Interval Based Finite Elements for Uncertainty Quantification in Engineering Mechanics

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Outline

- Introduction
- Interval Arithmetic
- Interval Finite Elements
- Overestimation in IFEM
- New Formulation
- Examples
- Conclusions

Introduction- Uncertainty

- Uncertainty is unavoidable in engineering system
  - Structural mechanics entails uncertainties in material, geometry and load parameters (aleatory-epistemic)
- Probabilistic approach is the traditional approach
  - Requires sufficient information to validate the probabilistic model
  - What if data is insufficient to justify a distribution?

Introduction- Uncertainty

Information

Available Information

Sufficient

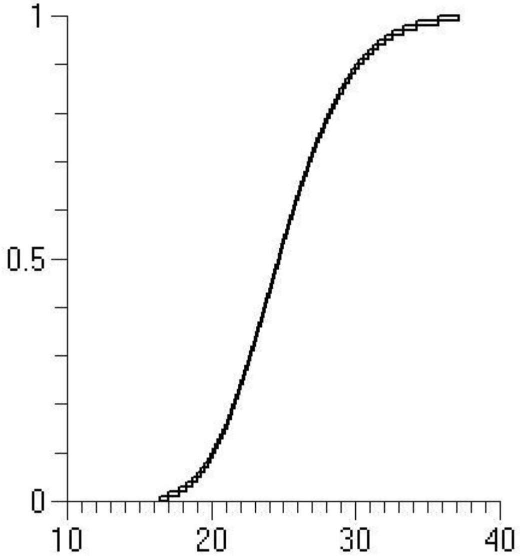
Probability

Incomplete

Probability Bounds

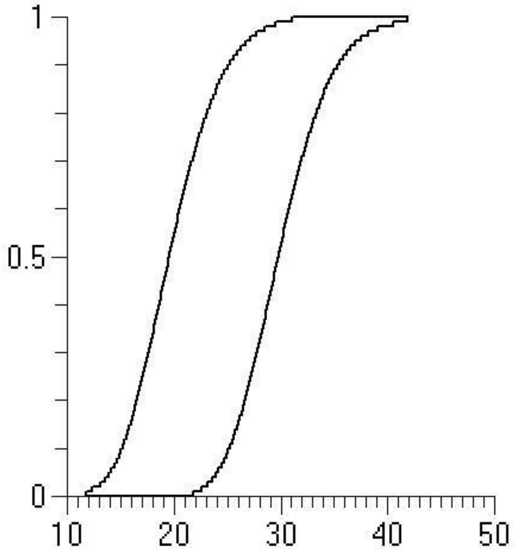
Introduction- Uncertainty

Lognormal



Probability

Lognormal with interval mean



Probability Bounds

Tucker, W. T. and Ferson, S. , Probability bounds analysis in environmental risk assessments, Applied Biomathematics, 2003. Mean = [20, 30], Standard deviation = 4, truncated at 0.5th and 99.5th.

Introduction- Uncertainty

What about functions of random variables?

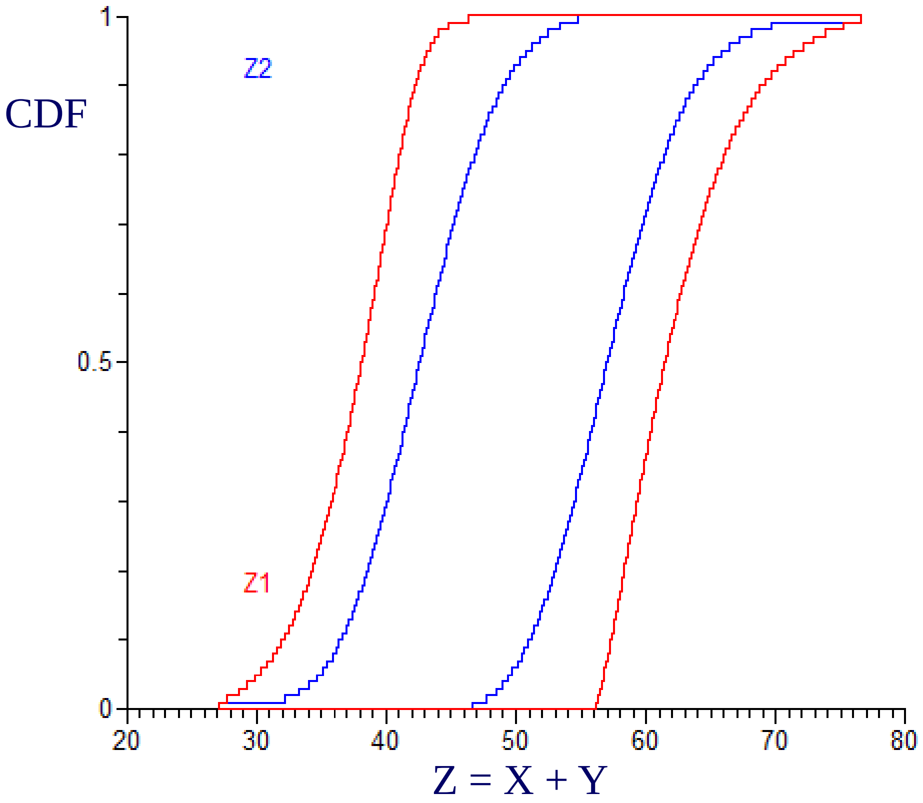
- If basic random variables are not all Gaussian, the probability distribution of the sum of two or more basic random variables may be not Gaussian.
- Unless all random variables are lognormally distributed, the products or quotients of several random variables may not be lognormal.
- More over, in the case when the function is a nonlinear function of several random variables, regardless of distributions, the distribution of the function is often difficult or nearly impossible to determine analytically.

