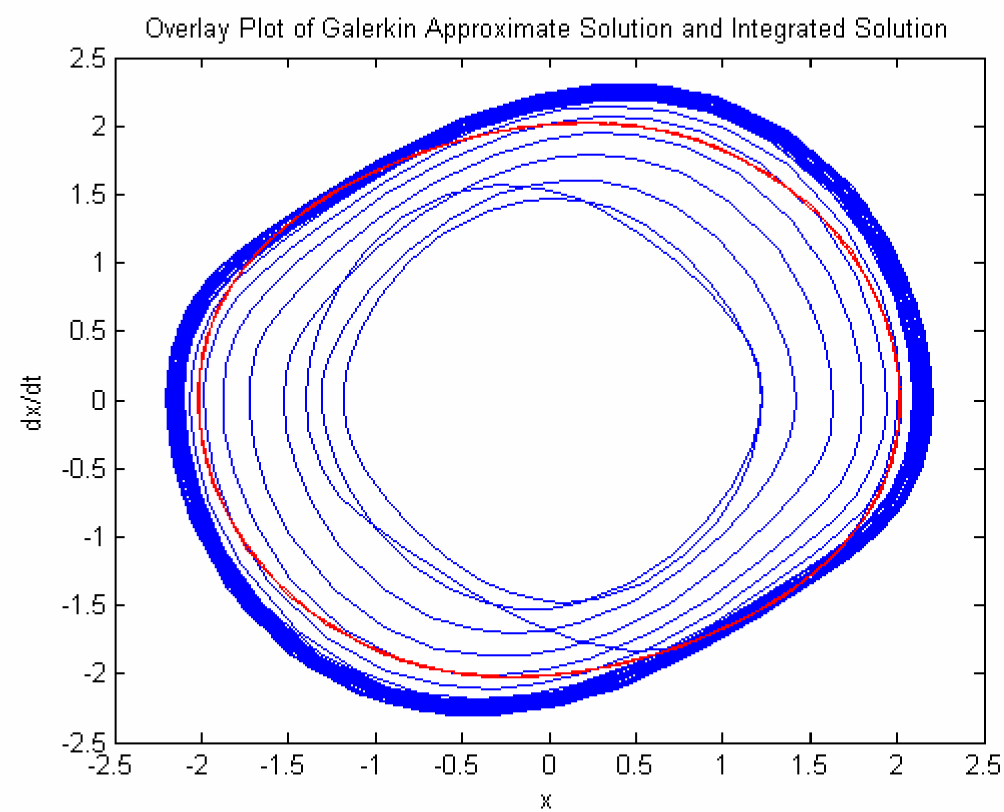
$$\left|J(\dot{x}, \bar{\omega}\right| = 2.6082$$

$$\lambda_1 = 38.3722$$

$$\lambda_2 = 113.2727$$

Calculation Data Flow




Overlay Plot of Galerkin Approximate Solution and Integrated Solution
dx/dt
x
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

Final Observations

- **Galerkin projection symbolic calculations are very lengthy.**

- **Lack of general form for** $Z(s,t)$ **required numerical integration by dde23**.

- **Spline interpolation reduced the number of times** $Z(s,t)$ **had to be computed.**

- **Integration rule possibly led to fine discretization of intervals.**

- **No reasonably computable Green's Function required numerical estimation of** M **.**

- **Numerical Procedures led to interesting algorithmic results as also demonstrated by Urabe and Reiter [9].**

- **Further study of convergence questions needed.**

- **Recent studies show that collocation might be more efficient for solving for characteristic multipliers.**

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