Introduction- Uncertainty

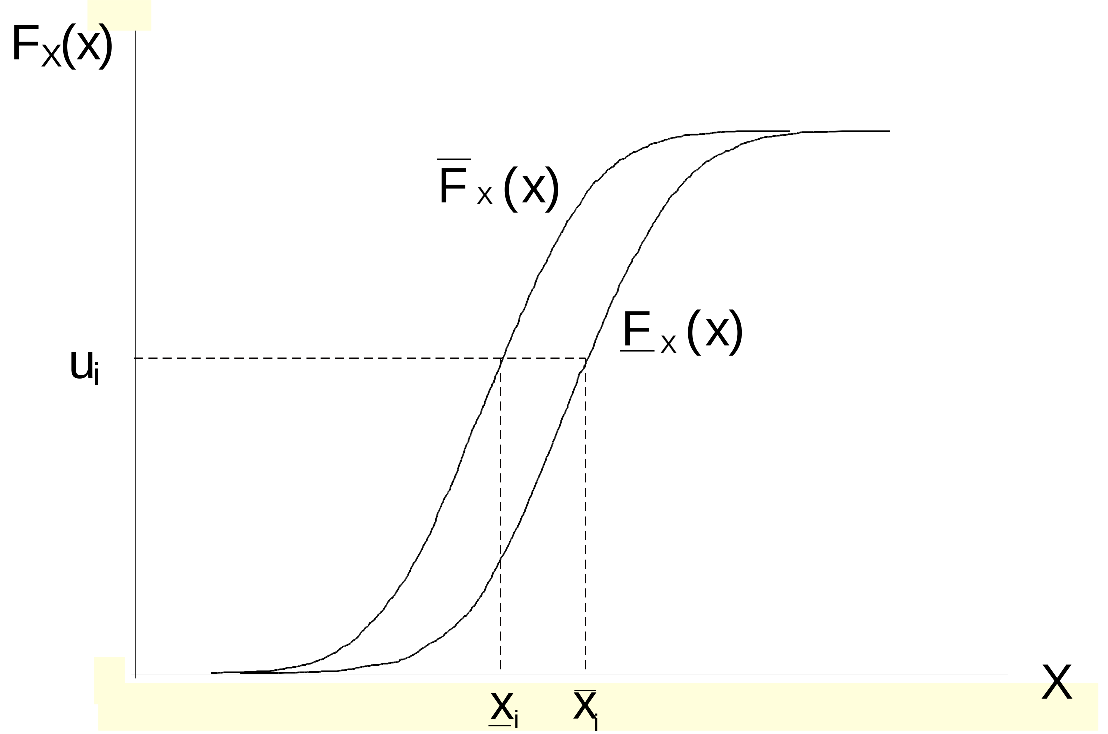
X: lognormal
mean = [20, 30]
sdv = 4

Y: normal
mean = [23, 27]
sdv = 3

Z1 = X + Y: any dependency
Z2 = X + Y: independent



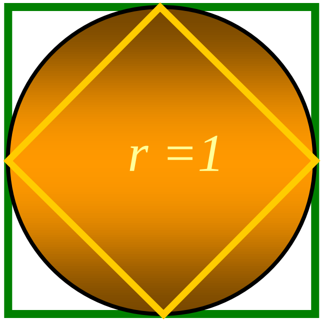
Introduction- Uncertainty



Zhang, H., Mullen, R. L. and Muhanna, R. L. “Interval Monte Carlo methods for structural reliability”, *Structural Safety*, Vol. 32,) 183–190, (2010)

Interval arithmetic – Background

- Archimedes (287 – 212 B.C.)
  - A circle of radius one has an area equal to π
  - $2 < \pi < 4$



$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

$\pi = [3.14085, 3.14286]$

Introduction- Interval Approach

- Only range of information (tolerance) is available

$$t = t_0 \pm \delta$$

- Represents an uncertain quantity by giving a range of possible values

$$t = [t_0 - \delta,\ t_0 + \delta]$$

- How to define bounds on the possible ranges of uncertainty?
  - experimental data, measurements, statistical analysis, expert knowledge

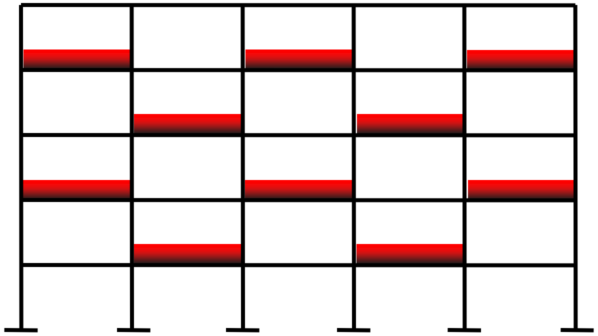
Introduction- Why Interval?

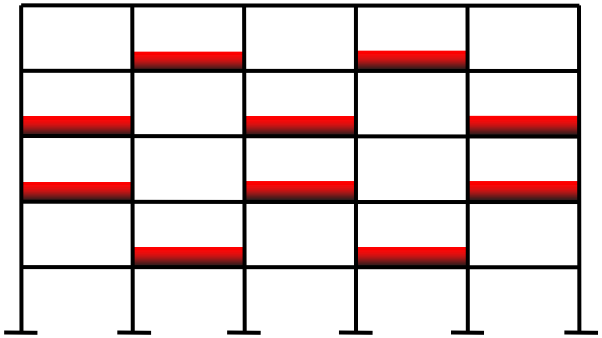
- Simple and elegant
- Conforms to practical tolerance concept
- Describes the uncertainty that can not be appropriately modeled by probabilistic approach
- Computational basis for other uncertainty approaches (e.g., fuzzy set, random set, probability bounds)

- **Provides guaranteed enclosures**

Examples- Load Uncertainty

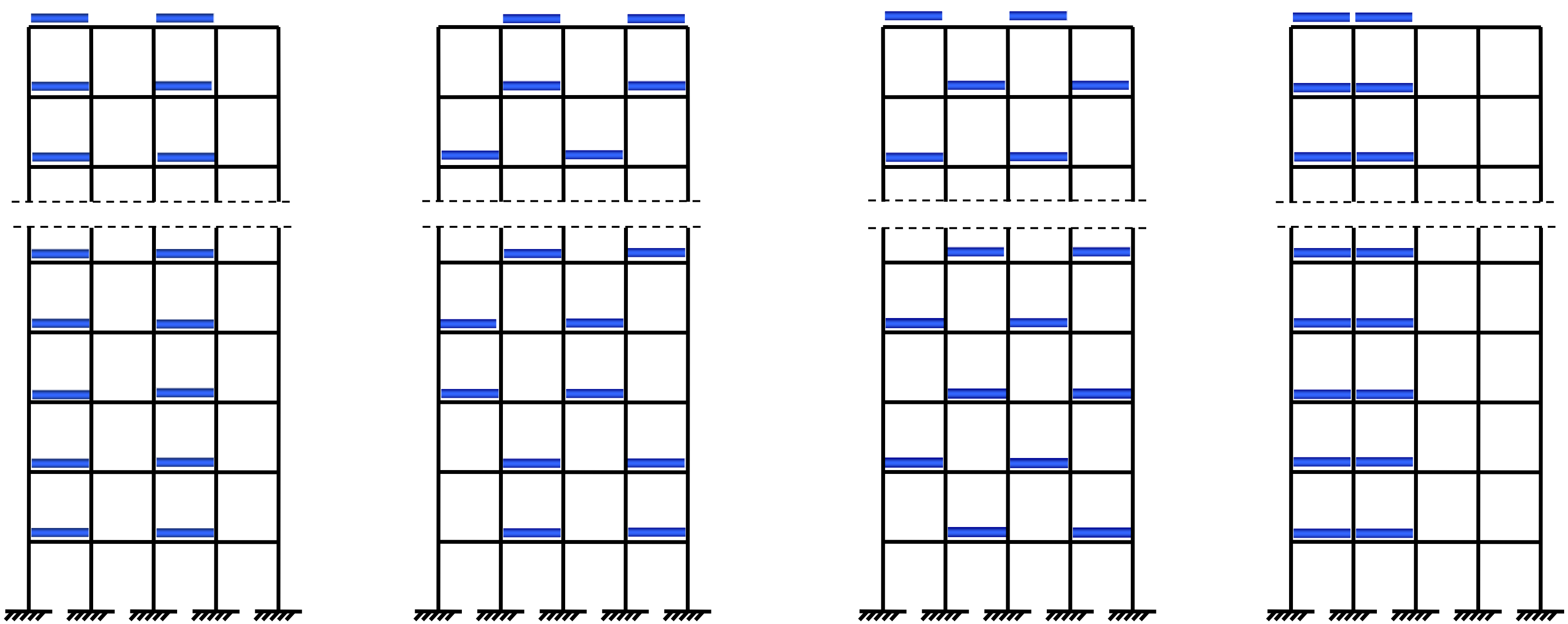
- Four-bay forty-story frame





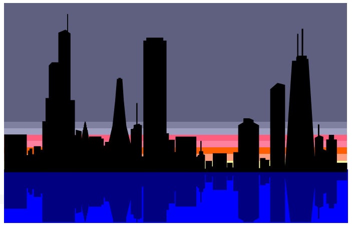
Examples- Load Uncertainty

- Four-bay forty-story frame



Loading A   Loading B   Loading C   Loading D

Examples- Load Uncertainty



- **Four-bay forty-story frame**

Total number of floor load patterns

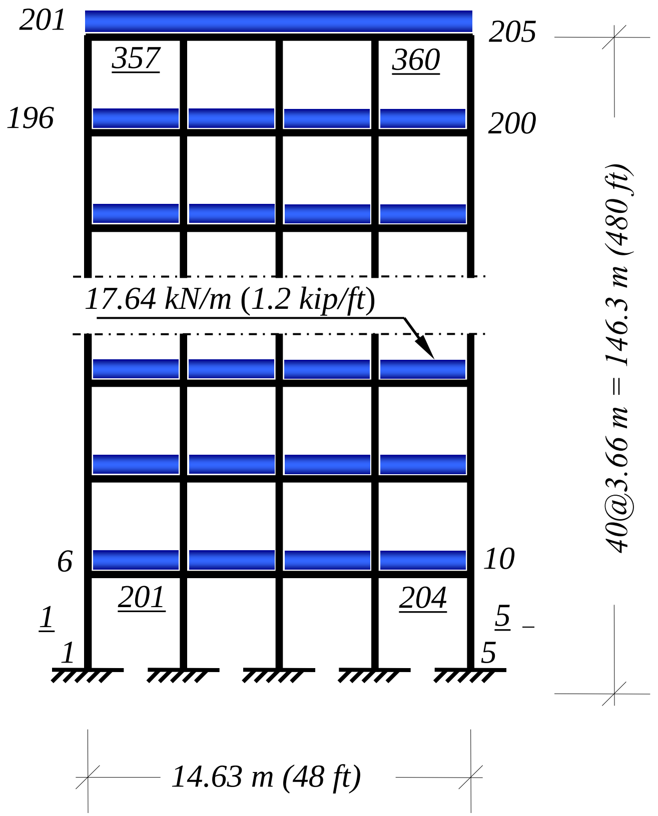
$$2^{160} = 1.46 \times 10^{48}$$

If one were able to calculate

10,000 *patterns / s*

there has not been sufficient time since the creation of the universe **(4-8)** billion years ? to solve all load patterns for this simple structure

Material *A36*, Beams *W24* x *55*, Columns *W14* x *398*



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Interval arithmetic

- Interval number represents a range of possible values within a closed set

$$\boldsymbol{x} \equiv [\underline{x}, \overline{x}] := \{x \in R \mid \underline{x} \le x \le \overline{x}\}$$

Properties of Interval Arithmetic

Let x, y and z be interval numbers

1. Commutative Law

$$x + y = y + x$$

$$xy = yx$$

2. Associative Law

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

3. ***Distributive Law does not always hold,*** *but*

$$x(y + z) \subseteq xy + xz$$

Sharp Results – Overestimation

- The ***DEPENDENCY*** problem arises when one or several variables occur more than once in an interval expression

  - $f(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}\,, \qquad\qquad \boldsymbol{x} = [1, 2]$
  - $f(\boldsymbol{x}) = [1-2,\ 2-1] = [-1, 1] \neq 0$
  - $\xcancel{f(\boldsymbol{x}, \boldsymbol{y}) = \{\, f(x, y) = x - y \mid x \in \boldsymbol{x},\ y \in \boldsymbol{y} \}}$
  - $f(\boldsymbol{x}) = \boldsymbol{x}\,(1-1) \qquad \Rightarrow \qquad f(\boldsymbol{x}) = 0$
  - $f(\boldsymbol{x}) = \{\, f(x) = x - x \mid x \in \boldsymbol{x} \}$

Sharp Results – Overestimation

- Let ***a***, ***b***, ***c*** and ***d*** be independent variables, each with interval [1, 3]

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a} & -\boldsymbol{b} \\ -\boldsymbol{c} & \boldsymbol{d} \end{pmatrix}, \quad A \times \boldsymbol{B} = \begin{pmatrix} [-2,2] & [-2,2] \\ [-2,2] & [-2,2] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B}_{phys} = \begin{pmatrix} \boldsymbol{b} & -\boldsymbol{b} \\ -\boldsymbol{b} & \boldsymbol{b} \end{pmatrix}, \quad A \times \boldsymbol{B}_{phys} = \begin{pmatrix} [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \\ [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B^{*}_{phys} = \boldsymbol{b} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A \times \boldsymbol{B}^{*}_{phys} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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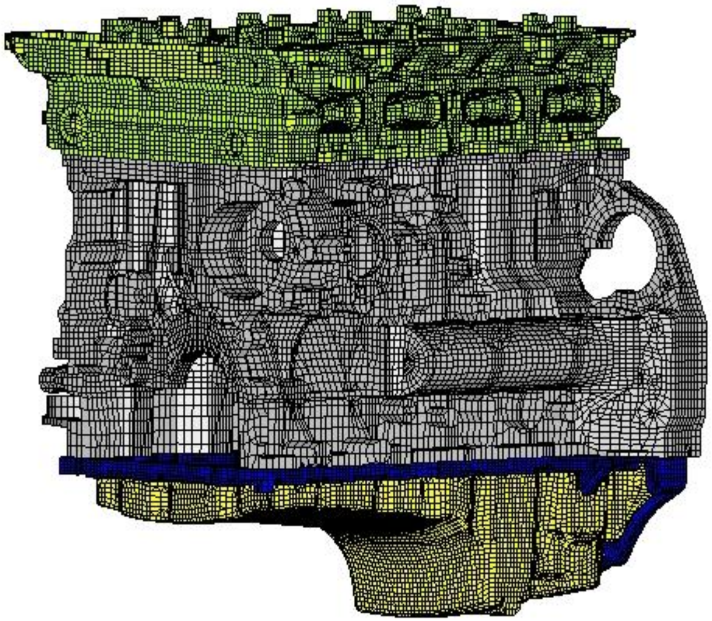
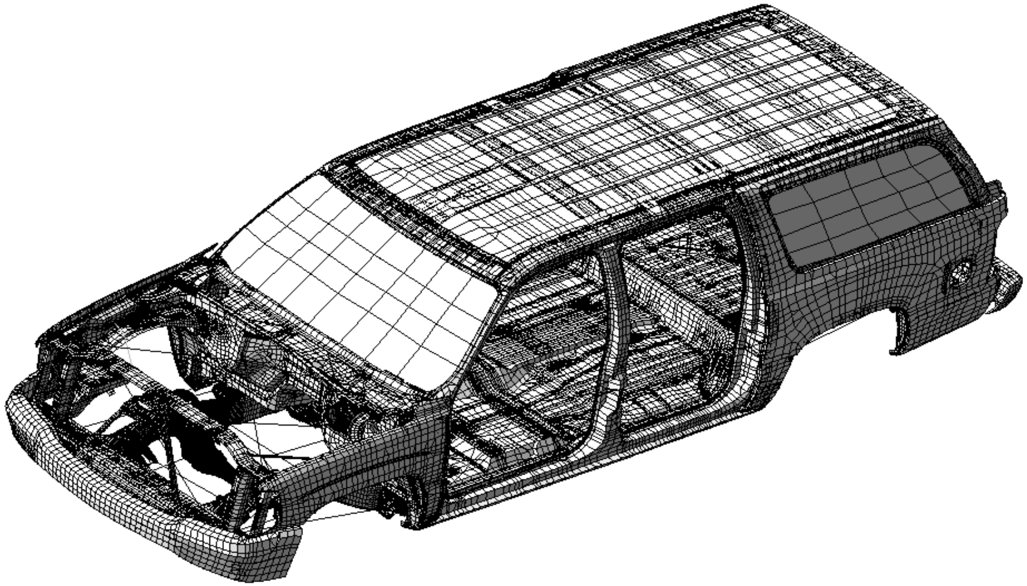
Finite Elements

Finite Element Methods (FEM) are numerical method that provide approximate solutions to differential equations (ODE and PDE)

Finite Elements

Finite Elements





Finite Element Model (courtesy of Prof. Mourelatous)

500,000-1,000,000 equations

Finite Elements- Uncertainty& Errors

- Mathematical model (validation)
- Discretization of the mathematical model into a computational framework (verification)
- Parameter uncertainty (loading, material properties)
- Rounding errors

Interval Finite Elements (IFEM)

- Follows conventional FEM
- Loads, geometry and material property are expressed as interval quantities
- System response is a function of the interval variables and therefore varies within an interval
- Computing the exact response range is proven NP-hard
- The problem is to estimate the bounds on the unknown exact response range based on the bounds of the parameters

FEM- Inner-Bound Methods

- Combinatorial method (Muhanna and Mullen 1995, Rao and Berke 1997)
- Sensitivity analysis method (Pownuk 2004)
- Perturbation (Mc William 2000)
- Monte Carlo sampling method
- Need for alternative methods that achieve
  - Rigorousness – guaranteed enclosure
  - Accuracy – sharp enclosure
  - Scalability – large scale problem
  - Efficiency

IFEM- Enclosure

- Linear static finite element
  - Muhanna, Mullen, 1995, 1999, 2001,and Zhang 2004
  - Popova 2003, and Kramer 2004
  - Corliss, Foley, and Kearfott 2004
  - Neumaier and Pownuk 2007
- Heat Conduction
  - Pereira and Muhanna 2004
- Dynamic
  - Dessombz, 2000
- Free vibration-Buckling
  - Modares, Mullen 2004, and Bellini and Muhanna 2005

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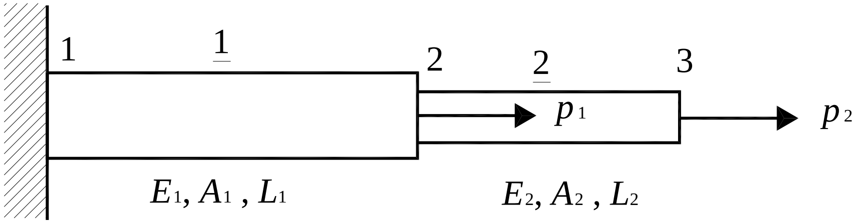
Overestimation in IFEM

- Multiple occurrences – element level
- Coupling – assemblage process
- Transformations – local to global and back
- Solvers – tightest enclosure
- Derived quantities – function of primary

Naïve interval FEA



$$E_1A_1/L_1 = \boldsymbol{k}_1 = [0.95,\ 1.05],$$
$$E_2A_2/L_2 = \boldsymbol{k}_2 = [1.9,\ 2.1],$$
$$p_1 = 0.5, \quad p_2 = 1$$

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Rightarrow \begin{pmatrix} [2.85,\ 3.15] & [-2.1,\ -1.9] \\ [-2.1,\ -1.9] & [1.9,\ 2.1] \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

- exact solution: $\boldsymbol{u}_2 = [1.429, 1.579]$, $\boldsymbol{u}_3 = [1.905, 2.105]$
- naïve solution: $\boldsymbol{u}_2 = [-0.052, 3.052]$, $\boldsymbol{u}_3 = [0.098, 3.902]$
- interval arithmetic assumes that all coefficients are independent
- response bounds are severely overestimated (up to 2000%)

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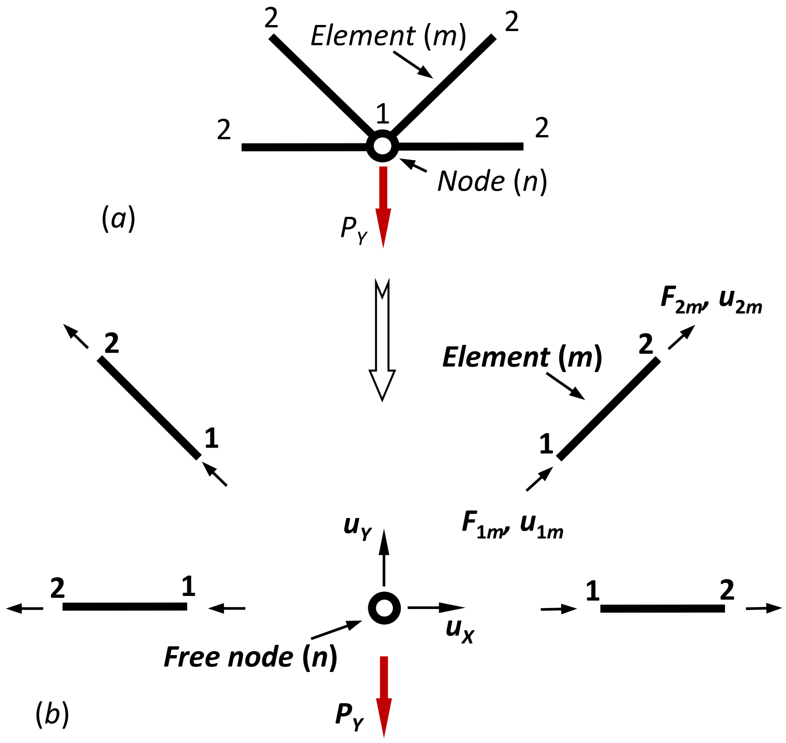
New Formulation



A typical node of a truss problem. (*a*) Conventional formulation. (*b*) Present formulation.

New Formulation

- Lagrange Multiplier Method

A method in which the minimum of a functional such as

$$I(u,v)=\int_a^b F(x,u,u',v,v')dx$$

with the linear equality constraints

$$G(u,u',v,v')=0$$

is determined

New Formulation

- Lagrange Multiplier Method

The Lagrange's method can be viewed as one of determining u, v and λ by setting the first variation of the *modified* functional

$$L(u,v,\lambda) \equiv I(u,v) + \int_a^b \lambda G(u,u',v,v')dx = \int_a^b (F + \lambda G)dx$$

to zero

New Formulation

- Lagrange Multiplier Method

The result is Euler Equations of the $L(u,v,\lambda) \equiv \int_a^b (F+\lambda G)dx$

$$\left.\begin{aligned} &\frac{\partial}{\partial u}(F+\lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial u'}(F+\lambda G)\right] = 0 \\ &\frac{\partial}{\partial v}(F+\lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial v'}(F+\lambda G)\right] = 0 \\ &G(u,u',v,v') = 0 \end{aligned}\right\}$$

from which the dependent variables *u*, *v*, and λ can be determined at the same time

New Formulation

In steady-state analysis, the variational formulation for a discrete structural model within the context of Finite Element Method (FEM) is given in the following form of the total potential energy functional when subjected to the constraints $CU = V$

$$\Pi^* = \frac{1}{2} U^T K U - U^T P + \lambda^T (CU - V)$$

New Formulation

Invoking the stationarity of Π^*, that is $\delta\Pi^* = 0$, we obtain

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} U \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ V \end{pmatrix}$$

In order to force unknowns associated with coincident nodes to have identical values, the constraint equation $CU=V$ takes the form $CU = 0$, and the above system will have the following form

New Formulation

$$\begin{pmatrix} \mathbf{k} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix}$$

or

$$\mathbf{KU} = \mathbf{P}$$

where

New Formulation

$$\mathbf{k} = \begin{pmatrix} \mathbf{k}_1 & -\mathbf{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{k}_1 & \mathbf{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}_n & -\mathbf{k}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{k}_n & \mathbf{k}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{1X} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{1Y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{mX} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{mY} \end{pmatrix}$$

$$\mathbf{k}_i = \frac{\mathbf{E}_i \mathbf{A}_i}{L_i}$$

New Formulation

$$\mathbf{u}_{1i} + \mathbf{u}_{jX} cos\varphi_i + \mathbf{u}_{jY} sin\varphi_i = 0$$

$$C^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ cos\,\varphi_1 & 0 & \cdots \\ sin\,\varphi_1 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & cos\,\varphi_1 & \cdots \\ 0 & sin\,\varphi_1 & \cdots \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} \mathbf{u}_{11} \\ \mathbf{u}_{21} \\ \vdots \\ \mathbf{u}_{1n} \\ \mathbf{u}_{2n} \\ \mathbf{u}_{1X} \\ \mathbf{u}_{1Y} \\ \vdots \\ \mathbf{u}_{mX} \\ \mathbf{u}_{mY} \end{pmatrix} \qquad \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_{11} \\ \boldsymbol{\lambda}_{21} \\ \vdots \\ \boldsymbol{\lambda}_{1n} \\ \boldsymbol{\lambda}_{2n} \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \mathbf{p}_{1X} \\ \mathbf{p}_{1Y} \\ \vdots \\ \mathbf{p}_{mX} \\ \mathbf{p}_{mY} \end{pmatrix}$$

New Formulation

- Iterative Enclosure (Neumaier 2007)

$$(K + B\,\mathbf{D}\,A)\mathbf{u} = a + F\,\mathbf{b}$$

$$\mathbf{v} = \{(ACa) + (ACF)\mathbf{b} + (ACB)\mathbf{d}\} \cap \mathbf{v}, \quad \mathbf{d} = \{(D_0 - \mathbf{D})\mathbf{v} \cap \mathbf{d}$$

$$\mathbf{u} = (Ca) + (CF)\mathbf{b} + (CB)\mathbf{d}$$

where

$$C := (K + BD_0A)^{-1}$$

$$\mathbf{u} = Ca + CF\mathbf{b} + CB\mathbf{d}$$

$$\mathbf{v} = ACa + ACF\mathbf{b} + ACB\mathbf{d}$$

$$\mathbf{d} = (D_0 - \mathbf{D})\mathbf{v}$$

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Numerical examples

- $Interval\ width = upper\ bound - lower\ bound$

- $Width\ error\% = \left(\frac{computed\ enclosure\ width}{exact\ enclosure\ width} - 1 \right) \times 100$

- $Bound\ error\% = \left(\frac{computed\ bound - exact\ bound}{exact\ bound} \right) \times 100$

Numerical examples

- Eleven bar truss

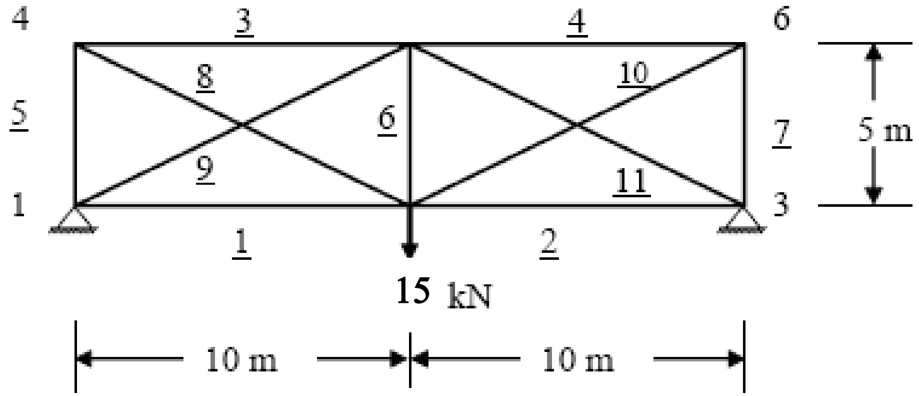
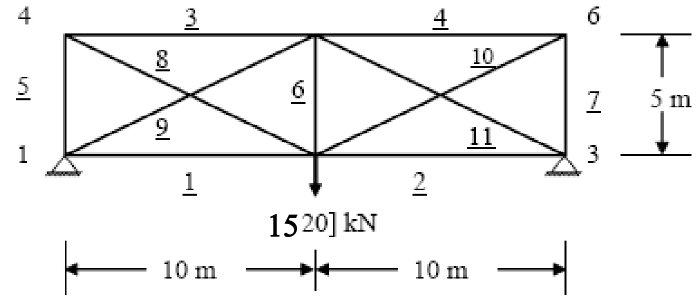


Table 2 Eleven bar truss -displacements for 12% uncertainty in the modulus of elasticity (E)						
	$V_2 \times 10^{-5}$		$U_4 \times 10^{-5}$		$V_4 \times 10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-15.903532	-14.103133	2.490376	3.451843	-0.843182	-0.650879
Krawczyk FPI	---	---	---	---	---	---
Neumaier's approach	-15.930764	-13.967877	2.431895	3.4943960	-0.848475	-0.633096
Error %(width)	9.02		10.50		11.99	
Present approach	-15.930764	-13.967877	2.431895	3.494396	-0.848475	-0.633096
Error %(width)	9.02		10.50		11.99	

Error in bounds%= 0.17 %



Numerical examples

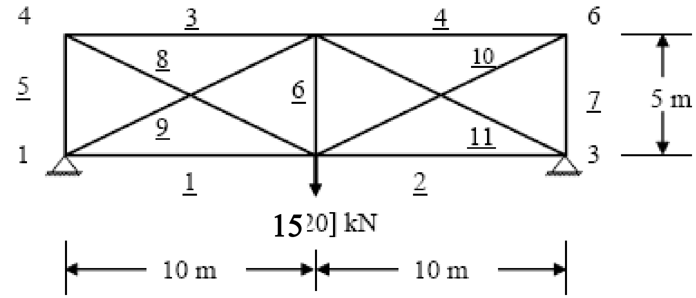
- Eleven bar truss

Table 4 Eleven bar truss - comparison of axial forces for 10% uncertainty in the modulus of elasticity (E) for various approaches

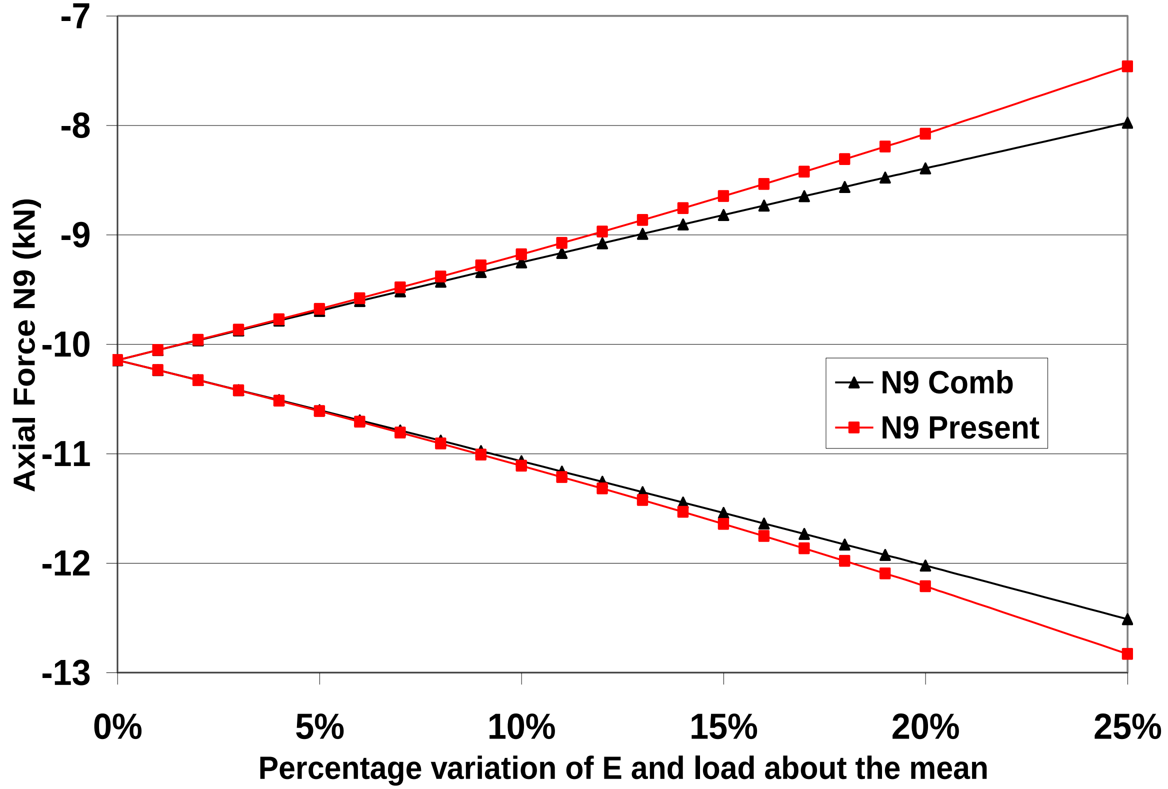
	$\underline{N}_3(kN)$	$\overline{N}_3(kN)$	$\underline{N}_9(kN)$	$\overline{N}_9(kN)$
Combinatorial approach	-6.28858	-5.57152	-10.54135	-9.73966
Simple enclosure $\mathbf{z}_1(\mathbf{u})$	-7.89043	-3.96214	-11.89702	-8.39240
Error %(width)	447.83		337.15	
Intersection $\mathbf{z}_2(\mathbf{u})$	-6.82238	-5.08732	-11.32576	-9.02784
Error %(width)	141.97		186.63	
Present approach	-6.31656	-5.53601	-10.58105	-9.70837
Error %(width)	8.85		8.85	

Error in bounds%= 0.45 %

Numerical examples

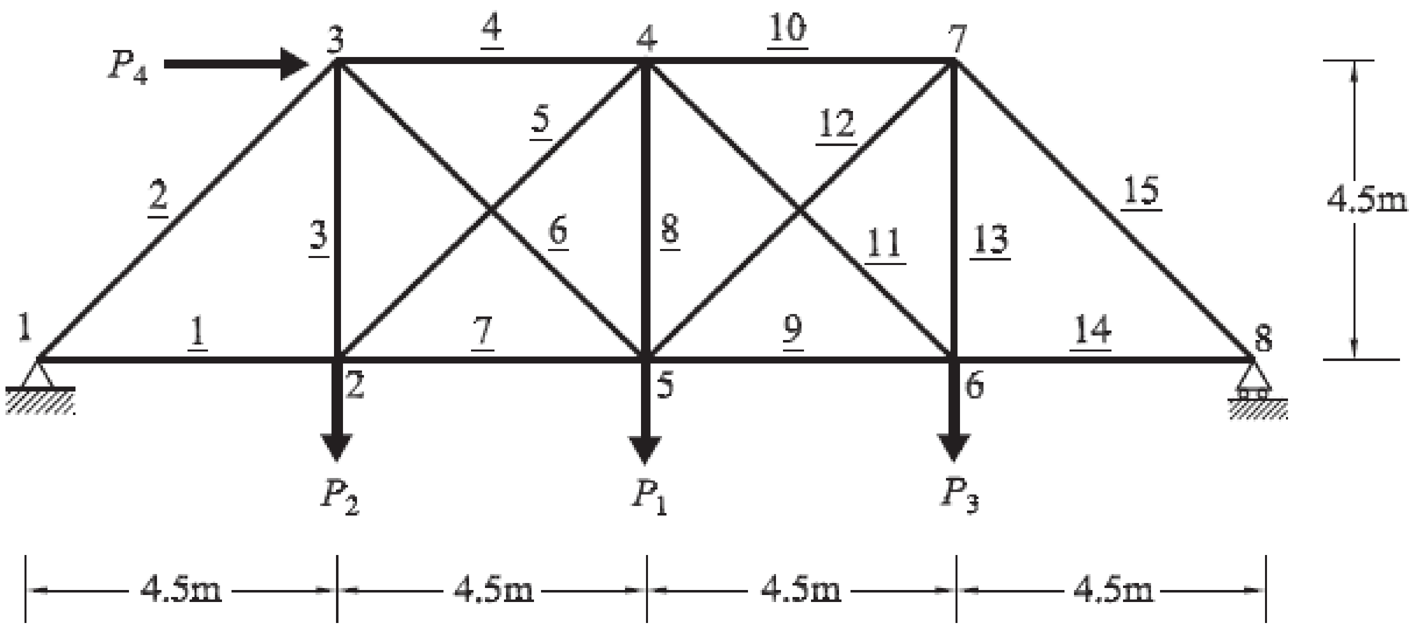


- Eleven bar truss – Bounds on axial forces

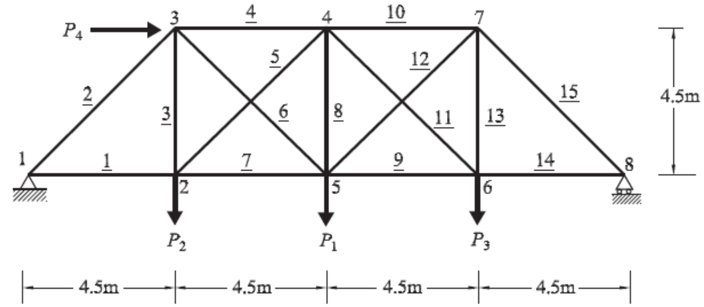


Numerical examples

- Fifteen bar truss – Bounds on axial forces



Numerical examples

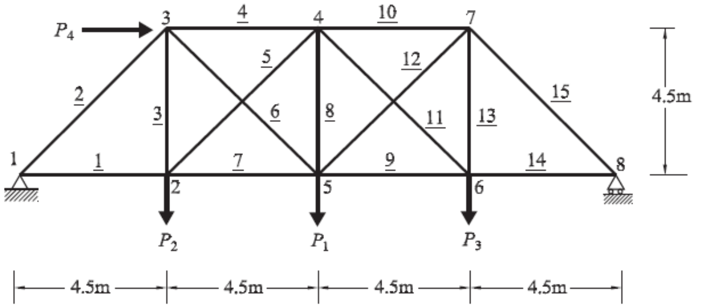


- Fifteen bar truss – Bounds on axial forces

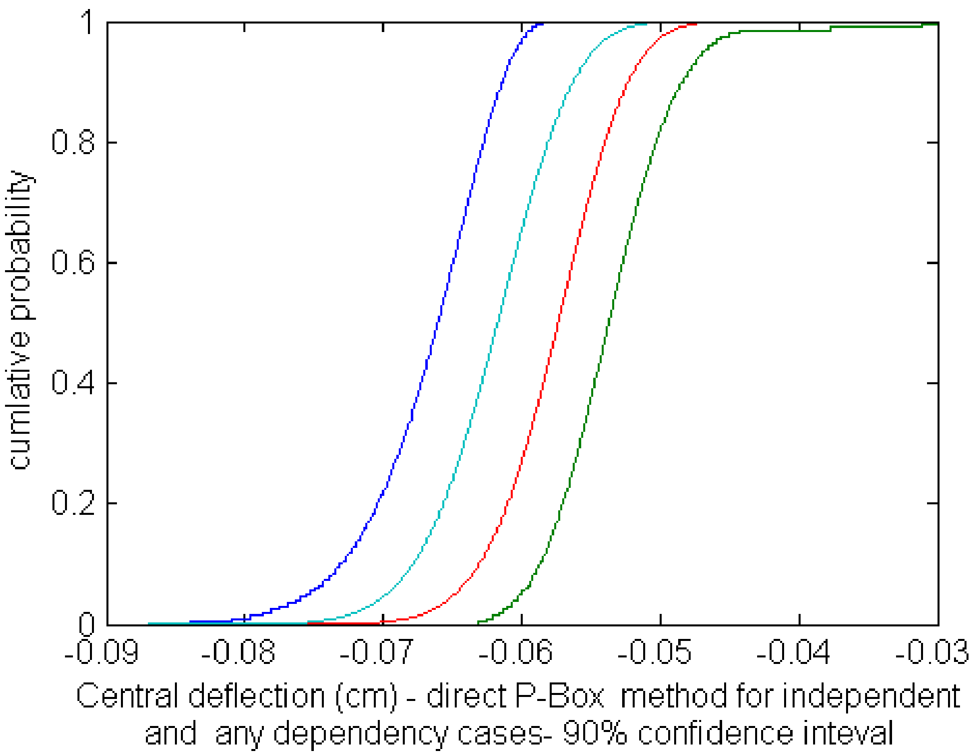
Table 12 Forces (kN) in elements of fifteen element truss for 10% uncertainty in modulus of elasticity (E) and load

Element	Combinatorial approach		Neumaier's approach		%Error in width	Present approach		%Error in width
	LB	UB	LB	UB		LB	UB	
1	254.125	280.875	227.375	310.440	210.53	254.125	280.875	0.000
2	-266.756	-235.289	-294.835	-210.187	169.01	-266.756	-235.289	0.000
3	108.385	134.257	95.920	148.174	101.97	107.098	134.987	7.797
4	-346.267	-302.194	-379.167	-272.461	142.12	-347.003	-300.909	4.585
5	-43.854	-16.275	-48.143	-12.985	27.48	-44.975	-14.543	10.344
14	211.375	233.625	189.125	258.217	210.53	211.375	233.625	0.000
15	-330.395	-298.929	-365.174	-267.463	210.53	-330.395	-298.929	0.000

Numerical examples



- Fifteen bar truss–Probability Bounds on mid-span displacement



Conclusions

- Development and implementation of IFEM
  - uncertain material, geometry and load parameters are described by interval variables
  - interval arithmetic is used to guarantee an enclosure of response
- Derived quantities obtained at the same accuracy of the primary ones
- The method is generally applicable to linear and nonlinear static FEM, regardless of element type
- IFEM forms a basis for generalized models of uncertainty in engineering

Center for Reliable Engineering Computing (REC)

We handle computations with care